HW#2 — Math 639 — Nonlinear Waves — Fall 2022

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1. Consider the nonlinear wave equation:

$$u_t + c(u)\, u_x = 0, \quad \text{with} \quad c(u) = \alpha\, u^n. \tag{1}$$

Repeat what we did in class to solve for the evolution of an initial wave $u(x,0) = A\, \exp(-x^2)$ for general n. Specifically:

a) Solve Eq. (1) using the method of characteristics.

b) Find the general solution for the breaking time t_B as a function of the initial x-position ξ.

c) Find the minimum breaking time, $t_{\min}$, for the whole wave.

d) For $n = 2$ and $\alpha = 1$ give the expressions for t_B and $t_{\min}$.

e) For $n = 2$ and $\alpha = 1$ plot the solution $u(x,t)$ in the (x,t) plane. [Hint: you might want to use the code `characteristics.m`.]

f) For $A = 1$, $n = 2$ and $\alpha = 1$ modify the code `charac_example_pde.m` to integrate the solution and to show that the $t_{\min}$ found in c) is correct.

g) Do the same as in d)–f) for $n = 3$ and $\alpha = 1$. Contrast the differences/similarities between $n = 2$ and $n = 3$ and explain them using physical arguments. In particular, which case ($n = 2$ or $n = 3$) breaks earlier for $A = 1$. What if $A < 1$ or $A > 1$. Can you find a value for A such that the two cases break at the same time?

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2. This exercise aims to analyze some generic aspects of the KdV equation:

$$\frac{1}{c} u_t + u_x + \frac{3}{2h} u u_x + \frac{h^2}{6} u_{xxx} = 0, \tag{2}$$

where $u(x,t)$ is the vertical displacement of the wave surface from its resting level, c is the speed of plane waves in the absence of nonlinearity and dispersion, and h is the resting depth of the water canal.

a) By the following transformations: $\xi = x - ct$, $\tilde{u} = \beta u$ and $y = \alpha \xi$, prove that the KdV equation (2) reduces to

$$\tilde{u}_t + 6\tilde{u}\tilde{u}_y + \tilde{u}_{yyy} = 0. \tag{3}$$

Give the explicit form for β and α in terms of h and c.

b) Prove that if $f(x - vt)$ is a solution for the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{4}$$

then $\lambda + f(x - \tilde{v}t)$ is also a solution of the KdV equation. Give the explicit expression for $\tilde{v}$ as a function of v and λ. Let us check numerically this invariance by taking an initial profile $u(x,0) = u_0(x)$ in the form of a `sech` solitary wave solution to the KdV and its transformed version $u(x,0) = \lambda + u_0(x)$ using some value for λ (for example $\lambda = 1$). Integrate *numerically* both of these initial conditions using the code `KdV.m`. Do you get what you expected? Elaborate.

c) Show that the superposition principle is not valid for the KdV. I.e. show that if $f(x,t)$ and $g(x,t)$ are general solutions to (4) then $f + g$ is *not* a solution. Are there any particular cases when $f + g$ is indeed a solution (where f and g are solutions) for special choices of f and g (or combinations thereof)?