HW#4 — Math 639 — Nonlinear Waves — Fall 2022

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1. The aim of this exercise is to find conservation laws for the KdV and mKdV equations.

a) For the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

use the following combination:

$$3u^2 \cdot \mathrm{KdV} - u_x \cdot (\mathrm{KdV})_x$$

to show that the energy

$$E(t) \equiv \int_{-\infty}^{+\infty} \left(u^3 - \frac{1}{2} u_x^2 \right) dx = \text{const},$$

is conserved. In the expression for the energy, the first term corresponds to the potential energy and the second one to the kinetic energy.

b) Use the code `KdV.m` for a few different initial conditions (a soliton, a cnoidal wave, and a "disordered" solutions [use the superposition of several, random, planes waves], etc.) to numerically show that indeed (i) the mass $M(t) = \int_{-\infty}^{+\infty} u(x,t)\,dx$, (ii) the momentum $P(t) = \int_{-\infty}^{+\infty} u^2(x,t)\,dx$, and (iii) the energy $E(t)$ (as defined above), are conserved. Use a few values of the integration step (`dt`) and grid size (`dx`) and comment on your results and how they vary as you change `dx` and `dt`.

Hints:

- Always, when trying to numerically check the constancy of a quantity $Q(t)$ you should plot, instead of $Q(t)$, the *relative error* of the conservation: $\Delta Q(t) = [Q(t) - Q(0)]/Q(0)$ vs. t.
- Note that your numerics and on a *finite* domain and, thus, the integrals above will be over your numerical domain and not over the whole real line.
- To compute the integrals at each time you can do something like: `mass = sum(u)*dx` for the mass in Matlab and something similar for the other conserved quantities.
- For the conservation of energy you need to compute u_x. To do this, you can get inspired by the part of the code in `PDE_integrator_higher_order.m` that computes spatial derivatives using finite differences.

c) For the modified KdV equation (mKdV) equation:

$$u_t + (n+1)(n+2)u^n u_x + u_{xxx} = 0.$$

Use the a similar treatment as the one done in class to find **two** independent conserved quantities.

d) For extra credit: Do the same as in b) for the two conserved quantities that you found in c).

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2. The aim of this exercise is to find approximate plane wave solutions to the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

a) Suppose that $u(x,t) = f(\xi) = f(x - vt)$ is a traveling wave solution for the KdV equation (1). Following the lecture notes, obtain a Newton's law-like equation for $f(\xi)$. Keep the integration constants intact in order to obtain the most general solution.

b) For this Newton's law-like equation, find the fixed points and study their stability. Perform a perturbation analysis around the neutrally stable fixed point f^*. Namely, consider $f(\xi) = f^* + \varepsilon g(\xi)$ with $|\varepsilon| \ll 1$. Write the differential equation for $g(\xi)$.

c) Suppose ε is small enough to neglect nonlinear terms. Write the linearized differential equation for $g(\xi)$ and solve it. Use this to find the oscillatory solution for f about the fixed point f^*. Write the approximate evolution for these waves.

d) Use `KdV.m` to integrate the approximate plane wave solution obtained in c) with amplitude=0.01 and $v = 2$ and the integration constant(s) equal to zero for $(x,t) \in [0,3\lambda] \times [0,5]$ (where λ is the wave length of your wave). Show all your plots. Does the wave evolve as expected (shape- and velocity-wise)? Perform the same simulation with amplitude=0.1 and amplitude=0.5. Notice any difference? Comment on your results. [Note: if the codes goes unstable (i.e. loads of spikes forming) reduce the time step size `dt`.]

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3. Consider the two-dimensional KdV equation:

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{2}$$

a) Show, using the traveling wave approach $[u(x,y,t) = f(\xi)$ with $\xi = x + by - ct]$, that the two-dimensional KdV equation has the solitary wave solutions of the form:

$$u(x,y,t) = \frac{k^2}{2} \operatorname{sech}^2\left[\frac{1}{2}(kx + ly - \omega t)\right],$$

with $\omega = k^3 + 3l^2/k$.

b) Describe the evolution of such a solitary wave. What does it represent?