HW#7 — Math 639 — Nonlinear Waves

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1. NLS with general power nonlinearity
Consider the following NLS equation with general nonlinearity:

$$iu_t = -\frac{1}{2}u_{xx} - |u|^{2\sigma}u,$$

where σ is a constant.

(a) Show that a standing wave solution $u(x,t) = f(x)\,e^{it}$ exists for arbitrary values of σ, acquiring the explicit form:

$$f(x) = (\sigma+1)^{1/(2\sigma)}\mathrm{sech}^{\frac{1}{\sigma}}(\sqrt{2}\sigma x). \tag{1}$$

(b) Modify the codes given in class/webpage to indeed check that above solution does behave as you expect. Do this for a handful of σ values. Explain your results.

(c) Can you extend solution (1) for a general frequency ω such that $u(x,t) = f(x)\,e^{i\omega t}$?

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2. NLS with saturation
Consider the following NLS equation with saturable nonlinearity:

$$iu_t + \frac{1}{2}u_{xx} + \frac{u|u|^2}{1+S|u|^2} = 0,$$

where $S > 0$.

(a) Seek solutions $u(x,t) = f(x)\,e^{i\omega t}$, with $\omega > 0$ and write down the ODE for $f(x)$.

(b) Write down the solution of the corresponding ODE for $f(x)$ by quadrature (do not attempt to evaluate the integral), i.e. the techniques we have used in class to solve the ODE resulting from the traveling wave reduction.

(c) Considering the relevant potential, identify a condition so that it can support homoclinic orbits. Under that condition, sketch various orbits of the system and their corresponding phase space representation.

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3. Redo all 1D theory *and* numerics of the paper: Kevrekidis et al. **70** (2004) 023602. [PDF]

Extra credit: Redo the 2D results as well.

Hints: Since we have not seen yet how to obtain the exact solutions numerically, when you feed the initial condition (IC) as the Thomas-Fermi (TF) approximation:

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u = real(sqrt((mu-((Omega^2)/2)*x.^2)))
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in the NLS with repulsive nonlinearity you'll find that the solution will oscillate a bit since you are not close enough to the exact solution. Thus, the plots you obtain will not be identical to the ones on the paper, but at least you should be obtaining something similar and, most importantly, you should notice a qualitative change for the dynamics before and after the critical value of Ω. Also, when computing the width of the cloud you could apply, for each saved snapshot, one of the following methods for the density $\rho = |u|^2$:

(a) a fitting procedure using an inverted parabola or a Gaussian,

(b) the Full width at half maximum (FWHM),

(c) the width of a distribution given by the second moment:

   mass of distributon : `M = int ( rho )`
   center of mass : `cm = int ( x.*rho )./M`
   width of the distribution : `w = int ( (x-cm).^2.*rho )./M`

For methods (a) and (c): you can get inspired by the "fitting" and the "moments distribution" subroutines that are posted in the lectures webpage (towards the bottom of the nonlinear waves section of the lectures). If you have any questions please do not hesitate to ask.