1

Linear Partial Differential Equations

"However varied may be the imagination of man, nature is still a thousand times richer, ... Each of the theories of physics ... presents (partial differential) equations under a new aspect ... without these theories, we should not know partial differential equations."

Henri Poincaré

"Since a general solution must be judged impossible from want of analysis, we must be content with the knowledge of some special cases, and that all the more, since the development of various cases seems to be the only way to bringing us at last to a more perfect knowledge."

Leonard Euler

1.1 Introduction

Partial differential equations arise frequently in the formulation of fundamental laws of nature and in the mathematical analysis of a wide variety of problems in applied mathematics, mathematical physics, and engineering science. This subject plays a central role in modern mathematical sciences, especially in physics, geometry, and analysis. Many problems of physical interest are described by partial differential equations with appropriate initial and/or boundary conditions. These problems are usually formulated as initial-value problems, boundary-value problems, or initial boundary-value problems. In order to prepare the reader for study and research in nonlinear partial differential equations, a broad coverage of the essential standard material on linear partial differential equations and their applications is required.

This chapter provides a review of basic concepts, principles, model equations, and their methods of solutions. This is followed by a systematic mathematical treatment of the theory and methods of solutions of second-order linear partial differential equations that gives the reader a clear understanding of the subject and its varied applications. Linear partial differential equations of the second order can be classified as one of the three types, hyperbolic, parabolic, and elliptic and reduced to an appropriate canonical or normal form. The classification and method of reduction are described in Section 1.5. Special emphasis is given to various methods of solution of the initial-value and/or boundary-value problems associated with the three types of linear equations, each of which shows an entirely different behavior in properties and construction of solutions. Section 1.6 deals with the solutions of linear partial differential equations using the method of separation of variables combined with the superposition principle. A brief discussion of Fourier, Laplace, and Hankel transforms is included in Sections 1.7–1.10. These integral transforms are then applied to solve a large variety of initial and boundary problems described by partial differential equations. The transform solution combined with the convolution theorem provides an elegant representation of the solution for initial-value and boundary-value problems. Section 1.11 is devoted to Green's functions for solving a wide variety of inhomogeneous partial differential equations of most common interest. This method can be made considerably easier by using generalized functions combined with appropriate integral transforms. The Sturm–Liouville systems and their general properties are discussed in Section 1.12. Section 1.13 deals with energy integrals, the law of conservation of energy, uniqueness theorems, and higher dimensional wave and diffusion equations. The final section contains some recent examples of fractional order diffusion-wave equations and their solutions.

1.2 Basic Concepts and Definitions

A partial differential equation for a function $u(x, y, \ldots)$ is a relationship between u and its partial derivatives $u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots$ and can be written as

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0, \tag{1.2.1}$$

where F is some function, $x, y, \ldots$ are independent variables and $u(x, y, \ldots)$ is called a *dependent variable*.

The *order* of a partial differential equation is defined in analogy with an ordinary differential equation as the highest-order derivative appearing in (1.2.1). The most general *first-order* partial differential equation can be written as

$$F(x, y, u, u_x, u_y) = 0. \tag{1.2.2}$$

Similarly, the most general *second-order* partial differential equation in two independent variables x, y has the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{1.2.3}$$

and so on for higher-order equations.

For example,

$$x\,u_x + y\,u_y = 0, \tag{1.2.4}$$
$$x\,u_x + y\,u_y = x^2 + y^2, \tag{1.2.5}$$
$$u\,u_x + u_t = u, \tag{1.2.6}$$
$$u_x^2 + u_y^2 = 1, \tag{1.2.7}$$

are first-order equations, and

$$u_{xx} + 2u_{xy} + u_{yy} = 0, \tag{1.2.8}$$
$$u_{xx} + u_{yy} = 0, \tag{1.2.9}$$
$$u_{tt} - c^2 u_{xx} = f(x, t), \tag{1.2.10}$$

are second-order equations. Finally,

$$u_t + u u_x + u_{xxx} = 0, \tag{1.2.11}$$
$$u_{tt} + u_{xxxx} = 0, \tag{1.2.12}$$

are examples of third-order and fourth-order equations, respectively.

A partial differential equation is called *linear* if it is linear in the unknown function and all its derivatives with coefficients depend only on the independent variables. It is called *quasi-linear*, if it is linear in the highest-order derivative of the unknown function. For example, (1.2.4), (1.2.5), (1.2.8)–(1.2.10) and (1.2.12) are linear equations, whereas (1.2.6) and (1.2.11) are quasi-linear equations.

It is possible to write a partial differential equation in the operator form

$$L_x u(\mathrm{x}) = f(\mathrm{x}), \tag{1.2.13}$$

where L_x is an operator. The operator L_x is called a *linear operator* if it satisfies the property

$$L_x(au + bv) = a\,L_x u + b\,L_x v \tag{1.2.14}$$

for any two functions u and v and for any two constants a and b.

Equation (1.2.13) is called *linear* if L_x is a linear operator. Equation (1.2.13) is called an *inhomogeneous* (or *nonhomogeneous*) linear equation. If $f(\mathrm{x}) \equiv 0$, (1.2.13) is called a *homogeneous* equation. Equations (1.2.4), (1.2.8), (1.2.9), and (1.2.12) are linear homogeneous equations, whereas (1.2.5) and (1.2.10) are linear inhomogeneous equations.

An equation which is not linear is called a *nonlinear equation.* If L_x is not linear, then (1.2.13) is called a *nonlinear* equation. Equations (1.2.6), (1.2.7) and (1.2.11) are examples of nonlinear equations.

A *classical solution* (or simply a *solution*) of (1.2.1) is an ordinary function $u = u(x, y, \ldots)$ defined in some domain D which is continuously differentiable such that all its partial derivatives involved in the equation exist and satisfy (1.2.1) identically.

However, this notion of classical solution can be extended by relaxing the requirement that u is continuously *differentiable* over D. The solution $u = u(x, y, \ldots)$ is called a weak (or *generalized*) *solution* of (1.2.1) if u or its partial derivatives are discontinuous in some or all points in D.

To introduce the idea of a *general solution* of a partial differential equation, we solve a simple equation for $u = u(x, y)$ of the form

$$u_{xy} = 0. \tag{1.2.15}$$

Integrating this equation with respect to x (keeping y fixed), we obtain

$$u_y = h(y),$$

where $h(y)$ is an arbitrary function of y. We then integrate it with respect to y to find

$$u(x, y) = \int h(y)\, dy + f(x),$$

where $f(x)$ is an arbitrary function. Or, equivalently,

$$u(x, y) = f(x) + g(y), \tag{1.2.16}$$

where $f(x)$ and $g(y)$ are arbitrary functions. The solution (1.2.16) is called the *general solution* of the second-order equation (1.2.15).

Usually, the general solution of a partial differential equation is an expression that involves arbitrary functions. This is a striking contrast to the general solution of an ordinary differential equation which involves arbitrary constants. Further, a simple equation (1.2.15) has infinitely many solutions. This can be illustrated by considering the problem of construction of partial differential equations from given arbitrary functions. For example, if

$$u(x, t) = f(x - ct) + g(x + ct), \tag{1.2.17}$$

where f and g are arbitrary functions of $(x - ct)$ and $(x + ct)$, respectively, then

$$u_{xx} = f''(x - ct) + g''(x + ct),$$
$$u_{tt} = c^2 f''(x - ct) + c^2 g''(x + ct) = c^2 u_{xx},$$

where primes denote differentiation with respect to the appropriate argument. Thus, we obtain the second-order linear equation, called the *wave equation*,

$$u_{tt} - c^2 u_{xx} = 0. \tag{1.2.18}$$

Thus, the function $u(x, t)$ defined by (1.2.17) satisfies (1.2.18) irrespective of the functional forms of $f(x - ct)$ and $g(x + ct)$, provided f and g are at least twice differentiable functions. Thus, the general solution of equation (1.2.18) is given by (1.2.17) which contains arbitrary functions.

In the case of only two independent variables x, y, the solution $u(x, y)$ of the equation (1.2.1) is visualized *geometrically as a surface*, called an *integral surface* in the (x, y, u) space.

1.3 The Linear Superposition Principle

The general solution of a linear homogeneous ordinary differential equation of order n is a linear combination of n linearly independent solutions with n arbitrary constants. In other words, if $u_1(x)$, $u_2(x)$, ..., $u_n(x)$ are n linearly independent solutions of an n-th order, linear, homogeneous, ordinary differential equation of the form

$$Lu(x) = 0, \tag{1.3.1}$$

then, for any arbitrary constants $c_1, c_2, \ldots, c_n$,

$$u(x) = \sum_{k=1}^{n} c_k u_k(x) \tag{1.3.2}$$

represents the most general solution of (1.3.1). This is called the *linear superposition principle* for ordinary differential equations. We note that the general solution of (1.3.1) depends on *exactly n arbitrary constants*.

In the case of linear homogeneous partial differential equations of the form

$$L_x u(\mathbf{x}) = 0, \tag{1.3.3}$$

the general solution depends on arbitrary functions rather than arbitrary constants. So there are infinitely many solutions of (1.3.3). If we represent this infinite set of solutions of (1.3.3) by $u_1(\mathbf{x})$, $u_2(\mathbf{x})$, ..., $u_n(\mathbf{x})$, ..., then the infinite linear combinations

$$u(\mathbf{x}) = \sum_{n=1}^{\infty} c_n u_n(\mathbf{x}), \tag{1.3.4}$$

where c_n are arbitrary constants, in general, may *not* be again a solution of (1.3.3) because the infinite series may not be convergent. So, for the case of partial differential equations, the superposition principle may not be true in general. However, if there are only a *finite* number of solutions $u_1(\mathbf{x})$, $u_2(\mathbf{x})$, ..., $u_n(\mathbf{x})$ of the partial differential equation (1.3.3), then

$$u(\mathbf{x}) = \sum_{n=1}^{n} c_n u_n(\mathbf{x}) \tag{1.3.5}$$

again is a solution of (1.3.3) as can be verified by direct substitution. As with linear homogeneous ordinary differential equations, the principle of superposition applies to linear homogeneous partial differential equations and $u(\mathbf{x})$ represents a solution of (1.3.3), provided that the infinite series (1.3.4) is convergent and the operator L_x can be applied to the series term by term.

