HW#5 — Math 639 — Nonlinear Waves — Fall 2022

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1. For the nonlinear repulive/defocusing Schrödinger equation:

$$iu_t + \frac{1}{2}u_{xx} - |u|^2 u = 0, \tag{1}$$

find the general form for a dark/grey soliton solution:

$$u(x,t) = u_0 \left\{B\, \tanh[u_0 B(x - Au_0 t - x_0)] + iA\right\} e^{-iu_0^2 t}, \tag{2}$$

where $A^2 + B^2 = 1$.

Hint: follow the notes where we split u as:

$$u(x,t) = [f(b\xi) + i\mathcal{A}]\, e^{i\phi(t)}, \tag{3}$$

where $\mathcal{A} = u_0\, A$, $\xi = x - ct$, and f and ϕ are real functions, and you will assume that $u(x = \pm\infty, t) = \pm u_0$.

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2. In this exercise you will obtain the basic conservation laws for the Nonlinear Schrödiger (NLS) equation:

$$iu_t + \frac{1}{2}u_{xx} \pm |u|^2 u = 0, \tag{4}$$

where $u = u(x,t)$ and $|u|^2$ describes a) the light intensity in nonlinear optics, or b) the atomic density in a Bose-Einstein condensate (BEC).

a) Show that the total power (optics) or mass (BEC), for localized solutions, is conserved. Namely, show that

$$M = \int_{-\infty}^{\infty} |u(x,t)|^2\, dx = \text{const.} \tag{5}$$

Hint: use $(4) \cdot u^* - (4)^* \cdot u$, where $(\cdot)^*$ means complex conjugation.

b) Show that the total energy is conserved. Namely, show that

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2}|u_x|^2 \mp \frac{1}{2}|u|^4\right) dx = \text{const.} \tag{6}$$

Hint: use $(4) \cdot u_t^* + (4)^* \cdot u_t$.

c) [**extra credit**] Using a couple of different initial conditions (ICs) that are not just constants nor single bright nor dark solitons (you can try a superposition of a few bright solitons or a sum of a few sines and cosines), numerically integrate the NLS and plot a time series of the above conserved quantities ($M(t)$ and $E(t)$) to verify they are conserved. Please also plot the relative error of the conservation (cf. $(M(t) - M(0))/M(0)$ and $(E(t) - E(0))/E(0)$) and discuss your results. You can use the spectral code `NLS.m` for this but remember that ICs have to be periodic in the domain!

d) [**extra-extra credit**] Show that the momentum is conserved. Namely, show that

$$P = i \int_{-\infty}^{\infty} (uu_x^* - u^* u_x)\, dx = \text{const.} \tag{7}$$

Hint: use $\{(4) \cdot u_x^* + (4)^* \cdot u_x\} - \{[(4)]_x \cdot u^* + [(4)^*]_x \cdot u\}$.