CHAPTER 7

Fourier Series and Transforms

1. INTRODUCTION

Problems involving vibrations or oscillations occur frequently in physics and engineering. You can think of examples you have already met: a vibrating tuning fork, a pendulum, a weight attached to a spring, water waves, sound waves, alternating electric currents, etc. In addition, there are many more examples which you will meet as you continue to study physics. Some of them—for example, heat conduction, electric and magnetic fields, light—do not appear in elementary work to have anything oscillatory about them, but will turn out in your more advanced work to involve the sines and cosines which are used in describing simple harmonic motion and wave motion.

In Chapter 1 we discussed the use of power series to approximate complicated functions. In many problems, series called Fourier series, whose terms are sines and cosines, are more useful than power series. In this chapter we shall see how to find and use Fourier series. Then, in Chapter 13 (Sections 2 to 4), we shall consider several of the physics problems which Fourier was trying to solve when he invented Fourier series.

Since sines and cosines are periodic functions, Fourier series can represent only periodic functions. We will see in Section 12 how to represent a non-periodic function by a Fourier integral (Fourier transform).

2. SIMPLE HARMONIC MOTION AND WAVE MOTION; PERIODIC FUNCTIONS

We shall need much of the notation and terminology used in discussing simple harmonic motion and wave motion. Let's discuss these two topics briefly.

Let particle P (Figure 2.1) move at constant speed around a circle of radius A. At the same time, let particle Q move up and down along the straight line segment RS in such a way that the y coordinates of P and Q are always equal. If ω is the angular velocity of P in radians per second, and

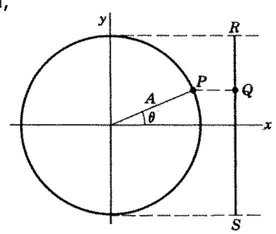


Figure 2.1

(Figure 2.1) $\theta = 0$ when $t = 0$, then at a later time t

(2.1) $$\theta = \omega t.$$

The y coordinate of Q (which is equal to the y coordinate of P) is

(2.2) $$y = A\sin\theta = A\sin\omega t.$$

The back and forth motion of Q is called *simple harmonic motion.* By definition, an object is executing simple harmonic motion if its displacement from equilibrium can be written as $A\sin\omega t$ [or $A\cos\omega t$ or $A\sin(\omega t+\phi)$, but these two functions differ from $A\sin\omega t$ only in choice of origin; such functions are called *sinusoidal functions*]. You can think of many physical examples of this sort of simple vibration: a pendulum, a tuning fork, a weight bobbing up and down at the end of a spring.

The x and y coordinates of particle P in Figure 2.1 are

(2.3) $$x = A\cos\omega t, \qquad y = A\sin\omega t.$$

If we think of P as the point $z = x+iy$ in the complex plane, we could replace (2.3) by a single equation to describe the motion of P:

(2.4) $$\begin{aligned} z = x + iy &= A(\cos\omega t + i\sin\omega t) \\ &= Ae^{i\omega t}. \end{aligned}$$

It is often worth while to use this complex notation even to describe the motion of Q; we then understand that the actual position of Q is equal to the imaginary part of z (or with different starting conditions the real part of z). For example, the velocity of Q is the imaginary part of

(2.5) $$\frac{dz}{dt} = \frac{d}{dt}(Ae^{i\omega t}) = Ai\omega e^{i\omega t} = Ai\omega(\cos\omega t + i\sin\omega t).$$

[The imaginary part of (2.5) is $A\omega\cos\omega t$, which is dy/dt from (2.2).]

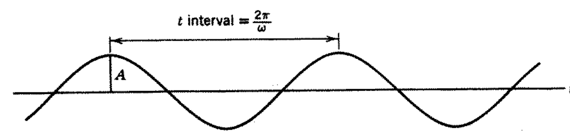


Figure 2.2

It is useful to draw a graph of x and y in (2.2) and (2.3) as a function of t. Figure 2.2 represents any of the functions $\sin\omega t$, $\cos\omega t$, $\sin(\omega t + \phi)$ if we choose the origin correctly. The number A is called the *amplitude of the vibration* or the *amplitude of the function.* Physically it is the maximum displacement of Q from its equilibrium position. The *period of the simple harmonic motion* or the *period of the function* is the time for one complete oscillation, that is, $2\pi/\omega$ (See Figure 2.2).

We could write the velocity of Q from (2.5) as

(2.6) $$\frac{dy}{dt} = A\omega\cos\omega t = B\cos\omega t.$$

Here B is the maximum value of the velocity and is called the *velocity amplitude*. Note that the velocity has the same period as the displacement. If the mass of the particle Q is m, its kinetic energy is:

$$\text{(2.7)}\qquad \text{Kinetic energy} = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 = \frac{1}{2}mB^2\cos^2\omega t.$$

We are considering an idealized harmonic oscillator which does not lose energy. Then the total energy (kinetic plus potential) must be equal to the largest value of the kinetic energy, that is, $\frac{1}{2}mB^2$. Thus we have:

$$\text{(2.8)}\qquad \text{Total energy} = \frac{1}{2}mB^2.$$

Notice that the energy is proportional to the square of the (velocity) amplitude; we shall be interested in this result later when we discuss sound.

Waves are another important example of an oscillatory phenomenon. The mathematical ideas of wave motion are useful in many fields; for example, we talk about water waves, sound waves, and radio waves.

▸ **Example 1.** Consider water waves in which the shape of the water surface is (unrealistically!) a sine curve. If we take a photograph (at the instant $t = 0$) of the water surface, the equation of this picture could be written (relative to appropriate axes)

$$\text{(2.9)}\qquad y = A\sin\frac{2\pi x}{\lambda},$$

where x represents horizontal distance and λ is the distance between wave crests. Usually λ is called the *wavelength*, but mathematically it is the same as the period of this function of x. Now suppose we take another photograph when the waves have moved forward a distance vt (v is the velocity of the waves and t is the time between photographs). Figure 2.3 shows the two photographs superimposed. Observe that the value of y at the point x on the graph labeled t, is just the same as the value of y at the point $(x - vt)$ on the graph labeled $t = 0$. If (2.9) is the equation representing the waves at $t = 0$, then

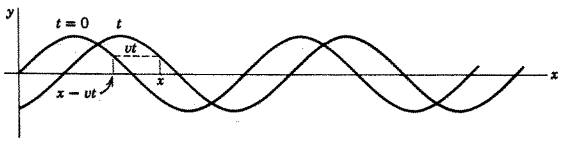


Figure 2.3

$$\text{(2.10)}\qquad y = A\sin\frac{2\pi}{\lambda}(x - vt)$$

represents the waves at time t. We can interpret (2.10) in another way. Suppose you stand at one point in the water [fixed x in (2.10)] and observe the up and down motion of the water, that is, y in (2.10) as a function of t (for fixed x). This is a simple harmonic motion of amplitude A and period λ/v. You are doing something analogous to this when you stand still and listen to a sound (sound waves pass your ear and you observe their frequency) or when you listen to the radio (radio waves pass the receiver and it reacts to their frequency).