In order to generate such an infinite set of solutions $u_n(\mathbf{x})$, the method of separation of variables is usually used. This method, combined with the superposition of solutions, is usually known as *Fourier's method*, which will be described in a subsequent section.

Another type of infinite linear combination is used to find the solution of a given partial differential equation. This is concerned with a family of solutions $u(\mathrm{x}, k)$ depending on a continuous real parameter k and a function $c(k)$ such that

$$\int_a^b c(k)\, u(\mathrm{x}, k)\, dk \text{ or } \int_{-\infty}^{\infty} c(k)\, u(\mathrm{x}, k)\, dk \tag{1.3.6}$$

is convergent. Then, under certain conditions, this integral, again, is a solution. This may also be regarded as the *linear superposition principle.*

In almost all cases, the general solution of a partial differential equation is of little use since it has to satisfy other supplementary conditions, usually called *initial* or *boundary* conditions. As indicated earlier, the general solution of a linear partial differential equation contains arbitrary functions. This means that there are infinitely many solutions and only by specifying the initial and/or boundary conditions can we determine a specific solution of interest.

Usually, both initial and boundary conditions arise from the physics of the problem. In the case of partial differential equations in which one of the independent variables is the time t, an initial condition(s) specifies the physical state of the dependent variable $u(\mathrm{x}, t)$ at a particular time $t = t_0$ or $t = 0$. Often $u(\mathrm{x}, 0)$ and/or $u_t(\mathrm{x}, 0)$ are specified to determine the function $u(\mathrm{x}, t)$ at later times. Such conditions are called the *Cauchy (or initial) conditions*. It can be shown that these conditions are necessary and sufficient for the existence of a unique solution. The problem of finding the solution of the initial-value problem with prescribed Cauchy data on the line $t = 0$ is called the *Cauchy problem* or the *initial-value problem.*

In each physical problem, the governing equation is to be solved within a given domain D of space with prescribed values of the dependent variable $u(\mathrm{x}, t)$ given on the boundary ∂D of D. Often, the boundary need not enclose a finite volume—in which case, part of the boundary is at infinity. For problems with a boundary at infinity, boundedness conditions on the behavior of the solution at infinity must be specified. This kind of problem is typically known as a *boundary-value problem,* and it is one of the most fundamental problems in applied mathematics and mathematical physics.

There are three important types of boundary conditions which arise frequently in formulating physical problems. These are

(a) *Dirichlet conditions*, where the solution u is prescribed at each point of a boundary ∂D of a domain D. The problem of finding the solution of a given equation $L_x u(\mathrm{x}) = 0$ inside D with prescribed values of u on ∂D is called the *Dirichlet boundary-value problem*;

(b) *Neumann conditions*, where values of normal derivative $\frac{\partial u}{\partial n}$ of the solution on the boundary ∂D are specified. In this case, the problem is called the *Neumann boundary-value problem*;

(c) *Robin conditions*, where $\left(\frac{\partial u}{\partial n} + au\right)$ is specified on ∂D. The corresponding problem is called the *Robin boundary-value problem.*

A problem described by a partial differential equation in a given domain with a set of initial and/or boundary conditions (or other supplementary conditions) is said to be *well-posed* (or *properly posed*) provided the following criteria are satisfied:

(i) *existence*: There exists at least one solution of the problem.

(ii) *uniqueness*: There is at most one solution.

(iii) *stability*: The solution must be stable in the sense that it depends continuously on the data. In other words, a small change in the given data must produce a small change in the solution.

The stability criterion is essential for physical problems. A mathematical problem is usually considered physically realistic if a small change in given data produces correspondingly a small change in the solution.

According to the Cauchy–Kowalewski theorem, the solution of an analytic Cauchy problem for partial differential equations exists and is unique. However, a Cauchy problem for Laplace's equation is *not* always *well-posed.* A famous example of a *non-well-posed* (or *ill-posed*) problem was first given by Hadamard. *Hadamard's example* deals with Cauchy's initial-value problem for the Laplace equation

$$\nabla^2 u \equiv u_{xx} + u_{yy} = 0, \quad 0 < y < \infty, \quad x \in R \tag{1.3.7}$$

with the Cauchy data

$$u(x, 0) = 0 \quad \text{and} \quad u_y(x, 0) = \left(\frac{1}{n}\right) \sin nx, \tag{1.3.8}$$

where n is an integer representing the wavenumber. These data tend to zero uniformly as $n \to \infty$.

It can easily be verified that the unique solution of this problem is given by

$$u(x, y) = \left(\frac{1}{n^2}\right) \sinh ny \sin nx. \tag{1.3.9}$$

As $n \to \infty$, this solution does not tend to the solution $u = 0$. In fact, solution (1.3.9) represents oscillations in x with unbounded amplitude $n^{-2} \sinh ny$ which tends to infinity as $n \to \infty$. In other words, although the data change by an arbitrarily small amount, the change in the solution is infinitely large. So the problem is certainly *not* well-posed, that is, the solution does not depend continuously on the initial data. Even if the wavenumber n is a fixed, finite quantity, the solution is clearly unstable in the sense that $u(x, y) \to \infty$ as $y \to \infty$ for any fixed x, such that $\sin nx \neq 0$.

On the other hand, the Cauchy problem (see Example 1.5.3) for the simplest hyperbolic equation (1.5.29) with the initial data (1.5.35ab) is well-posed. As to the domain of dependence for the solution, $u(x, t)$ depends *only* on those values of $f(\xi)$ and $g(\xi)$ for which $x - ct \leq \xi \leq x + ct$. Similarly, the Cauchy problems for parabolic equations are generally well-posed.

We conclude this section with a general remark. The existence, uniqueness, and stability of solutions are the basic requirements for a complete description of a physical problem with appropriate initial and boundary conditions. However, there are many situations in applied mathematics which deal with ill-posed problems. In recent years, considerable progress has been made on the theory of ill-posed problems, but the discussion of such problems is beyond the scope of this book.

1.4 Some Important Classical Linear Model Equations

We start with a special type of second-order linear partial differential equation for the following reasons. First, second-order equations arise more frequently in a wide variety of applications. Second, their mathematical treatment is simpler and easier to understand than that of first-order equations in general. Usually, in almost all physical phenomena, the dependent variable $u = u(x,y,z,t)$ is a function of three space variables and time variable t. Included here are only examples of equations of most common interest.

Example 1.4.1 The wave equation is

$$u_{tt} - c^2\nabla^2 u = 0, \tag{1.4.1}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{1.4.2}$$

and c is a constant. This equation describes the propagation of a wave (or disturbance), and it arises in a wide variety of physical problems. Some of these problems include a vibrating string, vibrating membrane, longitudinal vibrations of an elastic rod or beam, shallow water waves, acoustic problems for the velocity potential for a fluid flow through which sound can be transmitted, transmission of electric signals along a cable, and both electric and magnetic fields in the absence of charge and dielectric.

Example 1.4.2 The heat or diffusion equation is

$$u_t - \kappa\nabla^2 u = 0, \tag{1.4.3}$$

where κ is the constant of diffusivity. This equation describes the diffusion of thermal energy in a homogeneous medium. It can be used to model the flow of a quantity, such as heat, or a concentration of particles. It is also used as a model equation for growth and diffusion, in general, and growth of a solid tumor, in particular. The diffusion equation describes the unsteady boundary-layer flow in the Stokes and Rayleigh problems and also the diffusion of vorticity from a vortex sheet.

Example 1.4.3 The Laplace equation is

$$\nabla^2 u = 0. \tag{1.4.4}$$

This equation is used to describe electrostatic potential in the absence of charges, gravitational potential in the absence of mass, equilibrium displacement of an elastic membrane, velocity potential for an incompressible fluid flow, temperature in a steady-state heat conduction problem, and many other physical phenomena.

Example 1.4.4 The Poisson equation is

$$\nabla^2 u = f(x,y,z), \tag{1.4.5}$$

where $f(x,y,z)$ is a given function describing a source or sink. This is an inhomogeneous Laplace equation, and hence, the Poisson equation is used to study all phenomena described by the Laplace equation in the presence of external sources or sinks.

Example 1.4.5 The Helmholtz equation is

$$\nabla^2 u + \lambda u = 0, \tag{1.4.6}$$

where λ is a constant. This is a time-independent wave equation (1.4.1) with λ as a separation constant. In particular, its solution in acoustics represents an acoustic radiation potential.

Example 1.4.6 The telegraph equation is in general form

$$u_{tt} - c^2 u_{xx} + a\,u_t + b\,u = 0, \tag{1.4.7}$$

where a, b, and c are constants. This equation arises in the study of propagation of electrical signals in a cable of transmission line. Both current I and voltage V satisfy an equation of the form (1.4.7). This equation also arises in the propagation of pressure waves in the study of pulsatile blood flow in arteries and in one-dimensional random motion of bugs along a hedge.

Example 1.4.7 The Klein–Gordon (or KG) equation is

$$\Box\psi + \left(\frac{mc^2}{\hbar}\right)^2 \psi = 0, \tag{1.4.8}$$

where

$$\Box \equiv \frac{\partial^2}{\partial t^2} - c^2\nabla^2 \tag{1.4.9}$$

is the d'Alembertian operator, $h\,(= 2\pi\hbar)$ is the Planck constant, and m is a constant mass of the particle. Klein (1927) and Gordon (1926) derived a relativistic equation for a charged particle in an electromagnetic field. It is of conservative dispersive type and played an important role in our understanding of the elementary particles. This equation is also used to describe dispersive wave phenomena in general.

