HW#6 — Math 639 — Nonlinear Waves

---

1. Galilean boost for NLS

In this exercise we will see how *any* solution to the nonlinear Schrödinger (NLS) equation can be "kicked" and made to travel at any arbitrary speed. Let us start with an *arbitrary* solution, $u(x,t)$, to the NLS equation:

$$iu_t + \frac{1}{2}u_{xx} \pm |u|^2 u = 0, \tag{1}$$

where the $\pm$ corresponds to, respectively, an attractive and a repulsive nonlinearity. We now "kick" (or boost) $u(x,t)$ with a plane wave as follows:

$$v(x,t) = u(x,t)\, e^{i(kx-\omega t)}. \tag{2}$$

By forcing $v(x,t)$ to be a solution to the NLS in a co-moving reference frame of velocity c:

$$i(v_t + cv_x) + \frac{1}{2}v_{xx} \pm |v|^2 v = 0, \tag{3}$$

find the dependence of $c = c(k)$ and $\omega = \omega(k)$ on k. The above means that by initially "kicking" (i.e., multiplying) a solution by the term e^{ikx}, one obtains a solution that travels at velocity $c = c(k)$ and whose temporal frequency has to be adjusted by the term $e^{-i\omega t}$ where $\omega = \omega(k)$.

---

2. Oscillations of an NLS soliton inside a potential

This exercise aims at approximating the evolution of a bright NLS soliton trapped in a confining potential by exclusively using conserved quantities. As a reference please consult R. Scharf and A.R. Bishop, Phys. Rev. E **47** (1993) 1375 [Sections I–III].

Consider the self-focusing NLS with an external potential:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = V(x)\, u, \tag{4}$$

where the external potential $V(x)$ can represent (a) a transverse modulation of the index of refraction in an optical fiber trapping the optical beam or (b) an external magnetic or optical potential trapping a cloud of Bose-Einstein condensate (BEC) atoms. Consider that the evolution of a bright soliton placed inside the trap does not deviate too much from the shape of a standard NLS bright soliton. Then, one can approximate the shape of the evolving soliton by the following ansatz:

$$u(x,t) = a(t)\,\mathrm{sech}[a(t)\,(x-\xi(t))]\ \exp[i(c(t)\cdot x + d(t)\cdot t)], \tag{5}$$

where now the amplitude $a(t)$, position $\xi(t)$ and velocity, $c(t) = \dot{\xi}(t)$, and the so-called chirp, $d(t)$, are allowed to vary in time to accommodate for the perturbations in shape induced by the presence of the external potential.

A) Consider the following external potential

$$V(x) = -A\cos\left(\frac{2\pi}{\lambda}x\right). \tag{6}$$

This potential arises very naturally in a BEC trapped by a periodic potential induced by the interference pattern of two counter-propagating laser beams, the so-called optical lattice. By using the conservation of mass and the conservation of energy, that in this case with the additional trapping potential is:

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2}|u_x|^2 - \frac{1}{2}|u|^4 + V(x)\,|u|^2\right) dx = \text{const.} \tag{7}$$

perform the following tasks:

(a) Prove that the mass $M = \int_{-\infty}^{\infty} |u|^2\,dx$ and the energy (7) are indeed conserved quantities of the NLS when the external potential $V(x)$ is included [see Eq. (4)]. [Hint: use the results of previous HW exercises].

(b) Using the conservation of M and E, with the asantz (5) for the bright soliton, find an ordinary differential equation (ODE) describing the motion of the center of the soliton. What does this equation represent? (Hint: Newton's law with effective potential, compare external to effective potential, elaborate). What about the relations obtained for $a(t)$ and $d(t)$? (What do they represent? Elaborate.)

Hint: you'll probably need the following integrals:

$I_1 = \int_{-\infty}^{\infty} a^2 \text{sech}^2[a(x-\xi)]\cos(kx)\,dx = \dfrac{k\pi\cos(k\xi)}{\sinh(k\pi/(2a))}$, $I_3 = \int_{-\infty}^{\infty} \text{sech}^2(ax)\,dx = \dfrac{2}{a}$,

$I_2 = \int_{-\infty}^{\infty} \text{sech}^2(ax)\,\tanh(ax)^2\,dx = \dfrac{2}{3a}$, $I_4 = \int_{-\infty}^{\infty} \text{sech}^4(ax)\,dx = \dfrac{4}{3a}$.

(c) What is the dynamics of a trapped soliton centered at $\xi_0 = \xi(t=0)$ with zero initial velocity, $c_0 = c(t=0) = 0$?

(d) If a soliton initially centered at $\xi_0 = 0$ is given an initial velocity $c_0 = c(t=0) > 0$, what is the the critical velocity (i.e. escape velocity), c_e, such that the soliton permanently leaves the central valley of the trap?

(e) Use your previous results to compare them with results from directly integrating the NLS. Please show several examples including: $(\xi_0, c_0) = (0,0)$, $(\xi_0, c_0) = (0, c_e/2)$, $(\xi_0, c_0) = (\lambda/2, 0)$, $(\xi_0, c_0) = (0, c_e)$ (and slight above and below c_e), $(\xi_0, c_0) = (0, c_0)$ with $c_0 = 2c_e$. Please perform the above for the following six setup combinations: $\lambda = 2$, $a(0) = 4, 8$ and $A = 0.01, 0.1, 0.5$. Please be critical about the comparison between the ODE approximation and the full PDE runs. Elaborate.

B) Consider now the following external potential

$$V(x) = A\tanh^2(\lambda x). \tag{8}$$

Try to repeat an analysis as the one above. Can you obtain an ODE in closed form? Even if you are not successful in obtaining a closed form for the ODE, perform several direct numerical experiments on NLS to study the behavior of the trapped soliton for at least 4 orbits: $(\xi_0, c_0) = (0,0)$, $(\xi_0, c_0) = (0, c_0)$ such that $c_0 < c_e$, $(\xi_0, c_0) = (0, c_0)$ such that c_0 is close to c_e, $(\xi_0, c_0) = (0, c_0)$ with c_0 larger that c_e.