We see that y in (2.10) is a periodic function either of x (t fixed) or of t (x fixed); both interpretations are useful. It makes no difference in the basic mathematics, however, what letter we use for the independent variable. To simplify our notation we shall ordinarily use x as the variable, but if the physical problem calls for it, you can replace x by t.

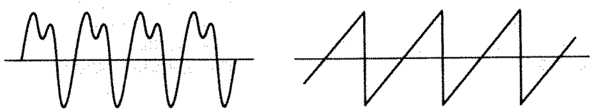


Figure 2.4

Sines and cosines are periodic functions; once you have drawn $\sin x$ from $x = 0$ to $x = 2\pi$, the rest of the graph from $x = -\infty$ to $x = +\infty$ is just a repetition over and over of the 0 to 2π graph. The number 2π is the period of $\sin x$. A periodic function need not be a simple sine or cosine, but may be any sort of complicated graph that repeats itself (Figure 2.4). The interval of repetition is the period.

▸ **Example 2.** If we are describing the vibration of a seconds pendulum, the period is 2 sec (time for one complete back-and-forth oscillation). The reciprocal of the period is the *frequency*, the number of oscillations per second; for the seconds pendulum, the frequency is $\frac{1}{2}$ sec^{-1}. When radio announcers say, "operating on a frequency of 780 kilohertz," they mean that 780,000 radio waves reach you per second, or that the period of one wave is (1/780,000) sec.

By definition, the function $f(x)$ is periodic if $f(x + p) = f(x)$ for every x; the number p is the period. The period of $\sin x$ is 2π since $\sin(x + 2\pi) = \sin x$; similarly, the period of $\sin 2\pi x$ is 1 since $\sin 2\pi(x + 1) = \sin(2\pi x + 2\pi) = \sin 2\pi x$, and the period of $\sin(\pi x/l)$ is $2l$ since $\sin(\pi/l)(x + 2l) = \sin(\pi x/l)$. In general, the period of $\sin 2\pi x/T$ is T.

▸ PROBLEMS, SECTION 2

In Problems 1 to 6 find the amplitude, period, frequency, and velocity amplitude for the motion of a particle whose distance s from the origin is the given function.

1. $s = 3\cos 5t$

2. $s = 2\sin(4t - 1)$

3. $s = \frac{1}{2}\cos(\pi t - 8)$

4. $s = 5\sin(t - \pi)$

5. $s = 2\sin 3t\cos 3t$

6. $s = 3\sin(2t + \pi/8) + 3\sin(2t - \pi/8)$

In Problems 7 to 10 you are given a complex function $z = f(t)$. In each case, show that a particle whose coordinate is (a) $x = \operatorname{Re} z$, (b) $y = \operatorname{Im} z$ is undergoing simple harmonic motion, and find the amplitude, period, frequency, and velocity amplitude of the motion.

7. $z = 5e^{it}$

8. $z = 2e^{-it/2}$

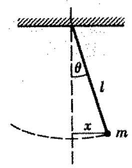
9. $z = 2e^{i\pi t}$

10. $z = -4e^{i(2t+3\pi)}$

11. The charge q on a capacitor in a simple a-c circuit varies with time according to the equation $q = 3\sin(120\pi t + \pi/4)$. Find the amplitude, period, and frequency of this oscillation. By definition, the current flowing in the circuit at time t is $I = dq/dt$. Show that I is also a sinusoidal function of t, and find its amplitude, period, and frequency.

12. Repeat Problem 11: (a) if $q = \text{Re}\, 4e^{30i\pi t}$; (b) if $q = \text{Im}\, 4e^{30i\pi t}$.

13. A simple pendulum consists of a point mass m suspended by a (weightless) cord or rod of length l, as shown, and swinging in a vertical plane under the action of gravity. Show that for small oscillations (small θ), both θ and x are sinusoidal functions of time, that is, the motion is simple harmonic. *Hint:* Write the differential equation $\mathbf{F} = m\mathbf{a}$ for the particle m. Use the approximation $\sin\theta = \theta$ for small θ, and show that $\theta = A\sin\omega t$ is a solution of your equation. What are A and ω?



14. The displacements x of two simple pendulums (see Problem 13) are $4\sin(\pi t/3)$ and $3\sin(\pi t/4)$. They start together at $x = 0$. How long will it be before they are together again at $x = 0$? *Hint:* Sketch or computer plot the graphs.

15. As in Problem 14, the displacements x of two simple pendulums are $x = -2\cos(t/2)$ and $3\sin(t/3)$. They are *not* together at $t = 0$; plot graphs to see when they are first together.

16. As in Problem 14, let the displacements be $y_1 = 3\sin(t/\sqrt{2})$ and $y_2 = \sin t$. The pendulums start together at $t = 0$. Make computer plots to estimate when they will be together again and then, by computer, solve the equation $y_1 = y_2$ for the root near your estimate.

17. Show that equation (2.10) for a wave can be written in all these forms:

$$y = A\sin\frac{2\pi}{\lambda}(x - vt) = A\sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$
$$= A\sin\omega\left(\frac{x}{v} - t\right) = A\sin\left(\frac{2\pi x}{\lambda} - 2\pi f t\right) = A\sin\frac{2\pi}{T}\left(\frac{x}{v} - t\right).$$

Here λ is the wavelength, f is the frequency, v is the wave velocity, T is the period, and $\omega = 2\pi f$ is called the *angular frequency. Hint:* Show that $v = \lambda f$.

In Problems 18 to 20, find the amplitude, period, frequency, wave velocity, and wavelength of the given wave. By computer, plot on the same axes, y as a function of x for the given values of t, and label each graph with its value of t. Similarly, plot on the same axes, y as a function of t for the given values of x, and label each curve with its value of x.

18. $y = 2\sin\frac{2}{3}\pi(x - 3t)$; $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$; $x = 0, 1, 2, 3$.

19. $y = \cos 2\pi(x - \frac{1}{4}t)$; $t = 0, 1, 2, 3$; $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

20. $y = 3\sin\pi(x - \frac{1}{2}t)$; $t = 0, 1, 2, 3$; $x = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$.

21. Write the equation for a sinusoidal wave of wavelength 4, amplitude 20, and velocity 6. (See Problem 17.) Make computer plots of y as a function of t for $x = 0, 1, 2, 3$, and of y as a function of x for $t = 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}$. If this wave represents the shape of a long rope which is being shaken back and forth at one end, find the velocity $\partial y/\partial t$ of particles of the rope as a function of x and t. (Note that this velocity has nothing to do with the wave velocity v, which is the rate at which crests of the wave move forward.)

22. Do Problem 21 for a wave of amplitude 4, period 6, and wavelength 3. Make computer plots of y as a function of x when $t = 0, 1, 2, 3$, and of y as a function of t when $x = \frac{1}{2}, 1, \frac{3}{2}, 2$.

23. Write an equation for a sinusoidal sound wave of amplitude 1 and frequency 440 hertz (1 hertz means 1 cycle per second). (Take the velocity of sound to be 350 m/sec.)

24. The velocity of sound in sea water is about 1530 m/sec. Write an equation for a sinusoidal sound wave in the ocean, of amplitude 1 and frequency 1000 hertz.