Example 1.4.8 The time-independent Schrödinger equation in quantum mechanics is

$$\left(\frac{\hbar^2}{2m}\right)\nabla^2\psi + (E - V)\,\psi = 0, \tag{1.4.10}$$

where $h\,(= 2\pi\hbar)$ is the Planck constant, m is the mass of the particle whose wave function is $\psi\,(x, y, z, t)$, E is a constant, and V is the potential energy. If $V = 0$, (1.4.10) reduces to the Helmholtz equation.

Example 1.4.9 The linear Korteweg–de Vries (or KdV) equation is

$$u_t + \alpha u_x + \beta u_{xxx} = 0, \tag{1.4.11}$$

where α and β are constants. This describes the propagation of linear, long, water waves and of plasma waves in a dispersive medium.

Example 1.4.10 The linear Boussinesq equation is

$$u_{tt} - \alpha^2\nabla^2 u - \beta^2\nabla^2 u_{tt} = 0, \tag{1.4.12}$$

where α and β are constants. This equation arises in elasticity for longitudinal waves in bars, long water waves, and plasma waves.

Example 1.4.11 The biharmonic wave equation is

$$u_{tt} + c^2\nabla^4 u = 0, \tag{1.4.13}$$

where c is a constant. In elasticity, the displacement of a thin elastic plate by small vibrations satisfies this equation. When u is independent of time t, (1.4.13) reduces to what is called the *biharmonic equation*

$$\nabla^4 u = 0. \tag{1.4.14}$$

This describes the equilibrium equation for the distribution of stresses in an elastic medium satisfied by Airy's stress function $u\,(x, y, z)$. In fluid dynamics, this equation is satisfied by the stream function $\psi\,(x, y, z)$ in a viscous fluid flow.

Example 1.4.12 The electromagnetic wave equations for the electric field E and the polarization P are

$$\mathcal{E}_0\left(E_{tt} - c_0^2 E_{xx}\right) + P_{tt} = 0, \tag{1.4.15}$$

$$\left(P_{tt} + \omega_0^2 P\right) - \mathcal{E}_0\omega_p^2 E = 0, \tag{1.4.16}$$

where $\mathcal{E}_0$ is the permittivity (or dielectric constant) of free space, ω_0 is the natural frequency of the oscillator, c_0 is the speed of light in a vacuum, and ω_p is the plasma frequency.

1.5 Second-Order Linear Equations and Method of Characteristics

The general second-order linear partial differential equation in two independent variables x, y is given by

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{1.5.1}$$

where A, B, C, D, E, F, and G are given functions of x and y or constants.

The classification of second-order equations is based upon the possibility of reducing equation (1.5.1) by a coordinate transformation to a canonical or standard form at a point. We consider the transformation from x, y to ξ, η defined by

$$\xi = \phi\,(x, y)\,, \quad \eta = \psi\,(x, y)\,, \tag{1.5.2ab}$$

where ϕ and ψ are twice continuously differentiable and the Jacobian $J\,(x, y) = \phi_x\psi_y - \psi_x\phi_y$ is nonzero in a domain of interest so that x, y can be determined uniquely from the system (1.5.2ab). Then, by the chain rule,

$$\begin{aligned}
u_x &= u_\xi\xi_x + u_\eta\eta_x, \qquad u_y = u_\xi\xi_y + u_\eta\eta_y,\\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx},\\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy},\\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy}.
\end{aligned}$$

Substituting these results in equation (1.5.1) gives

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*, \tag{1.5.3}$$

where

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2,\\
B^* &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y,\\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2,\\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y,\\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y,\\
F^* &= F \quad \text{and} \quad G^* = G.
\end{aligned}$$

Now, the problem is to determine ξ and η so that equation (1.5.3) takes the simplest possible form. We choose ξ and η such that $A^* = C^* = 0$ and $B^* \neq 0$. Or, more explicitly,

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0, \tag{1.5.4}$$

$$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0. \tag{1.5.5}$$

These two equations can be combined into a single quadratic equation for $\zeta = \xi$ or η

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0. \tag{1.5.6}$$

We consider level curves $\xi = \phi(x,y) =$ constant $= C_1$ and $\eta = \psi(x,y) =$ constant $= C_2$. On these curves

$$d\xi = \xi_x dx + \xi_y dy = 0, \quad d\eta = \eta_x dx + \eta_y dy, \tag{1.5.7ab}$$

that is, the slopes of these curves are given by

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}, \quad \frac{dy}{dx} = -\frac{\eta_x}{\eta_y}. \tag{1.5.8ab}$$

Thus, the slopes of both level curves are the roots of the same quadratic equation which is obtained from (1.5.6) as

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0, \tag{1.5.9}$$

and the roots of this equation are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right). \tag{1.5.10ab}$$

These equations are known as the *characteristic equations* for (1.5.1), and their solutions are called the *characteristic curves* or simply the *characteristics* of equation (1.5.1). The solution of the two ordinary differential equations (1.5.10ab) defines two distinct families of characteristics $\phi(x,y) = C_1$ and $\psi(x,y) = C_2$. There are three possible cases to consider.

Case I. $B^2 - 4AC > 0$.

Equations for which $B^2 - 4AC > 0$ are called *hyperbolic*. Integrating (1.5.10ab) gives two real and distinct families of characteristics $\phi(x,y) = C_1$ and $\psi(x,y) = C_2$, where C_1 and C_2 are constants of integration. Since $A^* = C^* = 0$, and $B^* \neq 0$, and dividing by B^*, equation (1.5.3) reduces to the form

$$u_{\xi\eta} = -\frac{1}{B^*}\left(D^* u_\xi + E^* u_\eta + F^* u - G^*\right) = H_1 \text{ (say)}. \tag{1.5.11}$$

This is called the *first canonical form of the hyperbolic equation.*

If the new independent variables

$$\alpha = \xi + \eta, \quad \beta = \xi - \eta \tag{1.5.12ab}$$

are introduced, then

$$u_\xi = u_\alpha \alpha_\xi + u_\beta \beta_\xi = u_\alpha + u_\beta, \qquad u_\eta = u_\alpha \alpha_\eta + u_\beta \beta_\eta = u_\alpha - u_\beta$$
$$(u_\eta)_\xi = (u_\eta)_\alpha \alpha_\xi + (u_\eta)_\beta \beta_\xi = (u_\alpha - u_\beta)_\alpha \cdot 1 + (u_\alpha - u_\beta)_\beta \cdot 1$$
$$= u_{\alpha\alpha} - u_{\beta\beta}.$$

Consequently, equation (1.5.11) becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = H_2(\alpha, \beta, u, u_\alpha, u_\beta). \tag{1.5.13}$$

This is called the *second canonical form of the hyperbolic equation.*

It is important to point out that characteristics play a fundamental role in the theory of hyperbolic equations.

Case II. $B^2 - 4AC = 0$.

There is only one family of real characteristics whose slope, due to (1.5.10ab), is given by

$$\frac{dy}{dx} = \frac{B}{2A}. \tag{1.5.14}$$

Integrating this equation gives $\xi = \phi(x,y) =$ const. (or $\eta = \psi(x,y) =$ const.).

Since $B^2 = 4AC$ and $A^* = 0$, we obtain

$$0 = A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = \left(\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y\right)^2.$$

It then follows that

$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y$$
$$= 2\left(\sqrt{A}\xi_x + \sqrt{C}\xi_y\right)\left(\sqrt{A}\eta_x + \sqrt{C}\eta_y\right) = 0$$

for an arbitrary value of η which is independent of ξ. For example, if $\eta = y$, the Jacobian is nonzero in the domain of parabolicity.

Dividing (1.5.3) by $C^* \neq 0$ yields

$$u_{\eta\eta} = H_3(\xi, \eta, u, u_\xi, u_\eta). \tag{1.5.15}$$

This is known as the *canonical form of the parabolic equation.*

On the other hand, if we choose $\eta = \psi(x,y) =$ constant as the integral of (1.5.14), equation (1.5.3) assumes the form

$$u_{\xi\xi} = H_3^*(\xi, \eta, u, u_\xi, u_\eta). \tag{1.5.16}$$

Equations for which $B^2 - 4AC = 0$ are called *parabolic.*

Case III. $B^2 - 4AC < 0$.

Equations for which $B^2 - 4AC < 0$ are called *elliptic*. In this case, equations (1.5.10ab) have no real solutions. So there are two families of complex characteristics. Since roots ξ, η of (1.5.10ab) are complex conjugates of each other, we introduce the new real variables as

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = \frac{1}{2i}(\xi - \eta), \tag{1.5.17ab}$$

so that $\xi = \alpha + i\beta$ and $\eta = \alpha - i\beta$.

We use (1.5.17ab) to transform (1.5.3) into the form

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} = H_4(\alpha, \beta, u, u_\alpha, u_\beta), \tag{1.5.18}$$

where the coefficients of this equation assume the same form as the coefficients of (1.5.3). It can easily be verified that $A^* = 0$ and $C^* = 0$ take the form

$$A^{**} - C^{**} \pm iB^{**} = 0$$

which are satisfied if and only if

$$A^{**} = C^{**} \quad \text{and} \quad B^{**} = 0.$$

Thus, dividing by A^{**}, equation (1.5.18) reduces to the form

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{H_4}{A^{**}} = H_5(\alpha, \beta, u, u_\alpha, u_\beta). \tag{1.5.19}$$

This is called the *canonical form of the elliptic equation*.

In summary, we state that the equation (1.5.1) is called *hyperbolic*, *parabolic*, or *elliptic* at a point (x_0, y_0) accordingly as

$$B^2(x_0, y_0) - 4A(x_0, y_0)\,C(x_0, y_0) > = < 0. \tag{1.5.20}$$

If it is true at all points in a given domain, then the equation is said to be *hyperbolic*, *parabolic*, or *elliptic* in that domain. Finally, it has been shown above that, for the case of two independent variables, a transformation can always be found to transform the given equation to the canonical form. However, in the case of several independent variables, in general, it is *not* possible to find such a transformation.

These three types of partial differential equations arise in many areas of mathematical and physical sciences. Usually, boundary-value problems are associated with elliptic equations, whereas the initial-value problems arise in connection with hyperbolic and parabolic equations.

