HW#8 — Math 639 — Nonlinear Waves

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1. Variational approximation for LEFT-RIGHT oscillations of a Gaussian ansatz soliton inside a parabolic trap

Consider the behavior of a bright soliton trapped by a harmonic potential. This is the model for an attractive Bose-Einstein condensate (BEC) trapped inside a magnetic trap. The meanfield approximation for the BEC close to absolute zero temperature indicates that the wave function of the BEC must obey the following NLS with a potential term:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = V(x)\, u, \tag{1}$$

where

$$V(x) = \frac{\Omega^2}{2}\, x^2. \tag{2}$$

By using the Lagrangian:

$$L = \frac{i}{2}(uu_t^* - u^* u_t) + \frac{1}{2}|u_x|^2 - \frac{1}{2}|u|^4 + V(x)|u|^2 \tag{3}$$

a) Obtain the equations of motion (ODEs) for the parameters $\{B(t), w(t), \xi(t), c(t), b(t), d(t)\}$ of the Gaussian ansatz:

$$u_G = B\,\exp\left[-\frac{(x-\xi)^2}{2w^2}\right]\,\exp\left[i(b(t) + c(x-\xi) + d(x-\xi)^2)\right]. \tag{4}$$

b) We are only interested in the ensuing LEFT-RIGHT oscillations of the soliton inside the harmonic potential. Write a single equation for the position $\xi(t)$ of the form:

$$\ddot{\xi}(t) = -\frac{d}{d\xi}V_{\text{eff}}(\xi) \qquad \text{where} \qquad V_{\text{eff}}(\xi) = \frac{\Omega_{\text{eff}}^2}{2}\,\xi^2, \tag{5}$$

where Ω_{eff} is the *effective* oscillation frequency felt by the bright soliton as a whole. What is the relation between Ω and Ω_{eff}?

c) Compare the PDE dynamics with the ODE dynamics for a few parameter values of Ω. Comment in detail what you observe!

Hints:

- Use the Maple code `VA_NLS_pot.mw` by modifying it accordingly.
- For the ODE numerics you can use `ode45` in Matlab (you can Google to find "simple example ode45" or go to this site for some basic examples to use `ode45`: http://12000.org/my_notes/matlab_ODE/
- For the PDE use the Matlab code `NLS_potential.m` posted in the Lectures (Nonlinear Waves) of our website.

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2. Variational approximation for WIDTH oscillations of a Gaussian ansatz soliton inside a magnetic trap

As in the previous problem, consider the behavior of a BEC bright soliton trapped by a harmonic potential:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = V(x)\, u, \tag{6}$$

where

$$V(x) = \frac{\Omega^2}{2}\, x^2. \tag{7}$$

a) Consider the equations of motion (ODEs) for the parameters $(B(t), w(t), \xi(t), c(t), b(t), d(t))$ of the Gaussian ansatz:

$$u_G = B \, \exp\left[-\frac{(x-\xi)^2}{2w^2}\right] \exp\left[i(b(t) + c(x-\xi) + d(x-\xi)^2)\right], \tag{8}$$

that you obtained in the previous problem. We are now only interested in the ensuing WIDTH oscillations of the soliton inside the harmonic potential. Write a single equation for the WIDTH $w(t)$ of the solitons of the form:

$$\frac{d^2}{dt^2}w(t) + k_1 w(t) = \frac{k_2}{w^2(t)} + \frac{k_3}{w^3(t)}, \tag{9}$$

where you need to determine k_1, k_2 and k_3 as functions of Ω. In order to get rid of the height parameter $B(t)$ please use conservation of mass and require that $\int |u|^2 = M_0$ [this means that some of the k_i's could depend on M_0].

b) Compare the PDE dynamics with the ODE dynamics for a few parameter values of Ω.

c) What happens if the harmonic trap is periodically varied according to:

$$\Omega(t) = \Omega_0 + \epsilon \sin(\omega t),$$

so that the forcing from the external potential matches the natural oscillation frequency of Eq. (9) [i.e. $\omega = \sqrt{k_1}$]? Do three cases: i) below, ii) on, and iii) above resonance.

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3. Variational approximation for sech soliton in the NLS with gain and loss.
The goal of this exercise is to obtain approximate solutions for the dynamics of soliton solutions in the NLS under the presence of gain and loss terms.

a) Consider the NLS equation with linear loss

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u + \epsilon i u = 0. \tag{10}$$

where $\epsilon > 0$. Employ the *perturbed* Lagrangian variational method to obtain the equations of motion (ODEs) for the parameters $\{a(t), \xi(t), c(t), b(t)\}$ of the ansatz

$$u_A = a \, \mathrm{sech}(a(x-\xi) \, \exp\left[i(c(x-\xi) + b)\right]. \tag{11}$$

Describe the evolution of a typical soliton under this perturbed equation.

b) Compare ODE and PDE dynamics corresponding to **a)** for a few values of the loss [cf. $\epsilon = 0.01$, $\epsilon = 0.1$, $\epsilon = 1$, and $\epsilon = 10$]. Comment in detail what you observe!

c) Same in **a)** but now for the NLS with linear loss and (nonlinear) saturable gain:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u + i\Gamma u - i\frac{g_0}{1 + P/P_s} u = 0, \tag{12}$$

where $P = \int_{-\infty}^{\infty} |u|^2 dx$, and Γ, g_0 and P_s are positive constants. Describe the evolution of a typical soliton under this perturbed equation.

d) Study the fixed points of the ODE. What do they represent?

e) Compare ODE and PDE dynamics corresponding to **c)** for a few parameter value combination.s Comment in detail what you observe!

Hints:

- Use the Maple code `VA_NLS_sech.mw` by modifying it accordingly.

- For PDE numerics you can use the code `NLS_potential.m` and replace, in the last subroutine (`NLS_potential_spectral`), the potential terms `v` (**two appearances**) by the appropriate loss/gain terms.

  For instance, for the loss and gain case, the **two appearances** of potential `v` need to be replaced by `v` $\rightarrow$ `-sqrt(-1)*(Gamma - g0/(1+P/Ps))`, where you need to numerically compute `P` at each time step [you will need to pass `dx` to the subroutine to compute the integral: `P = sum(u.*conj(u))*dx`].