Ex Let us try to approximate a NLS solution with a Gaussian profile

Ansatz: $\psi_A = A(t)\exp\left[-\frac{x^2}{2a^2(t)} + ib(t)x^2\right]$

parameters: $a(t)$: width; $A(t)$: height; $2b(t)x$ "chirp"

• Can I compute Average Lagrangian: $\int e^{-\frac{x^2}{a^2}}dx = \sqrt{\pi}\, a$, $\int x^2 e^{-x^2/a^2}dx = \frac{\sqrt{\pi}}{2}a^3$

$\frac{d}{dt}\frac{1}{t^2} = \frac{d}{dt}t^{-2} = -2t^{-3}$

$$\bar{L}_A = \int_{-\infty}^{\infty} \frac{i}{2}\left[\psi_A \psi_{At}^* - \psi_A^* \psi_{At}\right] - \alpha|\psi_{Ax}|^2 + \frac{k}{2}|\psi_A|^4 \, dx$$

$$= \int_{-\infty}^{\infty} \frac{i}{2}\left\{A^2 ee^* \frac{d}{dt}\left[-\frac{x^2}{2a^2} + ibx^2\right]^* - A^2 ee^* \frac{d}{dt}\left[-\frac{x^2}{2a^2} + ibx^2\right]\right\} + \frac{k}{2}A^4(ee^*)^2 dx$$

$$- \alpha A^2\left(-\frac{2x}{2a^2} + 2ibx\right)\left(-\frac{2x}{2a^2} + 2ibx\right)^* ee^*$$

$$= \int_{-\infty}^{\infty} \frac{i}{2}\left\{A^2 ee^*\left(+\frac{x^2}{a^3} - i\dot{b}x^2\right) - A^2 ee^*\left(\frac{x^2}{a^3} + i\dot{b}x^2\right)\right\}$$

$ee^* = \exp\left[\frac{x^2}{a^2}\right]$

$$= \int_{-\infty}^{\infty} \frac{i}{2}\left\{A^2 ee^* \left(+\frac{x^2}{a^3} - i\dot{b}x^2\right)\right\} - \alpha A^2\left(\frac{x^2}{a^4} + 4b^2x^2\right)ee^* + \frac{k}{2}A^4(ee^*)^2 dx$$

$$= \int_{-\infty}^{\infty} \left(A^2 ee^* \dot{b} x^2 dx\right) - \alpha A^2\left(\frac{x^2}{a^4} + 4b^2x^2\right)ee^* + \frac{k}{2}A^4\sqrt{\frac{\pi}{2}}\,a$$

$$= A^2\dot{b}\frac{\sqrt{\pi}}{2}a^3 - \alpha A^2\frac{\sqrt{\pi}a^3}{2}\left(\frac{1}{a^4} + 4b^2\right) + \frac{k}{2}A^4\sqrt{\frac{\pi}{2}}\,a \qquad -\alpha\frac{A^2}{a}$$

$$\boxed{\bar{L}_A = \frac{\sqrt{\pi}}{2}\left[|A|^2 a^3\dot{b} - \alpha a^3 A^2\left(\frac{1}{a^4} + 4b^2\right) + \frac{k a A^4}{\sqrt{2}}\right]}$$

• Euler-Lagrange Eqns:

$\boxed{A}$ $\underbrace{\frac{d}{dt}\left(\frac{\partial \bar{L}_A}{\partial \dot{A}}\right)}_{=0} = \frac{\partial \bar{L}_A}{\partial A} \Rightarrow \frac{\sqrt{\pi}}{2}\left[2Aa^3\dot{b} - 2\alpha a^3 A\left(4b^2 + \frac{1}{a^4}\right) + \frac{4kaA^3}{\sqrt{2}}\right] = 0$