Example 1.5.1 Show that

(a) the wave equation $u_{tt} - c^2 u_{xx} = 0$ is hyperbolic,

(b) the diffusion equation $u_t - \kappa u_{xx} = 0$ is parabolic,

(c) the Laplace equation $u_{xx} + u_{yy} = 0$ is elliptic,

(d) the Tricomi equation $u_{xx} + xu_{yy} = 0$ is elliptic for $x > 0$, parabolic for $x = 0$, and hyperbolic for $x < 0$.

For case (a) $A = -c^2$, $B = 0$, and $C = 1$. Hence, $B^2 - 4AC = c^2 > 0$ for all x and t. So, the wave equation is hyperbolic everywhere. Similarly, the reader can show (b) and (c). Finally, for case (d), $A = 1$, $B = 0$, $C = x$, hence, $B^2 - 4AC = -4x < 0, = 0$, or > 0 accordingly as $x > 0$, $x = 0$, or $x < 0$, and the result follows.

Example 1.5.2 Find the characteristic equations and characteristics and then reduce the equation

$$x\,u_{xx} + u_{yy} = x^2 \tag{1.5.21}$$

to canonical form.

In this problem, $A = x$, $B = 0$, $C = 1$, $B^2 - 4AC = -4x$. Thus, the equation is hyperbolic if $x < 0$, parabolic if $x = 0$, and elliptic if $x > 0$.

The characteristics equations are

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \pm\frac{1}{\sqrt{-x}}. \tag{1.5.22ab}$$

Hence,

$$y = \pm 2\sqrt{-x} = \text{constant} = \pm 2\sqrt{-x} + c,$$

or

$$\xi = y + 2\sqrt{-x} = \text{constant}, \quad \eta = y - 2\sqrt{-x} = \text{constant}. \tag{1.5.23ab}$$

These represent two branches of the parabolas $(y - c)^2 = -4x$ where c is a constant. The former equation ($\xi = \text{constant}$) gives a branch with positive slopes, whereas the latter equation ($\eta = \text{constant}$) represents a branch with negative slopes as shown in Figure 1.1. Both branches are tangent to the y-axis which is the single characteristic in the parabolic region. Indeed, the y-axis is the envelope of the characteristics for the hyperbolic region $x < 0$.

For $x < 0$, we use the transformations

$$\xi = y + 2\sqrt{-x}, \quad \eta = y - 2\sqrt{-x} \tag{1.5.24ab}$$

to reduce (1.5.21) to the canonical form.

We find

$$\xi_x = -\frac{1}{\sqrt{-x}}, \quad \xi_y = 1, \quad \xi_{xx} = -\frac{1}{2}\frac{1}{(-x)^{3/2}}, \quad \xi_{yy} = 0,$$

$$\eta_x = +\frac{1}{\sqrt{-x}}, \quad \eta_y = 1, \quad \eta_{xx} = \frac{1}{2}\frac{1}{(-x)^{3/2}}, \quad \xi_{yy} = 0,$$

$$(\xi - \eta) = 4\sqrt{-x} \quad \text{and} \quad (\xi - \eta)^4 = (16x)^2.$$

Consequently, the equation

$$x\,u_{xx} + u_{yy} = x^2$$

reduces to the form

$$x\left(u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}\right)$$
$$+ \left(u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}\right) = x^2.$$

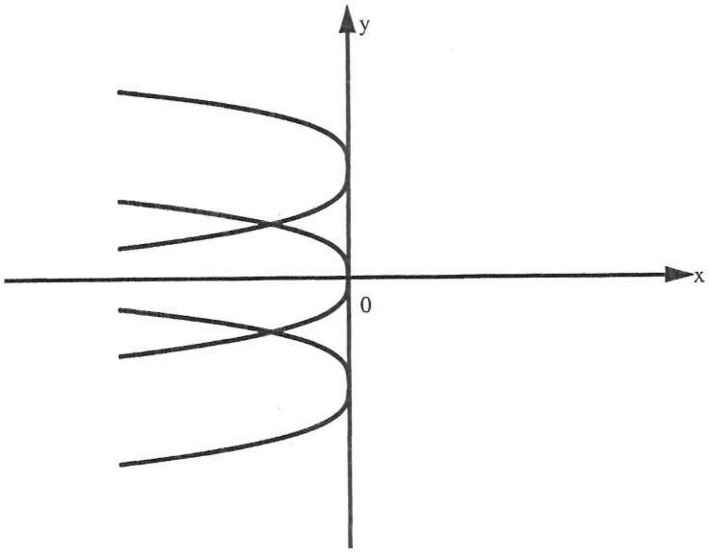

Figure 1.1 Characteristics are parabolas for $x < 0$.

Or,

$$x\left[u_{\xi\xi}\left(-\frac{1}{x}\right)+2u_{\xi\eta}\left(\frac{1}{x}\right)-u_{\eta\eta}\left(\frac{1}{x}\right)-\frac{1}{2}\frac{1}{(-x)^{3/2}}u_{\xi}+\frac{1}{2}\frac{1}{(-x)^{3/2}}u_{\eta}\right]$$
$$+\left[u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}\right]=x^2.$$

This leads to the transformed equation which is

$$4\,u_{\xi\eta}+\frac{1}{2}\frac{1}{\sqrt{-x}}\left(u_{\xi}-u_{\eta}\right)=\frac{1}{(16)^2}\left(\xi-\eta\right)^4,$$

or

$$u_{\xi\eta}=\frac{1}{4}\cdot\frac{1}{(16)^2}\left(\xi-\eta\right)^4-\frac{1}{2}\frac{1}{(\xi-\eta)}\left(u_{\xi}-u_{\eta}\right). \tag{1.5.25}$$

This is the first canonical form.

For $x > 0$, we use the transformations

$$\xi = y + 2i\sqrt{x}, \quad \eta = y - 2i\sqrt{x}$$

so that

$$\alpha=\frac{1}{2}\left(\xi+\eta\right)=y, \quad \beta=\frac{1}{2i}\left(\xi-\eta\right)=2\sqrt{x}. \tag{1.5.26ab}$$

Clearly,

$$\alpha_x = 0, \quad \alpha_y = 1, \quad \alpha_{xx} = 0, \quad \alpha_{yy} = 0, \quad \alpha_{xy} = 0,$$
$$\beta_x = \frac{1}{\sqrt{x}}, \quad \beta_y = 0, \quad \beta_{xx} = -\frac{1}{2}\frac{1}{x^{3/2}}, \quad \beta_{yy} = 0.$$

So, equation (1.5.21) reduces to the canonical form

$$x\left(u_{\alpha\alpha}\alpha_x^2+2u_{\alpha\beta}\alpha_x\beta_x+u_{\beta\beta}\beta_x^2+u_{\alpha}\alpha_{xx}+u_{\beta}\beta_{xx}\right)$$
$$+\left(u_{\alpha\alpha}\alpha_y^2+2u_{\alpha\beta}\alpha_y\beta_y+u_{\beta\beta}\beta_y^2+u_{\alpha}\alpha_{yy}+u_{\beta}\beta_{yy}\right)=\left(\frac{\beta}{2}\right)^4,$$

or

$$u_{\alpha\alpha}+u_{\beta\beta}-\frac{1}{2}\frac{1}{\sqrt{x}}u_{\beta}=\left(\frac{\beta}{2}\right)^4$$
$$u_{\alpha\alpha}+u_{\beta\beta}=\frac{1}{\beta}u_{\beta}+\left(\frac{\beta}{2}\right)^4. \tag{1.5.27}$$

This is the desired canonical form of the elliptic equation.

Finally, for the parabolic case ($x = 0$), equation (1.5.21) reduces to the canonical form

$$u_{yy} = 0. \tag{1.5.28}$$

In this case, the characteristic determined from $\frac{dx}{dy} = 0$ is $x = 0$. That is, the y-axis is the characteristic curve, and it represents a curve across which a transition from hyperbolic to elliptic form takes place.

Example 1.5.3 (*The Cauchy Problem for the Wave Equation*). The one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{1.5.29}$$

is a special case of (1.5.1) with $A = -c^2$, $B = 0$, and $C = 1$. Hence, $B^2 - 4AC = 4c^2 > 0$, and therefore, the equation is hyperbolic, as mentioned before. According to (1.5.10ab), the equations of characteristics are

$$\frac{dt}{dx} = \pm\frac{1}{c}. \tag{1.5.30}$$

Or

$$\xi = x - ct = \text{constant}, \quad \eta = x + ct = \text{constant}. \tag{1.5.31ab}$$

This shows that the characteristics are straight lines in the (x, t)-plane. The former represents a family of lines with positive slopes, and the latter a family of lines with negative slopes in the (x, t)-plane. In terms of new coordinates ξ and η, we obtain

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \quad u_{tt} = c^2 \left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right)$$

so that the wave equation (1.5.29) becomes

$$-4c^2 u_{\xi\eta} = 0. \tag{1.5.32}$$

Since $c \neq 0$, $u_{\xi\eta} = 0$ which can be integrated twice to obtain the solution

$$u(\xi, \eta) = \phi(\xi) + \psi(\eta), \tag{1.5.33}$$

where ϕ and ψ are arbitrary functions. Thus, in terms of the original variables, we obtain

$$u(x,t) = \phi(x - ct) + \psi(x + ct). \tag{1.5.34}$$

This represents the general solution provided ϕ and ψ are arbitrary but twice differentiable functions. The first term $\phi(x - ct)$ represents a wave (or disturbance) traveling to the right with constant speed c. Similarly, $\psi(x + ct)$ represents a wave moving to the left with constant speed c. Thus, the general solution $u(x,t)$ is a linear superposition of two such waves.