25. Write an equation for a sinusoidal radio wave of amplitude 10 and frequency 600 kilohertz. *Hint:* The velocity of a radio wave is the velocity of light, $c = 3\cdot 10^8$ m/sec.

3. APPLICATIONS OF FOURIER SERIES

We have said that the vibration of a tuning fork is an example of simple harmonic motion. When we hear the musical note produced, we say that a sound wave has passed through the air from the tuning fork to our ears. As the tuning fork vibrates it pushes against the air molecules, creating alternately regions of high and low

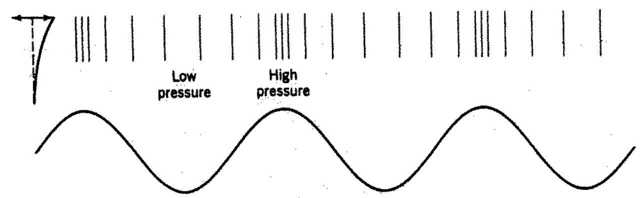


Figure 3.1

pressure (Figure 3.1). If we measure the pressure as a function of x and t from the tuning fork to us, we find that the pressure is of the form of (2.10); if we measure the pressure where we are as a function of t as the wave passes, we find that the pressure is a periodic function of t. The sound wave is a pure sine wave of a definite frequency (in the language of music, a pure tone). Now suppose that several pure tones are heard simultaneously. In the resultant sound wave, the pressure will not be a single sine function but a sum of several sine functions. If you strike a piano key you do not get a sound wave of just one frequency. Instead, you get a fundamental accompanied by a number of overtones (harmonics) of frequencies 2, 3, 4, $\cdots$, times the frequency of the fundamental. Higher frequencies mean shorter periods. If $\sin\omega t$ and $\cos\omega t$ correspond to the fundamental frequency, then $\sin n\omega t$ and $\cos n\omega t$ correspond to the higher harmonics. The combination of the fundamental and the harmonics is a complicated periodic function with the period of the fundamental (Problem 5). Given the complicated function, we could ask how to write it as a sum of terms corresponding to the various harmonics. In general it might require all the harmonics, that is, an infinite series of terms. This is called a Fourier series. Expanding a function in a Fourier series then amounts to breaking it down into its various harmonics. In fact, this process is sometimes called harmonic analysis.

There are applications to other fields besides sound. Radio waves, visible light, and x rays are all examples of a kind of wave motion in which the "waves" correspond to varying electric and magnetic fields. Exactly the same mathematical equations apply as for water waves and sound waves. We could then ask what light frequencies (these correspond to the color) are in a given light beam and in what proportions. To find the answer, we would expand the given function describing the wave in a Fourier series.

You have probably seen a sine curve used to represent an alternating current (a-c) or voltage in electricity. This is a periodic function, but so are the functions shown in Figure 3.2. Any of these and many others might represent signals (voltages or currents) which are to be applied to an electric circuit. Then we could ask

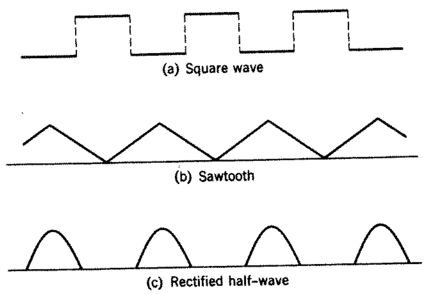


Figure 3.2

what a-c frequencies (harmonics) make up a given signal and in what proportions. When an electric signal is passed through a network (say a radio), some of the harmonics may be lost. If most of the important ones get through with their relative intensities preserved, we say that the radio possesses "high fidelity." To find out which harmonics are the important ones in a given signal, we expand it in a Fourier series. The terms of the series with large coefficients then represent the important harmonics (frequencies).

Since sines and cosines are themselves periodic, it seems rather natural to use series of them, rather than power series, to represent periodic functions. There is another important reason. The coefficients of a power series are obtained, you will recall (Chapter 1, Section 12), by finding successive derivatives of the function being expanded; consequently, only continuous functions with derivatives of all orders can be expanded in power series. Many periodic functions in practice are not continuous or not differentiable (Figure 3.2). Fortunately, Fourier series (unlike power series) can represent discontinuous functions or functions whose graphs have corners. On the other hand, Fourier series do not usually converge as rapidly as power series and much more care is needed in manipulating them. For example, a power series can be differentiated term by term (Chapter 1, Section 11), but differentiating a Fourier series term by term sometimes produces a series which doesn't converge. (See end of Section 9.)

Our problem then is to expand a given periodic function in a series of sines and cosines. We shall take this up in Section 5 after doing some preliminary work.

PROBLEMS, SECTION 3

For each of the following combinations of a fundamental musical tone and some of its overtones, make a computer plot of individual harmonics (all on the same axes) and then a plot of the sum. Note that the sum has the period of the fundamental (Problem 5).

1. $\sin t - \frac{1}{9}\sin 3t$

2. $2\cos t + \cos 2t$

3. $\sin \pi t + \sin 2\pi t + \frac{1}{3}\sin 3\pi t$

4. $\cos 2\pi t + \cos 4\pi t + \frac{1}{2}\cos 6\pi t$

5. Using the definition (end of Section 2) of a periodic function, show that a sum of terms corresponding to a fundamental musical tone and its overtones has the period of the fundamental.

In Problems 6 and 7, use a trigonometry formula to write the two terms as a single harmonic. Find the period and amplitude. Compare computer plots of your result and the given problem.

6. $\sin 2x + \sin 2(x + \pi/3)$

7. $\cos \pi x - \cos \pi(x - 1/2)$

8. A periodic modulated (AM) radio signal has the form

$$y = (A + B\sin 2\pi f t)\sin 2\pi f_c\left(t - \frac{x}{v}\right).$$

The factor $\sin 2\pi f_c(t - x/v)$ is called the carrier wave; it has a very high frequency (called radio frequency; f_c is of the order of 10^6 cycles per second). The amplitude of the carrier wave is $(A + B\sin 2\pi f t)$. This amplitude varies with time—hence the term "amplitude modulation"—with the much smaller frequency of the sound being transmitted (called audio frequency; f is of the order of 10^2 cycles per second). In order to see the general appearance of such a wave, use the following simple but unrealistic data to sketch a graph of y as a function of t for $x = 0$ over two periods of the *amplitude* function: $A = 3$, $B = 1$, $f = 1$, $f_c = 20$. Using trigonometric formulas, show that y can be written as a sum of three waves of frequencies f_c, $f_c + f$, and $f_c - f$; the first of these is the carrier wave and the other two are called side bands.

4. AVERAGE VALUE OF A FUNCTION

The concept of the average value of a function is often useful. You know how to find the average of a set of numbers: you add them and divide by the number of numbers.