$\boxed{B}$ $\underbrace{\frac{d}{dt}\left(\frac{\partial \bar{L}_A}{\partial \dot{a}}\right)}_{=0} = \frac{\partial \bar{L}_A}{\partial a} \Rightarrow \frac{\sqrt{\pi}}{2}\left[3A^2a^2\dot{b} - 3\alpha a^2 A^2 4b^2 + \frac{\alpha A^2}{a^2} + \frac{k}{\sqrt{2}}A^4\right] = 0$

$$\frac{d}{dt}\left(\frac{\sqrt{\pi}}{2}A^2a^3\right) = -8\alpha a^3 A^2 b$$

$\boxed{C}$ $\frac{d}{dt}\left(\frac{\partial \bar{L}_A}{\partial \dot{b}}\right) = \frac{\partial \bar{L}_A}{\partial b} \Rightarrow \frac{d}{dt}\left(\frac{\sqrt{\pi}}{2}A^2a^3\right)$

Since the ansatz does not have a variation for the phase let us let $A(t)$ be complex i.e. $A(t) = A_r(t) + iA_i(t)$ so we have 4 parameters A_r, A_i, a, b. Instead of using A_i & A_r note that $A^* = (A_r - iA_i)/2$ and $A = (A_r + iA)/2 \rightarrow$ so use $\boxed{A \text{ \& } A^*}$

So let is use A & A^*, thus

$$\vec{p} = \begin{pmatrix} A \\ A^* \\ a \\ b \end{pmatrix} \leftarrow \text{parameters.}$$

Compute averaged Lagrangian, use $\int e^{-x^2/a^2} = \sqrt{\pi}\, a$ & $\int x^2 e^{-x^2/a^2} = \frac{\sqrt{\pi}}{2} a^3$

$\rightarrow \bar{L}_A(x,t) = \int_{-\infty}^{\infty} L(x,t)\big|_{\psi_A} dx = $ [Eq(16) of Anderson]

*check in note

$$\bar{L}_A = \frac{\sqrt{\pi}}{2}\left[ia(A\dot{A}^* - A^*\dot{A}) + AA^* a^3 \dot{b} - \alpha a^3 AA^*\left(4b^2 + \frac{1}{a^4}\right) + \frac{1}{\sqrt{2}} k a AA^*AA^*\right]$$

Euler-Lagrange Equations

$\left(\frac{\delta \bar{L}_A}{\delta A^*} = \phi\right) \Rightarrow$ $\boxed{A^*}$ $\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{A}^*}\right)} = \frac{\partial L}{\partial A^*} \Rightarrow$ ODE (17)

$$\Rightarrow \frac{d}{dt}(iaA) = -ia\dot{A} + Aa^3\dot{b} - \alpha a^3 A\left[4b^2 + \frac{1}{a^4}\right] + \sqrt{2}\, k a AA^*A$$

these are reversed in Rosa notes

$\boxed{A}$ $\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{A}}\right)} = \frac{\partial L}{\partial A} \Rightarrow$ ODE(18)

$$\Rightarrow \frac{d}{dt}(-iaA^*) = -ia\dot{A}^* + A^* a^3 \dot{b} - \alpha a^3 A^*\left[4b^2 + \frac{1}{a^4}\right] + \sqrt{2}\, k a AA^*A$$

← I think this sign is wrong

$\boxed{a}$ $\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right)} = \frac{\partial L}{\partial a} \Rightarrow$ ODE(19)

$$0 = i(A\dot{A}^* - A^*\dot{A}) + 3AA^* a^2 \dot{b} \underbrace{-12\alpha a^2 AA^* b^2} + \alpha\frac{AA^*}{a^2} + \frac{1}{\sqrt{2}} k AA^*AA^*$$