The typical *initial-value problem* for the wave equation (1.5.29) is the *Cauchy problem* of an infinite string with initial data

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \tag{1.5.35ab}$$

where $f(x)$ and $g(x)$ are given functions representing the initial displacement and initial velocity, respectively. The conditions (1.5.35ab) imply that

$$\phi(x) + \psi(x) = f(x), \tag{1.5.36}$$

$$-c\phi'(x) + c\psi'(x) = g(x), \tag{1.5.37}$$

where the prime denotes the derivative with respect to the argument. Integrating equation (1.5.37) gives

$$-c\phi(x) + c\psi(x) = \int_{x_0}^{x} g(\tau)\, d\tau, \tag{1.5.38}$$

where x_0 is an arbitrary constant. Equations (1.5.36) and (1.5.38) now yield

$$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\tau)\, d\tau$$

$$\psi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\tau)\, d\tau.$$

Obviously, (1.5.34) gives the so called *d'Alembert solution* of the Cauchy problem as

$$u(x,t) = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau. \tag{1.5.39}$$

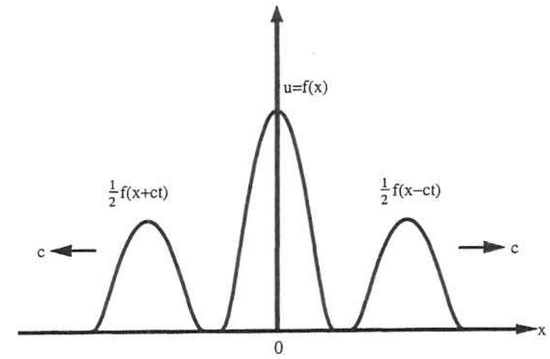

Figure 1.2 Splitting of initial data into equal waves.

It can be verified by direct substitution that $u(x,t)$ satisfies equation (1.5.29) provided f is twice differentiable and g is differentiable. Further, the d'Alembert solution (1.5.39) can be used to show that this problem is *well posed*. The solution (1.5.39) consists of terms involving $f(x \pm ct)$ and the term involving the integral of g. Both terms combined together suggest that the value of the solution at position x and time t depends only on the initial values of $f(x)$ at points $x \pm ct$ and the value of the integral of g between these points. The interval $(x - ct,\ x + ct)$ is called the *domain of dependence* of (x,t). The terms involving $f(x \pm ct)$ in (1.5.39) show that waves are propagated along the characteristics with constant velocity c.

In particular, if $g(x) = 0$, the solution is represented by the first two terms in (1.5.39), that is,

$$u(x,t) = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right]. \tag{1.5.40}$$

Physically, this solution shows that the initial data is split into two equal waves, similar in shape to the initial displacement, but of *half* the amplitude.

These waves propagate in the opposite direction with the same constant speed c as shown in Figure 1.2.

To investigate the physical significance of the d'Alembert solution, it is convenient to rewrite the solution in the form

$$u(x,t) = \frac{1}{2} f(x - ct) - \frac{1}{2c}\int_{0}^{x-ct} g(\tau)\, d\tau + \frac{1}{2} f(x + ct) + \frac{1}{2c}\int_{0}^{x+ct} g(\tau)\, d\tau, \tag{1.5.41}$$

$$= \Phi(x - ct) + \Psi(x + ct), \tag{1.5.42}$$

where

$$\Phi(\xi) = \frac{1}{2} f(\xi) - \frac{1}{2c}\int_{0}^{\xi} g(\tau)\, d\tau, \quad \Psi(\eta) = \frac{1}{2} f(\eta) + \frac{1}{2c}\int_{0}^{\eta} g(\tau)\, d\tau. \tag{1.5.43ab}$$

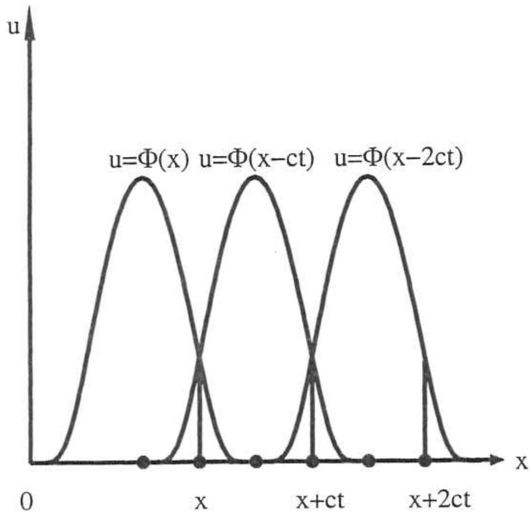

Figure 1.3 Graphical representation of solution.

Physically, $\Phi(x - ct)$ represents a progressive wave propagating in the positive x-direction with constant speed c without change of shape as shown in Figure 1.3. Similarly, $\Psi(x + ct)$ also represents a progressive wave traveling in the negative x-direction with the same speed without change of shape.

A more general form of the wave equation is

$$u_{tt} - a^2(x)\,u_{xx} = 0, \tag{1.5.44}$$

where a is a function of x only. The characteristic coordinates are now given by

$$\xi = t - \int^x \frac{d\tau}{a(\tau)}, \quad \eta = t + \int^x \frac{d\tau}{a(\tau)}. \tag{1.5.45ab}$$

Thus,

$$\begin{aligned}
u_x &= -\frac{1}{a}u_\xi + \frac{1}{a}u_\eta, \quad u_t = u_\xi + u_\eta,\\
u_{xx} &= \frac{1}{a^2}(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) - (u_\eta - u_\xi)\frac{a'(x)}{a^2},\\
u_{tt} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.
\end{aligned}$$

Consequently, equation (1.5.44) reduces to

$$4u_{\xi\eta} + a'(x)(u_\eta - u_\xi) = 0. \tag{1.5.46}$$

In order to express a' in terms of ξ and η, we observe that

$$\eta - \xi = 2\int^x \frac{d\tau}{a(\tau)}, \tag{1.5.47}$$

so that x is a function of $(\eta - \xi)$. Thus, $a'(x)$ will be some function of $(\eta - \xi)$.

In particular, if $a(x) = Ax^n$, where A is a constant, so that $a'(x) = n\,Ax^{n-1}$, and when $n \neq 1$, result (1.5.47) gives

$$\eta - \xi = -\frac{2}{A}\frac{1}{(n-1)}\frac{1}{x^{n-1}} \tag{1.5.48}$$

so that

$$a'(x) = -\frac{2n}{(n-1)} \cdot \frac{1}{\eta - \xi}.$$

Thus, equation (1.5.46) reduces to the form

$$4u_{\xi\eta} - \frac{2n}{(n-1)}\frac{1}{(\eta - \xi)}(u_\eta - u_\xi) = 0.$$

Finally, we find that

$$u_{\xi\eta} = \frac{n}{2(n-1)}\frac{1}{(\eta - \xi)}(u_\eta - u_\xi). \tag{1.5.49}$$

When $n = 1$, $a(x) = Ax$, and $a'(x) = A$, substituting $\xi = \frac{\alpha}{A}$ and $\eta = \frac{\beta}{A}$ can be used to reduce equation (1.5.46) to

$$u_{\alpha\beta} = \frac{1}{4}(u_\alpha - u_\beta). \tag{1.5.50}$$

Equation (1.5.49) is called the *Euler–Darboux equation* which has the hyperbolic form

$$u_{xy} = \frac{m}{x - y}(u_x - u_y), \tag{1.5.51}$$

where m is a positive integer.

We next note that

$$\frac{\partial^2}{\partial x \partial y}[(x - y)u] = \frac{\partial}{\partial x}\left[(x - y)\frac{\partial u}{\partial y} - u\right] = (x - y)u_{xy} + (u_y - u_x). \tag{1.5.52}$$

When $m = 1$, equation (1.5.51) becomes

$$(x - y)u_{xy} = u_x - u_y$$

so that (1.5.52) reduces to

$$\frac{\partial^2}{\partial x \partial y}\left[(x-y)\,u\right]=0. \tag{1.5.53}$$

This shows that the solution of (1.5.53) is $(x-y)\,u=\phi(x)+\psi(y)$. Hence, the solution of (1.5.51) with $m=1$ is

$$u(x,y)=\frac{\phi(x)+\psi(y)}{x-y}, \tag{1.5.54}$$

where ϕ and ψ are arbitrary functions.

We multiply (1.5.51) by $(x-y)$, and apply the derivative $\frac{\partial^2}{\partial x \partial y}$, so that the result is, due to (1.5.52),

$$(x-y)\frac{\partial^2}{\partial x \partial y}(u_{xy})+\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial x}\right)(u_{xy})=m\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)u_{xy}.$$

Or

$$(x-y)\frac{\partial^2}{\partial x \partial y}(u_{xy})=(m+1)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)(u_{xy}). \tag{1.5.55}$$

Hence, if u is a solution of (1.5.51), then u_{xy} is a solution of (1.5.51) with m replaced by $m+1$. When $m=1$, the solution is given by (1.5.54), and hence, the solution of (1.5.51) takes the form

$$u(x,y)=\frac{\partial^{2m-2}}{\partial x^{m-1}\partial y^{m-1}}\left[\frac{\phi(x)+\psi(y)}{x-y}\right], \tag{1.5.56}$$

where ϕ and ψ are arbitrary functions.

1.6 The Method of Separation of Variables

This method combined with the principle of superposition is widely used to solve initial boundary-value problems involving linear partial differential equations. Usually, the dependent variable $u(x,y)$ is expressed in the separable $u(x,y)=X(x)\,Y(y)$ where X and Y are functions of x and y, respectively. In many cases, the partial differential equation reduces to two ordinary differential equations for X and Y. A similar treatment can be applied to equations in three or more independent variables. However, the question of separability of a partial differential equation into two or more ordinary differential equations is by no means a trivial one. In spite of this question, the method is widely used in finding solutions of a large class of initial boundary-value problems. This method of solution is known as the *Fourier method* (or *the method of eigenfunction expansion*). Thus, the procedure outlined above leads to the important ideas of eigenvalues, eigenfunctions, and orthogonality, all of which are very general and powerful for dealing with linear problems. The following examples illustrate the general nature of this method.

