HW#3 — Math 639 — Nonlinear Waves — Fall 2022

For this assignment, you might need the following integrals:

$$\int \frac{df}{f\sqrt{B - Af^n}} = \frac{-2}{n\sqrt{B}} \tanh^{-1}\left(\frac{\sqrt{B - Af^n}}{\sqrt{B}}\right),$$

and

$$\int \frac{df}{f(f - A)} = \frac{-2}{A} \tanh^{-1}\left(\frac{2f - A}{A}\right).$$

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1. The aim of this exercise is to use the traveling wave reduction $[u(x,t) = f(x - ct)]$ to find *all* qualitatively different solutions (using Newton's equations of motion approach and its corresponding effective potential) and then also find analytically the exact localized solutions satisfying the given conditions.

a) Modified KdV equation (mKdV)

$$u_t + 6u^2 u_x + u_{xxx} = 0.$$

(i) Identify (i.e., plot), in phase space [f' vs. f] and profiles [$f(\xi)$ vs. ξ], *all* qualitatively different solutions using Newton's equations of motion approach and its corresponding effective potential.
(ii) For the boundary conditions: $u(\pm\infty, t) = 0$ and $u_x(\pm\infty, t) = u_{xx}(\pm\infty, t) = 0$, find the exact solution.

b) Generalized KdV equation

$$u_t + (n+1)(n+2)u^n u_x + u_{xxx} = 0, \quad \text{with} \quad n = 1, 2, \ldots$$

(i) Only for $n = 3$: identify (i.e., plot), in phase space [f' vs. f] and profiles [$f(\xi)$ vs. ξ], *all* qualitatively different solutions using Newton's equations of motion approach and its corresponding effective potential.
(ii) For the boundary conditions: $u(\pm\infty, t) = 0$ and $u_x(\pm\infty, t) = u_{xx}(\pm\infty, t) = 0$, find the exact solution for *arbitrary* n.

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2. Consider a KdV-mKdV equation:

$$u_t + 6uu_x + 12\,\delta\,u^2 u_x + u_{xxx} = 0. \tag{1}$$

with fixed $\delta \in \mathbb{R}$. Using geometrical arguments and the phase-portrait method (together with appropriate bounds for the traveling speed c and the boundary conditions) show that:

a) Periodic solutions exist for any value of δ.

b) For $\delta < 0$ solitary waves *and* kink solutions exist.

c) For $\delta > 0$ solitary waves exist but kink solutions do *not* exist.

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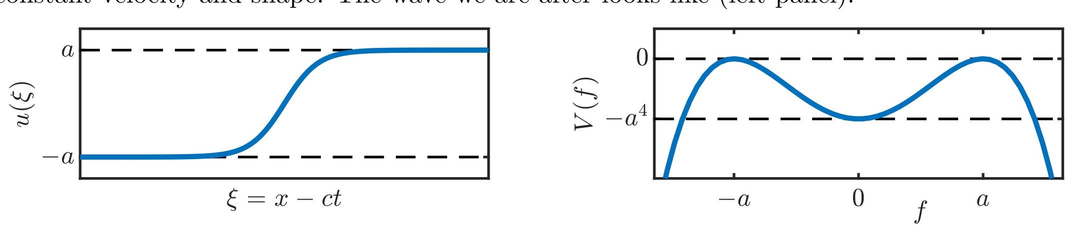
3. The aim of this exercise is to find a simple wave equation PDE model that propagates a *kink* solution with constant velocity and shape. The wave we are after looks like (left panel):



The method consists on defining a potential that generates a *heteroclinic* orbit connecting $\mp a$ with $\pm a$.

a) Consider the *quartic* potential $V(f)$ depicted on the right plot above.

   i) Sketch the phase portrait associated with $V(f)$.

   ii) Sketch all the qualitatively different orbits associated with $V(f)$.

   iii) Find the *simplest* quartic potential that satisfies the sketch in the right panel. Namely, find the coefficients A and B such that $V(f) = -f^4 + Af^2 + B$ passes through the particular points (with the right slopes) depicted in the right panel.

b) Consider now a particle of mass $m = 2$ subject to the potential $V(f)$ found in a.iii).

   i) Write the equation of motion using Newton's law ($mf'' = -dV/df$) for a particle of mass $m = 2$ at position f and time ξ under the potential $V(f)$ (i.e. f plays the role of x and ξ the role of t).

   ii) Differentiate this equation with respect to ξ to obtain a third order differential equation for f. Write a wave equation for $u(x,t) = f(\xi) = f(x - ct)$ with $c = 2a^2$ that leads to this third order differential equation for $f(\xi)$. Describe the terms you obtain (dissipation, dispersion, nonlinearity, etc..).

c) Now that you have the equation from b.ii), apply a similar analysis to the one done in class to obtain a kink solution connecting $u(-\infty, t) = -a$ and $u(+\infty, t) = +a$. Namely, you'll now check that the PDE you constructed does indeed support traveling kinks (as in the left panel of the figure). [Hint: every time you integrate w.r.t. ξ there is a constant that has to be determined by the boundary conditions. This is crucial to get the right solution. What are the boundary conditions for $f''(\pm\infty)$ and $f'(\pm\infty)$ and $f(\pm\infty)$?].

d) Check that the solution you obtained in c) does solve the wave equation obtained from b.ii).

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4. Consider the Boussinesq equation:

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0. \tag{2}$$

Note that there is a double time derivative and that $(u^2)_{xx} \neq (u_{xx})^2$. The Boussinesq equation is the analogue of the bidirectional wave equation for the KdV equation. Namely, the Boussinesq equation, supports wave propagation in both directions.

a) Follow the same methodology that we used in class for the KdV to find solitary wave solutions to the Boussinesq equation.

b) Show that these solitary waves can travel in both directions.