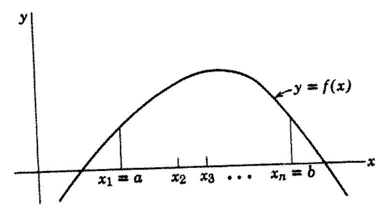


Figure 4.1

This process suggests that we ought to get an approximation to the average value of a function $f(x)$ on the interval (a, b) by averaging a number of values of $f(x)$ (Figure 4.1):

(4.1) Average of $f(x)$ on (a, b) is approximately equal to
$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

This should become a better approximation as n increases. Let the points $x_1, x_2, \cdots$ be Δx apart. Multiply the numerator and the denominator of the approximate average by Δx. Then (4.1) becomes:

(4.2) Average of $f(x)$ on (a, b) is approximately equal to
$$\frac{[f(x_1) + \cdots + f(x_n)]\Delta x}{n\,\Delta x}.$$

Now $n\,\Delta x = b - a$, the length of the interval over which we are averaging, no matter what n and Δx are. If we let $n \to \infty$ and $\Delta x \to 0$, the numerator approaches $\int_a^b f(x)\,dx$, and we have

(4.3) $$\text{Average of } f(x) \text{ on } (a, b) = \frac{\int_a^b f(x)\,dx}{b - a}.$$

In applications, it may happen that the average value of a given function is zero.

▸ **Example 1.** The average of $\sin x$ over any number of periods is zero. The average value of the velocity of a simple harmonic oscillator over any number of vibrations is zero. In such cases the average of the square of the function may be of interest.

▸ **Example 2.** If the alternating electric current flowing through a wire is described by a sine function, the square root of the average of the sine squared is known as the root-mean-square or effective value of the current, and is what you would measure with an a-c ammeter. In the example of the simple harmonic oscillator, the average kinetic energy (average of $\frac{1}{2}mv^2$) is $\frac{1}{2}m$ times the average of v^2.

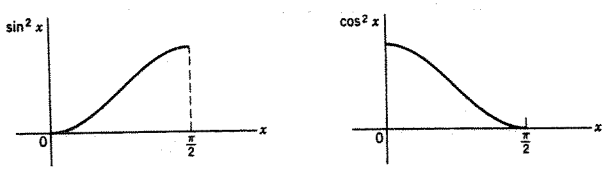


Figure 4.2

Now you can, of course, find the average value of $\sin^2 x$ over a period (say $-\pi$ to π) by evaluating the integral in (4.3). There is an easier way. Look at the graphs of $\cos^2 x$ and $\sin^2 x$ (Figure 4.2). You can probably convince yourself that the area under them is the same for any quarter-period from 0 to $\pi/2$, $\pi/2$ to π, etc. (Also see Problems 2 and 13.) Then

(4.4) $$\int_{-\pi}^{\pi} \sin^2 x\,dx = \int_{-\pi}^{\pi} \cos^2 x\,dx.$$

Similarly (for integral $n \neq 0$),

(4.5) $$\int_{-\pi}^{\pi} \sin^2 nx\,dx = \int_{-\pi}^{\pi} \cos^2 nx\,dx.$$

But since $\sin^2 nx + \cos^2 nx = 1$,

(4.6) $$\int_{-\pi}^{\pi} (\sin^2 nx + \cos^2 nx)\,dx = \int_{-\pi}^{\pi} dx = 2\pi.$$

Using (4.5), we get

(4.7) $$\int_{-\pi}^{\pi} \sin^2 nx\,dx = \int_{-\pi}^{\pi} \cos^2 nx\,dx = \pi.$$

Then using (4.3) we see that:

(4.8) The average value (over a period) of $\sin^2 nx$
= the average value (over a period) of $\cos^2 nx$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 nx\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2 nx\,dx = \frac{\pi}{2\pi} = \frac{1}{2}.$$

We can say all this more simply in words. By (4.5), the average value of $\sin^2 nx$ equals the average value of $\cos^2 nx$. The average value of $\sin^2 nx + \cos^2 nx = 1$ is 1. Therefore the average value of $\sin^2 nx$ or of $\cos^2 nx$ is $\frac{1}{2}$. (In each case the average value is taken over one or more periods.)

▸ PROBLEMS, SECTION 4

1. Show that if $f(x)$ has period p, the average value of f is the same over any interval of length p. *Hint:* Write $\int_a^{a+p} f(x)\,dx$ as the sum of two integrals (a to p, and p to $a + p$) and make the change of variable $x = t + p$ in the second integral.

2. (a) Prove that $\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx$ by making the change of variable $x = \frac{1}{2}\pi - t$ in one of the integrals.

(b) Use the same method to prove that the averages of $\sin^2(n\pi x/l)$ and $\cos^2(n\pi x/l)$ are the same over a period.

In Problems 3 to 12, find the average value of the function on the given interval. Use equation (4.8) if it applies. If an average value is zero, you may be able to decide this from a quick sketch which shows you that the areas above and below the x axis are the same.

3. $\sin x + 2\sin 2x + 3\sin 3x$ on $(0, 2\pi)$

4. $1 - e^{-x}$ on $(0, 1)$

5. $\cos^2 \frac{x}{2}$ on $\left(0, \frac{\pi}{2}\right)$

6. $\sin x$ on $(0, \pi)$

7. $x - \cos^2 6x$ on $\left(0, \frac{\pi}{6}\right)$

8. $\sin 2x$ on $\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$

9. $\sin^2 3x$ on $(0, 4\pi)$

10. $\cos x$ on $(0, 3\pi)$

11. $\sin x + \sin^2 x$ on $(0, 2\pi)$

12. $\cos^2 \frac{7\pi x}{2}$ on $\left(0, \frac{8}{7}\right)$

13. Using (4.3) and equations similar to (4.5) to (4.7), show that

$$\int_a^b \sin^2 kx\,dx = \int_a^b \cos^2 kx\,dx = \frac{1}{2}(b-a)$$

if $k(b-a)$ is an integral multiple of π, or if kb and ka are both integral multiples of $\pi/2$.

Use the results of Problem 13 to evaluate the following integrals without calculation.

14. (a) $\displaystyle\int_0^{4\pi/3} \sin^2\left(\frac{3x}{2}\right) dx$ (b) $\displaystyle\int_{-\pi/2}^{3\pi/2} \cos^2\left(\frac{x}{2}\right) dx$

15. (a) $\displaystyle\int_{-1/4}^{11/4} \cos^2 \pi x\,dx$ (b) $\displaystyle\int_{-1}^{2} \sin^2\left(\frac{\pi x}{3}\right) dx$

16. (a) $\displaystyle\int_0^{2\pi/\omega} \sin^2 \omega t\,dt$ (b) $\displaystyle\int_0^{2} \cos^2 2\pi t\,dt$

5. FOURIER COEFFICIENTS

We want to expand a given periodic function in a series of sines and cosines. To simplify our formulas at first, we start with functions of period 2π; that is, we shall expand periodic functions of period 2π in terms of the functions $\sin nx$ and $\cos nx$. (Later we shall see how we can change the formulas to fit a different period—see Section 8.) The functions $\sin x$ and $\cos x$ have period 2π; so do $\sin nx$ and $\cos nx$ for any integral n since $\sin n(x+2\pi) = \sin(nx + 2n\pi) = \sin nx$. (It is true that $\sin nx$ and $\cos nx$ also have shorter periods, namely $2\pi/n$, but the fact that they repeat every 2π is what we are interested in here, for this makes them reasonable functions to use in an expansion of a function of period 2π.) Then, given a function $f(x)$ of period 2π, we write

(5.1)
$$f(x) = \frac{1}{2}a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots,$$

and derive formulas for the coefficients a_n and b_n. (The reason for writing $\frac{1}{2}a_0$ as the constant term will be clear later—it makes the formulas for the coefficients simpler to remember—but you must not forget the $\frac{1}{2}$ in the series!)