Example 1.6.1 (*Transverse Vibration of a String*). We consider the one-dimensional linear wave equation

$$u_{tt}=c^2u_{xx},\qquad 0<x<\ell,\ t>0, \tag{1.6.1}$$

where $c^2=T^*/\rho$, T^* is a constant tension, and ρ is the constant line density of the string. The initial and boundary conditions are

$$u(x,0)=f(x),\quad u_t(x,0)=g(x),\quad 0\le x\le\ell, \tag{1.6.2ab}$$
$$u(0,t)=0=u(\ell,t),\qquad t>0, \tag{1.6.3ab}$$

where f and g are the initial displacement and initial velocity, respectively.

According to the method of separation of variables, we assume a separable solution of the form

$$u(x,t)=X(x)\,T(t)\neq 0, \tag{1.6.4}$$

where X is a function of x only, and T is a function of t only.

Substituting this solution in equation (1.6.1) yields

$$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{c^2T}\frac{d^2T}{dt^2}. \tag{1.6.5}$$

Since the left side of this equation is a function of x only and the right-hand side is a function of t only, it follows that (1.6.5) can be true only if both sides are equal to the same constant value. We then write

$$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{c^2T}\frac{d^2T}{dt^2}=\lambda, \tag{1.6.6}$$

where λ is an arbitrary separation constant. Thus, this leads to the pair of ordinary differential equations

$$\frac{d^2X}{dx^2}=\lambda X,\qquad \frac{d^2T}{dt^2}=\lambda c^2T. \tag{1.6.7ab}$$

We solve this pair of equations by using the boundary conditions which are obtained from (1.6.3ab) as

$$u(0,t)=X(0)\,T(t)=0\quad\text{for}\quad t>0, \tag{1.6.8}$$
$$u(\ell,t)=X(\ell)\,T(t)=0\quad\text{for}\quad t>0. \tag{1.6.9}$$

Hence, we take $T(t)\neq 0$ to obtain

$$X(0)=0=X(\ell). \tag{1.6.10ab}$$

There are three possible cases: (i) $\lambda>0$, (ii) $\lambda=0$, (iii) $\lambda<0$.

For case (i), $\lambda=\alpha^2>0$. The solution of (1.6.7a) is

$$X(x) = A\,e^{\alpha x} + B\,e^{-\alpha x}, \tag{1.6.11}$$

which together with (1.6.10ab) leads to $A = B = 0$. This leads to a trivial solution $u(x,t) = 0$.

For case (ii), $\lambda = 0$. In this case, the solution of (1.6.7a) is

$$X(x) = Ax + B. \tag{1.6.12}$$

Then. we use (1.6.10ab) to obtain $A = B = 0$. This also leads to the trivial solution $u(x,t) = 0$.

For case (iii), $\lambda < 0$, and hence, we write $\lambda = -\alpha^2$ so that the solution of equation (1.6.7a) gives

$$X = A\cos\alpha x + B\sin\alpha x, \tag{1.6.13}$$

whence, using (1.6.10ab), we derive the nontrivial solution

$$X(x) = B\sin\alpha x, \tag{1.6.14}$$

where B is an arbitrary nonzero constant. Clearly, since $B \neq 0$ and $X(\ell) = 0$,

$$\sin\alpha\ell = 0, \tag{1.6.15}$$

which gives the solution for the parameter α

$$\alpha = \alpha_n = \left(\frac{n\pi}{\ell}\right), \quad n = 1, 2, 3, \ldots. \tag{1.6.16}$$

Note that $n = 0$ $(\alpha = 0)$ leads to the trivial solution $u(x,t) = 0$, and hence, the case $n = 0$ is to be excluded.

Evidently, we see from (1.6.16) that there exists an infinite set of discrete values of α for which the problem has a nontrivial solution. These values α_n are called the *eigenvalues*, and the corresponding solutions are

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{\ell}\right). \tag{1.6.17}$$

We next solve (1.6.7a) with $\lambda = -\alpha_n^2$ to find the solution for $T_n(t)$ as

$$T_n(t) = C_n\cos(\alpha_n ct) + D_n\sin(\alpha_n ct), \tag{1.6.18}$$

where C_n and D_n are constants of integration. Combining (1.6.17) with (1.6.18) yields the solution from (1.6.4) as

$$u_n(x,t) = \left[a_n\cos\left(\frac{n\pi ct}{\ell}\right) + b_n\sin\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.19}$$

where $a_n = C_nB_n$, $b_n = B_nD_n$ are new arbitrary constants and $n = 1, 2, 3, \ldots$. These solutions $u_n(x,t)$, corresponding to eigenvalues $\alpha_n = \left(\frac{n\pi}{\ell}\right)$, are called the *eigenfunctions*. Finally, since the problem is linear, the most general solution is obtained by the principle of superposition in the form

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{\ell} + b_n\sin\frac{n\pi ct}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.20}$$

provided the series converges and it is twice continuously differentiable with respect to x and t. The arbitrary coefficients a_n and b_n are determined from the initial conditions (1.6.2ab) which give

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n\sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.21}$$

$$u_t(x,0) = g(x) = \left(\frac{\pi c}{\ell}\right)\sum_{n=1}^{\infty} n\,b_n\sin\left(\frac{n\pi x}{\ell}\right). \tag{1.6.22}$$

Either by a Fourier series method or by direct multiplication of (1.6.21) and (1.6.22) by $\sin\left(\frac{m\pi x}{\ell}\right)$ and integrating from 0 to ℓ, we can find a_n and b_n as

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx,\quad b_n = \frac{2}{n\pi c}\int_0^{\ell} g(x)\sin\left(\frac{n\pi x}{\ell}\right)dx, \tag{1.6.23ab}$$

in which we have used the result

$$\int_0^{\ell}\sin\left(\frac{m\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{\ell}{2}\delta_{mn}, \tag{1.6.24}$$

where δ_{mn} are *Kronecker* delta. Thus, (1.6.20) represents the solution where a_n and b_n are given by (1.6.23ab). Hence, the problem is completely solved.

We examine the physical significance of the solution (1.6.19) in the context of the free vibration of a string of length ℓ. The eigenfunctions

$$u_n(x,t) = (a_n\cos\omega_n t + b_n\sin\omega_n t)\sin\left(\frac{n\pi x}{\ell}\right), \quad \left(\omega_n = \frac{n\pi c}{\ell}\right), \tag{1.6.25}$$

are called the nth *normal modes* of vibration or the nth harmonic, and ω_n represents the discrete spectrum of *circular (or radian) frequency* or $\nu_n = \frac{\omega_n}{2\pi} = \frac{nc}{2\ell}$, which are called the *angular frequencies*. The first harmonic $(n = 1)$ is called the *fundamental harmonic* and all other harmonics $(n > 1)$ are called *overtones*. The frequency of the fundamental mode is given by

$$\omega_1 = \frac{\pi c}{\ell}, \quad \nu_1 = \frac{1}{2\ell}\sqrt{\frac{T^*}{\rho}}. \tag{1.6.26ab}$$

Result (1.6.26ab) is considered the fundamental law (or *Mersenne law*) of a stringed musical instrument. The angular frequency of the fundamental mode of transverse vibration of a string varies as the square root of the tension, inversely as

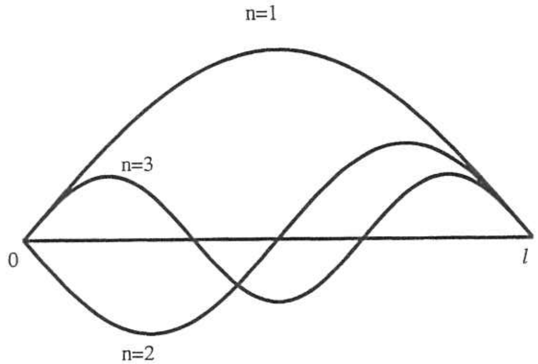

Figure 1.4 Several modes of vibration in a string.

the length, and inversely as the square root of the density. The period of the fundamental mode is $T_1 = \frac{2\pi}{\omega_1} = \frac{2\ell}{c}$, which is called the *fundamental period*. Finally, the solution (1.6.20) describes the motion of a plucked string as a superposition of all normal modes of vibration with frequencies which are all integral multiples ($\omega_n = n\omega_1$ or $\nu_n = n\nu_1$) of the fundamental frequency. This is the main reason for the fact that stringed instruments produce more sweet musical sounds (or tones) than drum instruments.

In order to describe waves produced in the plucked string with zero initial velocity $(u_t(x,0) = 0)$, we write the solution (1.6.25) in the form

$$u_n(x,t) = a_n \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi ct}{\ell}\right), \quad n = 1, 2, 3, \ldots. \tag{1.6.27}$$

These solutions are called *standing waves* with amplitude $a_n \sin\left(\frac{n\pi x}{\ell}\right)$, which vanishes at

$$x = 0, \frac{\ell}{n}, \frac{2\ell}{n}, \ldots, \ell.$$

These are called the *nodes* of the *n*th harmonic. The string displays n loops separated by the nodes as shown in Figure 1.4.

It follows from elementary trigonometry that (1.6.27) takes the form

$$u_n(x,t) = \frac{1}{2} a_n \left[\sin\frac{n\pi}{\ell}(x - ct) + \sin\frac{n\pi}{\ell}(x + ct)\right]. \tag{1.6.28}$$

This shows that a standing wave is expressed as a sum of two progressive waves of equal amplitude traveling in opposite directions. This result is in agreement with the d'Alembert solution.

Finally, we can rewrite the solution (1.6.19) of the *n*th normal modes in the form

$$u_n(x,t) = c_n \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi ct}{\ell} - \varepsilon_n\right), \tag{1.6.29}$$

where $c_n = \left(a_n^2 + b_n^2\right)^{\frac{1}{2}}$ and $\tan\,\varepsilon_n = \left(\frac{b_n}{a_n}\right)$.