← factor different than anderson

$\boxed{b}$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{b}}\right) = \frac{\partial L}{\partial b} \Rightarrow$ ODE (20)

$$\frac{d}{dt}(AA^* a^3) = \underbrace{-8}\alpha b a^3 AA^*$$

→ factor different than Anderson

Combining (17) & (18) $\Rightarrow \frac{d}{dt}(a|A|^2) = \phi$ $\qquad |A|^2 = AA^*$

$$i[A^*\dot{A} - A\dot{A}^*] = |A|^2\left[a^2\dot{b} - \alpha a^2\left[4b^2 + \frac{1}{a^4}\right] + \sqrt{2}\, k|A|^2\right]$$

$\therefore a|A|^2 = \text{constant} = a_0|A_0|^2$ = Mass

this recovers conservation of mass:

$$\int |\psi_A|^2 dx = \text{const}$$

$$\Rightarrow N = \int_{-\infty}^{\infty} |A|^2 e\, dx = |A|^2 \int e^{-x^2/2a^2} = |A|^2\sqrt{2\pi}\, a = \underbrace{\text{constant}}_{N}$$

$$\frac{N}{\sqrt{2\pi}} = |A|^2 a$$

*(20): $\frac{d}{dt}\left(a^3|A|^2\right) = -8\alpha b a^3|A|^2$ (still not sure why this isn't 8)

$\frac{d}{dt}\left(a|A|^2\cdot a^2\right) = a|A|^2 2\dot{a}a = -8\alpha b a^3|A|^2$

$\Rightarrow \boxed{\dot{a} = -4\alpha b a}$ (25)

This is the factor of 2 in the notes

* Combining (17) & (18) in (19)

$$a\dot{b} - 4\alpha b^2 a + \frac{\alpha}{a^3} - \frac{k}{2\sqrt{2}}\frac{|A|^2}{a} = 0 \quad (26)$$

now $\frac{d}{dt}(25) \Rightarrow \ddot{a} = -\overset{\text{or }2}{4}\alpha(\dot{b}a + b\dot{a}) = -4\alpha(\overset{\text{or }2}{\dot{b}a} - \overset{\text{or }2}{b^2}4\alpha a)$

$(26) \Rightarrow \ddot{a} = -4\alpha\left(4\alpha b^2 a - \frac{\alpha}{a^3} + \frac{k}{2\sqrt{2}}\frac{|A|^2}{a}\right) + 8\alpha^2 b^2 a$

$= -\cancel{8\alpha^2 b^2 a} + \frac{4\alpha^2}{a^3} - \frac{2}{\sqrt{2}}k\alpha\frac{|A|^2}{a} + \cancel{8\alpha^2 b^2 a}$

$$\boxed{\ddot{a} = \frac{4\alpha^2}{a^3} - \alpha k\sqrt{2}\frac{|A|^2}{a}} \quad (27)$$

It is convenient to eliminate $a|A|^2$ since $\frac{N}{\sqrt{2\pi}} = |A|^2 a$

$\therefore \ddot{a} = \frac{4\alpha^2}{a^3} - \frac{\alpha k\cancel{\sqrt{2}}}{a^2}\frac{N}{\sqrt{\cancel{2}\pi}}$

$\rightarrow \ddot{a} = \frac{4\alpha^2}{a^3} - \frac{\alpha kN}{\sqrt{\pi}a^2} = -\frac{d}{da}\left[\frac{\overset{?}{4}_{2}\alpha^2}{a^2} - \frac{\alpha kN}{\sqrt{\pi}a}\right]$

$\Rightarrow \boxed{\ddot{a} = -\frac{d}{da}\left[V_{eff}(a)\right]}$ (28) $\quad \boxed{V_{eff}(a) = \frac{2\alpha^2}{a^2} - \frac{\alpha kN}{\sqrt{\pi}a}}$ (29)

The ODE is solved for a and then b is obtained through (25) and $|A|^2 a$ = constant

• b Eqn: $\frac{\dot{a}}{a} = -4\alpha b \Rightarrow \frac{d}{dt}\ln(a) = -4\alpha b$

$$\boxed{b(t) = -\frac{1}{4\alpha}\frac{d}{dt}(\ln(a))}$$

phase of A: $A(t) = \underbrace{B(t)e^{i\phi(t)}}$ → This is another choice for ansatz