In finding formulas for a_n and b_n in (5.1) we need the following integrals:

(5.2) The average value of $\sin mx \cos nx$ (over a period)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0.$$

The average value of $\sin mx \sin nx$ (over a period)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n \neq 0, \\ 0, & m = n = 0. \end{cases}$$

The average value of $\cos mx \cos nx$ (over a period)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0, & m \neq n, \\ \frac{1}{2}, & m = n \neq 0, \\ 1, & m = n = 0. \end{cases}$$

We have already shown that the average values of $\sin^2 nx$ and $\cos^2 nx$ are $\frac{1}{2}$. The last integral in (5.2) is the average value of 1 which is 1. To show that the other average values in (5.2) are zero (unless $m = n \neq 0$), we could use the trigonometry formulas for products like $\sin\theta\cos\phi$ and then integrate. An easier way is to use the formulas for the sines and cosines in terms of complex exponentials. [See (7.1) or Chapter 2, Section 11.] We shall show this method for one integral

$$\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = \int_{-\pi}^{\pi} \frac{e^{imx} - e^{-imx}}{2i} \cdot \frac{e^{inx} + e^{-inx}}{2}\,dx. \tag{5.3}$$

We can see the result without actually multiplying these out. All terms in the product are of the form e^{ikx}, where k is an integer $\neq 0$ (except for the cross-product terms when $n = m$, and these cancel). We can show that the integral of each such term is zero:

$$\int_{-\pi}^{\pi} e^{ikx}\,dx = \left.\frac{e^{ikx}}{ik}\right|_{-\pi}^{\pi} = \frac{e^{ik\pi} - e^{-ik\pi}}{ik} = 0 \tag{5.4}$$

because $e^{ik\pi} = e^{-ik\pi} = \cos k\pi$ (since $\sin k\pi = 0$). The other integrals in (5.2) may be evaluated similarly (Problem 12).

We now show how to find a_n and b_n in (5.1). To find a_0, we find the average value on $(-\pi, \pi)$ of each term of (5.1).

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx &= \frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} dx + a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\,dx \\ &\quad + a_2\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos 2x\,dx + \cdots + b_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin x\,dx + \cdots. \end{aligned} \tag{5.5}$$

By (5.2), all the integrals on the right-hand side of (5.5) are zero except the first, because they are integrals of $\sin mx \cos nx$ or of $\cos mx \cos nx$ with $n = 0$ and $m \neq 0$ (that is, $m \neq n$). Then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} dx = \frac{a_0}{2},$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{5.6}$$

Given $f(x)$ to be expanded in a Fourier series, we can now evaluate a_0 by calculating the integral in (5.6).

To find a_1, multiply both sides of (5.1) by $\cos x$ and again find the average value of each term:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\cos x\,dx &= \frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\,dx + a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx\\ &+ a_2\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos 2x\cos x\,dx+\cdots\\ &+ b_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin x\,\cos x\,dx+\cdots.\end{aligned} \tag{5.7}$$

This time, by (5.2), all terms on the right are zero except the a_1 term and we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\cos x\,dx = a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx = \frac{1}{2}a_1.$$

Solving for a_1, we have

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx.$$

The method should be clear by now, so we shall next find a general formula for a_n. Multiply both sides of (5.1) by $\cos nx$ and find the average value of each term:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx &= \frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos nx\,dx + a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos x\,\cos nx\,dx\\ &+ a_2\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos 2x\,\cos nx\,dx+\cdots\\ &+ b_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin x\,\cos nx\,dx+\cdots.\end{aligned} \tag{5.8}$$

By (5.2), all terms on the right are zero except the a_n term and we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_n\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 nx\,dx = \frac{1}{2}a_n.$$

Solving for a_n, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,dx. \tag{5.9}$$

Notice that this includes the $n=0$ formula, but only because we called the constant term $\frac{1}{2}a_0$.

To obtain a formula for b_n, we multiply both sides of (5.1) by $\sin nx$ and take average values just as we did in deriving (5.9). We find (Problem 13)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,dx. \tag{5.10}$$

The formulas (5.9) and (5.10) will be used repeatedly in problems and should be memorized.

▸ **Example 1.** Expand in a Fourier series the function $f(x)$ sketched in Figure 5.1. This function might represent, for example, a periodic voltage pulse. The terms of our Fourier series would then correspond to the different a-c frequencies which are combined in this "square wave" voltage, and the magnitude of the Fourier coefficients would indicate the relative importance of the various frequencies.

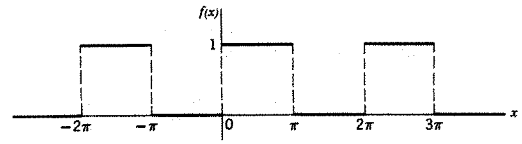


Figure 5.1

Note that $f(x)$ is a function of period 2π. Often in problems you will be given $f(x)$ for only one period; you should always sketch several periods so that you see clearly the periodic function you are expanding. For example, in this problem, instead of a sketch, you might have been given

$$f(x) = \begin{cases} 0, & -\pi < x < 0,\\ 1, & 0 < x < \pi.\end{cases} \tag{5.11}$$

It is then understood that $f(x)$ is to be continued periodically with period 2π outside the interval $(-\pi,\pi)$.

We use equations (5.9) and (5.10) to find a_n and b_n:

$$\begin{aligned}a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx\,dx + \int_{0}^{\pi} 1\cdot\cos nx\,dx\right]\\ &= \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx = \begin{cases}\dfrac{1}{\pi}\cdot\dfrac{1}{n}\sin nx\Big|_0^{\pi} = 0 & \text{for } n\neq 0,\\[2mm] \dfrac{1}{\pi}\cdot\pi = 1 & \text{for } n = 0.\end{cases}\end{aligned}$$

Thus $a_0 = 1$, and all other $a_n = 0$.

$$\begin{aligned}b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\sin nx\,dx + \int_{0}^{\pi} 1\cdot\sin nx\,dx\right]\\ &= \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = -\frac{1}{n\pi}[(-1)^n - 1]\\ &= \begin{cases} 0 & \text{for even } n,\\ \dfrac{2}{n\pi} & \text{for odd } n.\end{cases}\end{aligned}$$

Putting these values for the coefficients into (5.1), we have

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right). \tag{5.12}$$

▶ **Example 2.** We can now find the Fourier series for some other functions without more evaluation of coefficients. For example, consider

$$g(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases} \tag{5.13}$$

Sketch this and verify that $g(x) = 2f(x) - 1$, where $f(x)$ is the function in Example 1. Then from (5.12), the Fourier series for $g(x)$ is

$$g(x) = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right). \tag{5.14}$$

Similarly, verify that $h(x) = f(x + \pi/2)$ is Fig. 5.1 shifted $\pi/2$ to the left (sketch it), and its Fourier series is (replace x in (5.12) by $x + \pi/2$)

$$h(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \cdots\right)$$

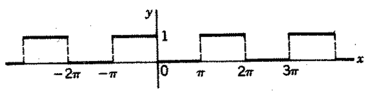
since $\sin(x + \pi/2) = \cos x$, $\sin(x + 3\pi/2) = -\cos 3x$, etc.