This solution represents transverse vibrations of the string at any point x and at any time t with amplitude $c_n \sin\left(\frac{n\pi x}{\ell}\right)$ and circular frequency $\omega_n = \frac{n\pi c}{\ell}$. This form of the solution enables us to calculate the kinetic and potential energies of the transverse vibrations. The total kinetic energy (K.E.) is obtained by integrating with respect to x from 0 to ℓ, that is,

$$K_n = K.E. = \int_0^\ell \frac{1}{2}\rho\left(\frac{\partial u_n}{\partial t}\right)^2 dx, \tag{1.6.30}$$

where ρ is the line density of the string. Similarly, the total potential energy (P.E.) is given by

$$V_n = P.E. = \frac{1}{2}T^* \int_0^\ell \left(\frac{\partial u_n}{\partial x}\right)^2 dx. \tag{1.6.31}$$

Substituting (1.6.29) in (1.6.30) and (1.6.31) gives

$$\begin{aligned} K_n &= \frac{1}{2}\rho\left(\frac{n\pi c}{\ell}c_n\right)^2 \sin^2\left(\frac{n\pi ct}{\ell} - \varepsilon_n\right) \int_0^\ell \sin^2\left(\frac{n\pi x}{\ell}\right) dx \\ &= \frac{\rho c^2\pi^2}{4\ell}(n\,c_n)^2 \sin^2\left(\frac{n\pi ct}{\ell} - \varepsilon_n\right) = \frac{1}{4}\rho\ell\omega_n^2 c_n^2 \sin^2(\omega_n t - \varepsilon_n), \end{aligned} \tag{1.6.32}$$

where $\omega_n = \frac{n\pi c}{\ell}$.

Similarly,

$$\begin{aligned} V_n &= \frac{1}{2}T^*\left(\frac{n\pi c_n}{\ell}\right)^2 \cos^2\left(\frac{n\pi ct}{\ell} - \varepsilon_n\right) \int_0^\ell \cos^2\left(\frac{n\pi x}{\ell}\right) dx \\ &= \frac{\pi^2 T^*}{4\ell}(n\,c_n)^2 \cos^2\left(\frac{n\pi ct}{\ell} - \varepsilon_n\right) = \frac{1}{4}\rho\ell\omega_n^2 c_n^2 \cos^2(\omega_n t - \varepsilon_n). \end{aligned} \tag{1.6.33}$$

Thus, the total energy of the *n*th normal modes of vibrations is given by

$$E_n = K_n + V_n = \frac{1}{4}\rho\ell(\omega_n c_n)^2 = \text{constant}. \tag{1.6.34}$$

For a given string oscillating in a normal mode, the total energy is proportional to the square of the circular frequency and to the square of the amplitude.

Finally, the total energy of the system is given by

$$E = \sum_{n=1}^{\infty} E_n = \frac{1}{4}\rho\ell \sum_{n=1}^{\infty} \omega_n^2 c_n^2, \tag{1.6.35}$$

which is constant because $E_n = \text{constant}$.

Example 1.6.2 (One-Dimensional Diffusion Equation). The temperature distribution $u(x,t)$ in a homogeneous rod of length ℓ satisfies the diffusion equation

$$u_t = \kappa u_{xx}, \quad 0 < x < \ell, \quad t > 0, \tag{1.6.36}$$

with the boundary and initial conditions

$$u(0,t) = 0 = u(\ell,t), \qquad t \geq 0 \tag{1.6.37ab}$$

$$u(x,0) = f(x), \qquad 0 \leq x \leq \ell, \tag{1.6.38}$$

where κ is a diffusivity constant.

We assume a separable solution of (1.6.36) in the form

$$u(x,t) = X(x)\,T(t) \neq 0. \tag{1.6.39}$$

Substituting (1.6.39) in (1.6.36) gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\kappa T}\frac{dT}{dt}. \tag{1.6.40}$$

Since the left-hand side depends only on x and the right-hand side is a function of time t only, result (1.6.40) can be true only if both sides are equal to the same constant λ. Thus, we obtain two ordinary differential equations

$$\frac{d^2X}{dx^2} - \lambda X = 0, \quad \frac{dT}{dt} - \lambda\kappa T = 0. \tag{1.6.41ab}$$

For $\lambda \geq 0$, the only solution of the form (1.6.39) consistent with the given boundary conditions is $u(x,t) \equiv 0$. Hence, for negative $\lambda = -\alpha^2$,

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0, \quad \frac{dT}{dt} + \kappa\alpha^2 T = 0, \tag{1.6.42ab}$$

which admit solutions as

$$X(x) = A\cos\alpha x + B\sin\alpha x \tag{1.6.43}$$

and

$$T(t) = C\exp\left(-\kappa\alpha^2 t\right), \tag{1.6.44}$$

where A, B, and C are constants of integration.

The boundary conditions for $X(x)$ are

$$X(0) = 0 = X(\ell), \tag{1.6.45}$$

which are used to find A and B in solution (1.6.43). It turns out that $A = 0$ and $B \neq 0$. Hence,

$$\sin\alpha\ell = 0, \tag{1.6.46}$$

which gives the eigenvalues

$$\alpha = \alpha_n = \frac{n\pi}{\ell}, \quad n = 1, 2, 3, \ldots. \tag{1.6.47}$$

The value $n = 0$ is excluded because it leads to a trivial solution. Thus, the eigenfunctions are given by

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.48}$$

where B_n are nonzero constants.

With $\alpha = \alpha_n = \frac{n\pi}{\ell}$, we combine (1.6.44) with (1.6.48) to obtain the solution for $u_n(x,t)$ as

$$u_n(x,t) = a_n \exp\left[-\left(\frac{n\pi}{\ell}\right)^2 \kappa t\right] \sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.49}$$

where $a_n = B_n C_n$ is a new constant. Thus, (1.6.47) and (1.6.49) constitute an infinite set of eigenvalues and eigenfunctions. Thus, the most general solution is obtained by the principle of superposition in the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left[-\left(\frac{n\pi}{\ell}\right)^2 \kappa t\right] \sin\left(\frac{n\pi x}{\ell}\right). \tag{1.6.50}$$

Now, the initial condition implies that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.51}$$

which determines a_n, in view of (1.6.24), as

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right) dx. \tag{1.6.52}$$

Thus, the final form of the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{\ell}\int_0^\ell f(x')\sin\left(\frac{n\pi x'}{\ell}\right)dx'\right]\exp\left[-\left(\frac{n\pi}{\ell}\right)^2\kappa t\right]\sin\left(\frac{n\pi x}{\ell}\right). \tag{1.6.53}$$

It follows from the series solution (1.6.53) that the series satisfies the given boundary and initial conditions. It also satisfies the equation (1.6.36) because the series is convergent for all x $(0 \leq x \leq \ell)$ and $t \geq 0$ and can be differentiated term by term. Physically, the temperature distribution decays exponentially with time t. This shows a striking contrast to the wave equation, whose solution oscillates in time t. The time scale of decay for the nth mode is $T_d \sim \frac{1}{\kappa}\left(\frac{\ell}{n\pi}\right)^2$ which is directly proportional to ℓ^2 and inversely proportional to the thermal diffusivity.

The method of separation of variables is applicable to the wave equation and the diffusion equation, and also to problems involving Laplace's equation and other equations in two or three dimensions with a wide variety of initial and boundary conditions. We consider the following examples.

Example 1.6.3 (Two-Dimensional Diffusion Equation). We consider

$$u_t = \kappa\left(u_{xx} + u_{yy}\right), \quad 0 < x < a, \quad 0 < y < b, \quad t > 0, \tag{1.6.54}$$
$$u(x,y,t) = f(x,y) \quad \text{at} \quad t = 0, \tag{1.6.55}$$
$$u(x,y,t) = 0 \quad \text{on} \quad \partial D, \tag{1.6.56}$$

where ∂D is the boundary of the rectangle defined by $0 \le x \le a$, $0 \le y \le b$.

The method here is precisely the same as in the previous examples except that we seek a solution of (1.6.54) in the form

$$u(x,y,z) = S(x,y)\,T(t) \neq 0, \tag{1.6.57}$$

so that S and T satisfy the equations

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \lambda S = 0, \tag{1.6.58}$$
$$\frac{\partial T}{\partial t} + \kappa\lambda T = 0. \tag{1.6.59}$$

For $\lambda \le 0$, the separable solution (1.6.57) with the given boundary data leads only to a trivial solution $u(x,y,t) \equiv 0$. Hence, for positive λ, we solve (1.6.58), (1.6.59) subject to the given boundary and initial conditions. Equation (1.6.58) is an elliptic equation, and here we seek a solution $S(x,y)$ which satisfies the boundary conditions

$$S(0,y) = 0 = S(a,y) \quad \text{for} \quad 0 \le y \le b, \tag{1.6.60}$$
$$S(x,0) = 0 = S(x,b) \quad \text{for} \quad 0 \le x \le a. \tag{1.6.61}$$

We also seek a separable solution of (1.6.58) in the form

$$S(x,y) = X(x)\,Y(y) \neq 0 \tag{1.6.62}$$

and find that $X(x)$ and $Y(y)$ satisfy the equation

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0,$$

that is,

$$\frac{X''}{X} = -\mu = -\left(\frac{Y''}{Y} + \lambda\right). \tag{1.6.63}$$

Or

$$X'' + \mu X = 0, \quad Y'' + (\lambda - \mu)\,Y = 0. \tag{1.6.64ab}$$

These equations have to be solved with the boundary conditions

$$\left.\begin{array}{l} X(0) = 0 = X(a) \\ Y(0) = 0 = Y(b) \end{array}\right\}. \tag{1.6.65ab}$$

The eigenvalue problem (1.6.64ab) with (1.6.65ab) gives the eigenvalues

$$\mu_m = \left(\frac{m\pi}{a}\right)^2, \tag{1.6.66}$$

and the corresponding eigenfunctions

$$X_m(x) = A_m \sin\left(\frac{m\pi x}{a}\right), \tag{1.6.67}$$

when $m = 1, 2, 3, \ldots$. Thus, equation (1.6.64b) becomes

$$Y + (\lambda - \mu_m)\,Y = 0, \tag{1.6.68}$$

which has to be solved with (1.6.65b). This is another eigenvalue problem similar to that already considered and leads to the eigenvalues

$$\lambda_n - \mu_m = \left(\frac{n\pi}{b}\right)^2 \tag{1.6.69}$$

and the corresponding eigenfunctions

$$Y_n(y) = B_n \sin\left(\frac{n\pi y}{b}\right), \tag{1.6.70}$$

where $n = 1, 2, 3, \ldots$. In other words, the solution of equation (1.6.58) becomes

$$S_{mn}(x,y) = X_m(x)\,Y_n(y) = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \tag{1.6.71}$$

where $A_{mn} = A_m B_n$ are constants together with the eigenvalues

$$\lambda_{mn} = \mu_m + \left(\frac{n\pi}{b}\right)^2 = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\pi^2, \tag{1.6.72}$$

where $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$.