$\psi_A = B(t)e^{\left(-\frac{x^2}{2a^2} + ibx^2 + i\phi(t)\right)}$, This is more standard

Combining (17) & (18) ⇒ eq(22) and use (eq) 26

$$\Rightarrow \boxed{\dot{\phi}(t) = \frac{\alpha}{a^2} - \frac{5\sqrt{2}}{8}k|A|^2}$$

Steady State: minimum of Veff $\Rightarrow \dfrac{4\alpha^2}{a^2} = \dfrac{\alpha k N}{\sqrt{\pi}\, a} \Rightarrow \dfrac{4\alpha}{a} = \dfrac{kN}{\sqrt{\pi}}$

$$\therefore\quad a = \frac{4\alpha\sqrt{\pi}}{kN}$$

compare $\phi = E\,\mathrm{sech}\,Ex$ & a standing soliton w/ $\phi_A = Ae^{-x^2/2a^2}$

where $N = \sqrt{2\pi}\, a|A|^2$ and $N = \int E^2\,\mathrm{sech}^2 Ex = 2E$

and $a = \dfrac{4\alpha\sqrt{\pi}}{kn} \Rightarrow$ see Fig 5 of Anderson

Variational Approach in general

- nonlinear PDE: $PDE(\psi) = \emptyset$
- cast PDE as a variational Lagrangian

$$\delta \iint L(\psi(x,t))\,dx\,dt = \emptyset \iff PDE(\psi) = \emptyset$$

- restrict solutions to ansatz: $L_A = L(\psi_A(x,\vec{p}))$
- Average over x: $\bar{L}_A(x,\vec{p}) = \int L_A\,dx$
- Euler-Lagrange $\dfrac{d}{dt}\dfrac{\partial \bar{L}_A}{\partial \dot{p}_i} = \dfrac{\partial \bar{L}_A}{\partial p_i}$

- Find a system of ∂DE's for params:

most of the time E-L $\Rightarrow \boxed{M(\vec{p})\dfrac{d\vec{p}}{dt} = f(\vec{p})}$

mass matrix — focusing term

Caution, the system of ODEs can be degenerate (non-invertible M)

$\Rightarrow$ one has to try a $\neq$ ansatz

Ex. Adding the chirp removes degeneracy!

$$u(x,t) = a\,\mathrm{sech}\,\omega(x-d)\,e^{i(b+cx)} \qquad p = \begin{bmatrix} a \\ b \\ c \\ d \\ \omega \end{bmatrix}$$

non-chirped ansatz: $\boxed{V=0}$

E-L .. $M(p)\dot{p} = f(p)$

$$M = \begin{bmatrix} 0 & & \\ & \ddots & \\ & & \end{bmatrix} = \begin{bmatrix} 0 & 4\alpha/\omega & 0 & 0 & 2a^2/\omega^2 \\ -4\alpha/\omega & 0 & 0 & -2a^2/\omega & 0 \\ 0 & 0 & 2a^2/\omega & 0 & 0 \\ 0 & \frac{2a^2}{\omega^2} & -\frac{2a^2 d}{\omega^2} & 0 & 0 \end{bmatrix} \quad 4ad/\omega \quad 0 \quad 0$$

$$f = \begin{bmatrix} f_1 \\ 0 \\ f_2 \\ 0 \\ f_3 \end{bmatrix}$$

① $\begin{cases} \dot{b} + \dot{c} = \\ \dot{a} + \dot{\omega} = \\ \dot{d} = \\ \dot{c} = \\ \dot{b} + \dot{c} = \end{cases}$ 1 eqn for $\dot{a},\dot{\omega}$; 4 eqns for $\dot{b},\dot{c},\dot{d}$

$\Rightarrow$ non invertible