▶ PROBLEMS, SECTION 5

In each of the following problems you are given a function on the interval $-\pi < x < \pi$. Sketch several periods of the corresponding periodic function of period 2π. Expand the periodic function in a sine-cosine Fourier series.

1. $f(x) = \begin{cases} 1, & -\pi < x < 0, \\ 0, & 0 < x < \pi. \end{cases}$

In this case the sketch is:



Your answer for the series is: $f(x) = \frac{1}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} \cdots\right)$.

Can you use the ideas of Example 2 to find this result without computation?

2. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < x < \pi. \end{cases}$

Answer: $f(x) = \frac{1}{4} + \frac{1}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} \cdots\right)$
$+ \frac{1}{\pi}\left(\frac{\sin x}{1} + \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} \cdots\right)$.

3. $f(x) = \begin{cases} 0, & -\pi < x < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < x < \pi. \end{cases}$

Answer: $f(x) = \frac{1}{4} - \frac{1}{\pi}\left(\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} \cdots\right)$
$+ \frac{1}{\pi}\left(\frac{\sin x}{1} - \frac{2\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \frac{2\sin 6x}{6} \cdots\right)$.

4. $f(x) = \begin{cases} -1, & -\pi < x < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < x < \pi. \end{cases}$

Could you use Problem 3 to solve Problem 4 without computation?

5. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ -1, & 0 < x < \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < x < \pi. \end{cases}$

6. $f(x) = \begin{cases} 1, & -\pi < x < -\frac{\pi}{2}, \text{ and } 0 < x < \frac{\pi}{2}; \\ 0, & -\frac{\pi}{2} < x < 0, \text{ and } \frac{\pi}{2} < x < \pi. \end{cases}$

7. $f(x) = \begin{cases} 0, & -\pi < x < 0; \\ x, & 0 < x < \pi. \end{cases}$

Answer: $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$
$+ \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$.

8. $f(x) = 1 + x$, $\quad -\pi < x < \pi$.

Answer: $f(x) = 1 + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right)$.

9. $f(x) = \begin{cases} -x, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$

Answer: $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$.

10. $f(x) = \begin{cases} \pi + x, & -\pi < x < 0, \\ \pi - x, & 0 < x < \pi. \end{cases}$

11. $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin x, & 0 < x < \pi. \end{cases}$

Answer: $f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left(\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1} + \cdots\right)$.

12. Show that in (5.2) the average values of $\sin mx \sin nx$ and of $\cos mx \cos nx$, $m \neq n$, are zero (over a period), by using the complex exponential forms for the sines and cosines as in (5.3).

13. Write out the details of the derivation of equation (5.10).

▶ 6. DIRICHLET CONDITIONS

Now we have a series, but there are still some questions that we ought to get answered. Does it converge, and if so, does it converge to the values of $f(x)$? You will find, if you try, that for most values of x the series in (5.12) does not respond to any of the tests for convergence that we discussed in Chapter 1. What is the sum of the series at $x = 0$ where $f(x)$ jumps from 0 to 1? You can see from the series (5.12) that the sum at $x = 0$ is $\frac{1}{2}$, but what does this have to do with $f(x)$?

These questions would not be easy for us to answer for ourselves, but they are answered for us for most practical purposes by the *theorem of Dirichlet*:

> If $f(x)$ is periodic of period 2π, and if between $-\pi$ and π it is single-valued, has a finite number of maximum and minimum values, and a finite number of discontinuities, and if $\int_{-\pi}^{\pi}|f(x)|\,dx$ is finite, then the Fourier series (5.1) [with coefficients given by (5.9) and (5.10)] converges to $f(x)$ at all the points where $f(x)$ is continuous; at jumps the Fourier series converges to the midpoint of the jump. (This includes jumps that occur at $\pm\pi$ for the periodic function.)

To see what all this means, we shall consider some special functions. We have already discussed what a periodic function means. A function $f(x)$ is single-valued if there is just one value of $f(x)$ for each x. For example, if $x^2 + y^2 = 1$, y is not a single-valued function of x, unless we select just $y = +\sqrt{1-x^2}$ or just $y = -\sqrt{1-x^2}$. An example of a function with an infinite number of maxima and minima is $\sin(1/x)$, which oscillates infinitely many times as $x \to 0$. If we imagine a function constructed from $\sin(1/x)$ by making $f(x) = 1$ for every x for which $\sin(1/x) > 0$, and $f(x) = -1$ for every x for which $\sin(1/x) < 0$, this function would have an infinite number of discontinuities. Now most functions in applied work do not behave like these, but will satisfy the Dirichlet conditions.

Finally, if $y = 1/x$, we find

$$\int_{-\pi}^{\pi}\left|\frac{1}{x}\right| dx = 2\int_0^{\pi}\frac{1}{x}\,dx = 2\ln x\Big|_0^1 = \infty,$$

so the function $1/x$ is ruled out by the Dirichlet conditions. On the other hand, if $f(x) = 1/\sqrt{|x|}$, then

$$\int_{-\pi}^{\pi}\frac{1}{\sqrt{|x|}}\,dx = 2\int_0^{\pi}\frac{dx}{\sqrt{x}} = 4\sqrt{x}\Big|_0^{\pi} = 4\sqrt{\pi},$$

so the periodic function which is $1/\sqrt{|x|}$ between $-\pi$ and π can be expanded in a Fourier series. In most problems it is not necessary to find the value of $\int_{-\pi}^{\pi}|f(x)|\,dx$; let us see why. If $f(x)$ is bounded (that is, all its values lie between $\pm M$ for some positive constant M), then

$$\int_{-\pi}^{\pi}|f(x)|dx \le \int_{-\pi}^{\pi} M\,dx = M\cdot 2\pi$$

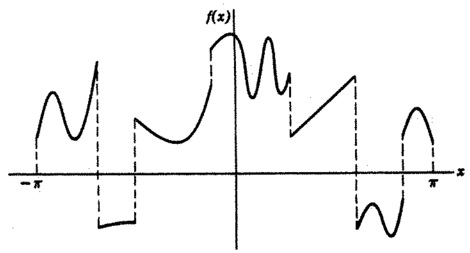


Figure 6.1

and so is finite. Thus you can simply verify that the function you are considering is bounded (if it is) instead of evaluating the integral. Figure 6.1 is an (exaggerated!) example of a function which satisfies the Dirichlet conditions on $(-\pi, \pi)$.