With λ_{mn} as eigenvalues, we solve (1.6.59) to obtain

$$T_{mn}(t) = B_{mn} \exp\left(-\lambda_{mn}\kappa t\right), \tag{1.6.73}$$

where B_{mn} are integrating constants.

Finally, the solution (1.6.57) can be expressed as a double series

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \exp\left(-\lambda_{mn}\kappa t\right), \tag{1.6.74}$$

where a_{mn} are constants to be determined from the initial condition so that

$$f(x,y) = u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right). \quad (1.6.75)$$

To find constants a_{mn}, we multiply (1.6.75) by $\sin\left(\frac{r\pi x}{a}\right)$ and integrate the result with respect to x from 0 to a with fixed y, so that

$$\frac{a}{2}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{n\pi y}{b}\right) = \int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)dx. \quad (1.6.76)$$

The right-hand side is a function of y and is set equal to $g(y)$, so that

$$\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)dx = g(y). \quad (1.6.77)$$

Then, the coefficients a_{mn} (m fixed) in (1.6.76) are found by multiplying it by $\sin\left(\frac{n\pi y}{b}\right)$ and integrating with respect to y from 0 to b, so that

$$\left(\frac{ab}{4}\right)a_{mn} = \int_0^b g(y)\sin\left(\frac{n\pi y}{b}\right)dy, \quad (1.6.78)$$

whence

$$a_{mn} = \left(\frac{4}{ab}\right)\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy. \quad (1.6.79)$$

Thus, the solution of the problem is given by (1.6.74) where a_{mn} is represented by (1.6.79). The method of construction of the solution shows that the initial and boundary conditions are satisfied by the solution. Moreover, the uniform convergence of the double series justifies differentiation of the series, and this, in turn, permits us to verify the solution by direct substitution in the original diffusion equation (1.6.54).

Example 1.6.4 (Dirichlet's Problem for a Circle). We consider the Laplace equation in polar coordinates (r,θ,z) as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \qquad 0 < r < a, \quad 0 \le \theta < 2\pi, \quad (1.6.80)$$

with the boundary condition

$$u(a,\theta) = f(\theta) \text{ for all } \theta. \quad (1.6.81)$$

According to the method of separation of variables, we seek a solution in the form

$$u(r,\theta) = R(r)\,\Theta(\theta) \ne 0. \quad (1.6.82)$$

Substituting this solution in equation (1.6.80) gives

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

Hence,

$$r^2R'' + rR' - \lambda R = 0 \quad \text{and} \quad \Theta'' + \lambda\Theta = 0. \quad (1.6.83ab)$$

For $\Theta(\theta)$, we naturally require periodic boundary conditions

$$\Theta(\theta + 2\pi) = \Theta(\theta) \qquad \text{for} \quad -\infty < \theta < \infty. \quad (1.6.84)$$

Due to the periodicity condition, for $\lambda < 0$, the solution (1.6.82) leads to a trivial solution. So, there are two cases: (i) $\lambda = 0$ and (ii) $\lambda > 0$.

For case (i), we have the solution

$$u(r,\theta) = (A + B\log r)(C\theta + D). \quad (1.6.85)$$

Since $\log r$ is singular at $r = 0$, hence, $B = 0$. For u to be periodic with period 2π, $C = 0$. Hence, the solution u must be constant for $\lambda = 0$.

For $\lambda > 0$, the solution of equation (1.6.83b) is

$$\Theta(\theta) = A\cos\sqrt{\lambda}\,\theta + B\sin\sqrt{\lambda}\,\theta. \quad (1.6.86)$$

Since $\Theta(\theta)$ is periodic with period 2π, $\sqrt{\lambda}$ must be an integer n so that $\lambda = n^2$, $n = 1, 2, 3, \ldots$. Thus, solution (1.6.86) becomes

$$\Theta(\theta) = A\cos n\theta + B\sin n\theta. \quad (1.6.87)$$

The equation (1.6.83a) is the Euler equation with $\lambda = n^2$. It gives solutions of the form $R(r) = r^{\alpha} \ne 0$ so that (1.6.83a) gives

$$\left[\alpha(\alpha - 1) + \alpha - n^2\right]r^{\alpha} = 0$$

whence, $\alpha = \pm n$. Thus, the solution for $R(r)$is given by

$$R(r) = C\,r^n + D\,r^{-n}. \quad (1.6.88)$$

Since $R(r) \to \infty$ as $r \to 0$ because of the term r^{-n}, hence, $D = 0$. Thus, the solution (1.6.82) reduces to

$$u(r,\theta) = C\,r^n(A\cos n\theta + B\sin n\theta). \quad (1.6.89)$$

By the superposition principle, the solution of the Laplace equation within a circular region including the origin $r = 0$ is

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n\sin n\theta), \quad (1.6.90)$$

where a_0, a_n, and b_n are constants to be determined from the boundary conditions, and the first term $\frac{1}{2}a_0$ represents the constant solution for $\lambda = 0$ $(n = 0)$.

Finally, using the boundary condition (1.6.81), we derive

$$f(\theta) = u(a,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a^n (a_n \cos n\theta + b_n \sin n\theta). \tag{1.6.91}$$

This is exactly the Fourier series representation for $f(\theta)$, and hence, the coefficients are given by

$$a_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\phi)\cos n\phi \, d\phi, \qquad n = 0,1,2,3,\ldots,$$

$$b_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\phi)\sin n\phi \, d\phi, \qquad n = 1,2,3,\ldots.$$

Substituting the values for a_n and b_n in (1.6.91) yields the solution

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi \\ &\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \int_0^{2\pi} (\cos n\theta \, \cos n\phi + \sin n\theta \, \sin n\phi)\, f(\phi)\,d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right\} d\phi, \end{aligned} \tag{1.6.92}$$

where the term inside the set of braces in the above integral can be summed by writing it as a geometric series, that is,

$$\begin{aligned} &1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \exp\{in(\theta-\phi)\} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \exp\{-in(\theta-\phi)\} \\ &= 1 + \frac{r\,\exp\{i(\theta-\phi)\}}{a - r\,\exp\{i(\theta-\phi)\}} + \frac{r\,\exp\{-i(\theta-\phi)\}}{a - r\,\exp\{-i(\theta-\phi)\}} \\ &= \frac{(a^2-r^2)}{a^2 - 2ar\,\cos(\theta-\phi) + r^2}. \end{aligned}$$

Thus, the final form of the solution is

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(a^2-r^2)\, f(\phi)\,d\phi}{a^2 - 2ar\,\cos(\theta-\phi) + r^2}. \tag{1.6.93}$$

This formula is known as *Poisson's integral formula* representing the solution of the Laplace equation within the circle of radius a in terms of values prescribed on the circle. It has several important consequences. First, we set $r = 0$ and $\theta = 0$ in (1.6.93) to obtain

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi. \tag{1.6.94}$$

This states that the value of u at the center is equal to the mean value of u on the boundary of the circle. This is called the *mean value property*.

We rewrite (1.6.93) in the form

$$u(r,\theta) = \int_0^{2\pi} P(r,\theta-\phi)\, f(\phi)\,d\phi, \tag{1.6.95}$$

where $P(r,\theta-\phi)$ is called the *Poisson kernel* given by

$$P(r,\theta-\phi) = \frac{1}{2\pi}\cdot\frac{(a^2-r^2)}{a^2 - 2ar\cos(\theta-\phi) + r^2}, \tag{1.6.96}$$

which is zero for $r = a$ but $\theta \neq \phi$. Further,

$$f(\theta) = \lim_{r\to a-} u(r,\theta) = \int_0^{2\pi}\left[\lim_{r\to a-} P(r,\theta-\phi)\right] f(\phi)\,d\phi,$$

which implies that

$$\lim_{r\to a-} P(r,\theta-\phi) = \delta(\theta-\phi), \tag{1.6.97}$$

where $\delta(x)$ is the Dirac delta function.

1.7 Fourier Transforms and Initial Boundary-Value Problems

The Fourier transform of $u(x,t)$ with respect to $x \in R$ is denoted by $\mathcal{F}\{u(x,t)\} = U(k,t)$ and is defined by the integral

$$\mathcal{F}\{u(x,t)\} = U(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} u(x,t)\,dx, \tag{1.7.1}$$

where k is real and is called the *transform variable*. The *inverse Fourier transform*, denoted by $\mathcal{F}^{-1}\{U(k,t)\} = u(x,t)$, is defined by

$$\mathcal{F}^{-1}\{U(k,t)\} = u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\, U(k,t)\,dk. \tag{1.7.2}$$

Example 1.7.1

$$\text{(a)}\quad \mathcal{F}\{\exp(-ax^2)\} = \frac{1}{\sqrt{2a}}\exp\left(-\frac{k^2}{4a}\right), \quad a > 0, \tag{1.7.3}$$

$$\text{(b)}\quad \mathcal{F}\{\exp(-a|x|)\} = \sqrt{\frac{1}{\pi}}\cdot\frac{a}{(k^2+a^2)}, \quad a > 0. \tag{1.7.4}$$

If $u(x,t) \to 0$, as $|x| \to \infty$, then