We see, then, that rather than testing Fourier series for convergence as we did power series, we instead check the given function; if it satisfies the Dirichlet conditions we are then sure that the Fourier series, when we get it, will converge to the function at points of continuity and to the midpoint of a jump. For example, consider the function $f(x)$ in Figure 5.1. Between $-\pi$ and π the given $f(x)$ is single-valued (one value for each x), bounded (between $+1$ and 0), has a finite number of maximum and minimum values (one of each), and a finite number of discontinuities (at $-\pi$, 0, and π), and therefore satisfies the Dirichlet conditions. Dirichlet's theorem then assures us that the series (5.12) actually converges to the function $f(x)$ in Figure 5.1 at all points except $x = n\pi$ where it converges to $1/2$.

In Chapter 3, Sections 10 and 14, we defined a *basis* for ordinary 3-dimensional space as a set of linearly independent vectors (like $\mathbf{i}, \mathbf{j}, \mathbf{k}$) in terms of which we could write every vector in the space. We then extended this idea to an n-dimensional space and to a space in which the basis vectors were functions. By analogy, we say here that the functions $\sin nx$, $\cos nx$ are a set of basis functions for the (infinite dimensional) space of all functions (satisfying Dirichlet conditions) defined on $(-\pi, \pi)$ or any 2π interval. (Also see "completeness relation" in Section 11. And for more examples of such sets of basis functions, see Chapters 12 and 13.)

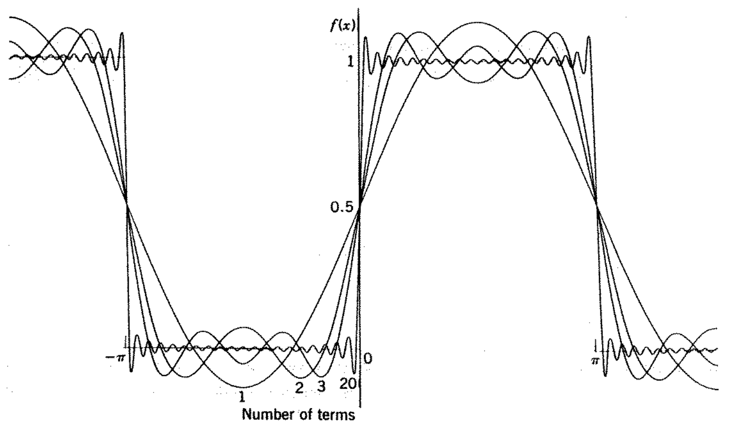


Figure 6.2

It is interesting to see a graph of the sum of a large number of terms of a Fourier series. Figure 6.2 shows several different partial sums of the series in (5.12) for the function in Figure 5.1. We can see that the sum of many terms of the series closely approximates the function away from the jumps and goes through the midpoint of the jump. The "overshoot" on either side of a jump bears comment. It does not disappear as we add more and more terms of the series. It simply becomes a narrower and narrower spike of height equal to about 9% of the jump. This fact is called the *Gibbs phenomenon*.

We ought to say here that the converse of Dirichlet's theorem is not true—if a function fails to satisfy the Dirichlet conditions, it still *may* be expandable in a Fourier series. The periodic function which is $\sin(1/x)$ on $(-\pi,\pi)$ is an example of such a function. However, such functions are rarely met with in practice.

▸ **Example.** Fourier series can be useful in summing numerical series. Look at Problem 5.2 (sketch it). From Dirichlet's theorem, we see that the Fourier series converges to $1/2$ at $x=0$. Let $x=0$ in the Fourier series to get

$$\frac{1}{2}=\frac{1}{4}+\frac{1}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)$$

since $\sin 0=0$ and $\cos 0=1$. Thus

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}.$$

▸ PROBLEMS, SECTION 6

1 to 11. For each of the periodic functions in Problems 5.1 to 5.11, use Dirichlet's theorem to find the value to which the Fourier series converges at $x=0,\ \pm\pi/2,\ \pm\pi,\ \pm 2\pi$.

12. Use a computer to produce graphs like Fig. 6.2 showing Fourier series approximations to the functions in Problems 5.1 to 5.3, and 5.7 to 5.11. You might like to set up a computer animation showing the Gibbs phenomenon as the number of terms increases.

13. Repeat the example using the same Fourier series but at $x=\pi/2$.

14. Use Problem 5.7 to show that $\sum_{\text{odd } n} 1/n^2=\pi^2/8$. Try $x=0$, and $x=\pi$. What do you find at $x=\pi/2$?

15. Use Problem 5.11 to show that $\dfrac{1}{2^2-1}+\dfrac{1}{4^2-1}+\dfrac{1}{6^2-1}+\cdots=\dfrac{1}{2}$.

▸ 7. COMPLEX FORM OF FOURIER SERIES

Recall that real sines and cosines can be expressed in terms of complex exponentials by the formulas [Chapter 2, (11.3)]

$$\sin nx=\frac{e^{inx}-e^{-inx}}{2i},\qquad \cos nx=\frac{e^{inx}+e^{-inx}}{2}. \tag{7.1}$$

If we substitute equations (7.1) into a Fourier series like (5.12), we get a series of terms of the forms e^{inx} and e^{-inx}. This is the complex form of a Fourier series. We can also find the complex form directly; this is often easier than finding the sine-cosine form. We can then, if we like, work back the other way and [using Euler's formula, Chapter 2, (9.3)] get the sine-cosine form from the exponential form.

We want to see how to find the coefficients in the complex form directly. We assume a series

$$\begin{aligned} f(x)&=c_0+c_1e^{ix}+c_{-1}e^{-ix}+c_2e^{2ix}+c_{-2}e^{-2ix}+\cdots\\ &=\sum_{n=-\infty}^{n=+\infty}c_ne^{inx} \end{aligned} \tag{7.2}$$

and try to find the c_n's. From (5.4) we know that the average value of e^{ikx} on $(-\pi,\pi)$ is zero when k is an integer not equal to zero. To find c_0, we find the average values of the terms in (7.2):

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx&=c_0\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}dx+\left\{\begin{matrix}\text{average values of terms of the}\\ \text{form } e^{ikx}\text{ with } k \text{ an integer}\neq 0\end{matrix}\right.\\ &=c_0+0,\end{aligned} \tag{7.3}$$

$$c_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx. \tag{7.4}$$

To find c_n, we multiply (7.2) by e^{-inx} and again find the average value of each term. Note the minus sign in the exponent. In finding a_n, the coefficient of $\cos nx$ in equation (5.1), we multiplied by $\cos nx$; but here in finding the coefficient c_n of e^{inx}, we multiply by the complex conjugate e^{-inx}.

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\,dx&=c_0\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-inx}\,dx+c_1\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-inx}e^{ix}\,dx\\ &\quad+c_{-1}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-inx}e^{-ix}\,dx+\cdots.\end{aligned} \tag{7.5}$$

The terms on the right are the average values of exponentials e^{ikx}, where the k values are integers. Therefore all these terms are zero except the one where $k=0$; this is the term containing c_n. We then have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\,dx=c_n\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-inx}e^{inx}\,dx=c_n\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}dx=c_n,$$

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\,dx. \tag{7.6}$$

Note that this formula contains the one for c_0 (no $\frac{1}{2}$ to worry about here!). Also, since (7.6) is valid for negative as well as positive n, you have only one formula to memorize here! You can easily show that for *real* $f(x)$, $c_{-n}=\bar{c}_n$ (Problem 12).

▸ **Example.** Let us expand the same $f(x)$ we did before, namely (5.11). We have from (7.6)

$$\begin{aligned}c_n&=\frac{1}{2\pi}\int_{-\pi}^{0}e^{-inx}\cdot 0\cdot dx+\frac{1}{2\pi}\int_{0}^{\pi}e^{-inx}\cdot 1\cdot dx\\ &=\frac{1}{2\pi}\left.\frac{e^{-inx}}{-in}\right|_0^{\pi}=\frac{1}{-2\pi in}(e^{-in\pi}-1)=\begin{cases}\frac{1}{\pi in}, & n \text{ odd},\\ 0, & n \text{ even}\neq 0,\end{cases}\\ c_0&=\frac{1}{2\pi}\int_0^{\pi}dx=\frac{1}{2}.\end{aligned} \tag{7.7}$$

Then

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} = \frac{1}{2} + \frac{1}{i\pi}\left(\frac{e^{ix}}{1} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + \cdots\right) \tag{7.8}$$
$$+ \frac{1}{i\pi}\left(\frac{e^{-ix}}{-1} + \frac{e^{-3ix}}{-3} + \frac{e^{-5ix}}{-5} + \cdots\right).$$

It is interesting to verify that this is the same as the sine-cosine series we had before. We *could* use Euler's formula for each exponential, but it is easier to collect terms like this:

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{e^{ix} - e^{-ix}}{2i} + \frac{1}{3}\frac{e^{3ix} - e^{-3ix}}{2i} + \cdots\right) \tag{7.9}$$
$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \cdots\right)$$

which is the same as (5.12).

▸ PROBLEMS, SECTION 7

1 to 11. Expand the same functions as in Problems 5.1 to 5.11 in Fourier series of complex exponentials e^{inx} on the interval $(-\pi, \pi)$ and verify in each case that the answer is equivalent to the one found in Section 5.

12. Show that if a real $f(x)$ is expanded in a complex exponential Fourier series $\sum_{-\infty}^{\infty} c_n e^{inx}$, then $c_{-n} = \bar{c}_n$, where $\bar{c}_n$ means the complex conjugate of c_n.

13. If $f(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx = \sum_{-\infty}^{\infty} c_n e^{inx}$, use Euler's formula to find a_n and b_n in terms of c_n and c_{-n}, and to find c_n and c_{-n} in terms of a_n and b_n.

▸ 8. OTHER INTERVALS

The functions $\sin nx$ and $\cos nx$ and e^{inx} have period 2π. We have been considering $(-\pi, \pi)$ as the basic interval of length 2π. Given $f(x)$ on $(-\pi, \pi)$, we have first sketched it for this interval, and then repeated our sketch for the intervals $(\pi, 3\pi)$, $(3\pi, 5\pi)$, $(-3\pi, -\pi)$, etc. There are (infinitely) many other intervals of length 2π, any one of which could serve as the basic interval. If we are given $f(x)$ on *any* interval of length 2π, we can sketch $f(x)$ for that given basic interval and then repeat it periodically with period 2π. We then want to expand the periodic function so obtained, in a Fourier series. Recall that in evaluating the Fourier coefficients, we used average values *over a period.* The formulas for the coefficients are then unchanged (except for the limits of integration) if we use other basic intervals of length 2π. In practice, the intervals $(-\pi, \pi)$ and $(0, 2\pi)$ are the ones most frequently used. For $f(x)$ defined on $(0, 2\pi)$ and then repeated periodically, (5.9), (5.10), and (7.6) would read

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \tag{8.1}$$
$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx,$$

and (5.1) and (7.2) are unchanged.

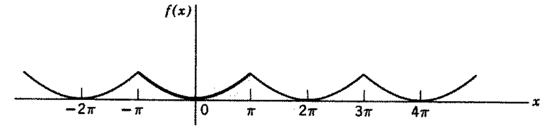


Figure 8.1

Notice how important it is to sketch a graph to see clearly what function you are talking about. For example, given $f(x) = x^2$ on $(-\pi, \pi)$, the extended function of period 2π is shown in Figure 8.1. But given $f(x) = x^2$ on $(0, 2\pi)$, the extended periodic function is different (see Figure 8.2). On the other hand, given $f(x)$ as in our example (5.11), or given $f(x) = 1$ on $(0, \pi)$, $f(x) = 0$ on $(\pi, 2\pi)$, you can easily verify by sketching that the graphs of the extended functions are identical. In this case you would get the same answer from either formulas (5.9), (5.10), and (7.6) or formulas (8.1).

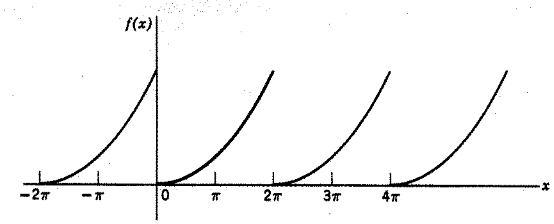


Figure 8.2

Physics problems do not always come to us with intervals of length 2π. Fortunately, it is easy now to change to other intervals. Consider intervals of length $2l$, say $(-l, l)$ or $(0, 2l)$. The function $\sin(n\pi x/l)$ has period $2l$, since

$$\sin\frac{n\pi}{l}(x + 2l) = \sin\left(\frac{n\pi x}{l} + 2n\pi\right) = \sin\frac{n\pi x}{l}.$$

Similarly, $\cos(n\pi x/l)$ and $e^{in\pi x/l}$ have period $2l$. Equations (5.1) and (7.2) are now replaced by

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{l} + a_2\cos\frac{2\pi x}{l} + \cdots$$
$$+ b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + \cdots$$
$$= \frac{a_0}{2} + \sum_1^\infty\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right), \tag{8.2}$$
$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}.$$

We have already found the average values *over a period* of all the functions we need to use to find a_n, b_n, and c_n here. The period is now of length $2l$, say $-l$ to l, so in finding average values of the terms we replace

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \quad \text{by} \quad \frac{1}{2l}\int_{-l}^{l}.$$

Recall that the average of the square of either the sine or the cosine over a period is $\frac{1}{2}$ and the average of $e^{in\pi x/l}\cdot e^{-in\pi x/l}=1$ is 1. Then the formulas (5.9), (5.10), and (7.6) for the coefficients become

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,\\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx,\\ c_n &= \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx. \end{aligned} \tag{8.3}$$

For the basic interval $(0,2l)$ we need only change the integration limits to 0 to $2l$. The Dirichlet theorem just needs π replaced by l in order to apply here.

▸ **Example.** Given $f(x)=\begin{cases}0, & 0<x<l,\\ 1, & l<x<2l.\end{cases}$

Expand $f(x)$ in an exponential Fourier series of period $2l$. [The function is given by the same formulas as (5.11) but on a different interval.]

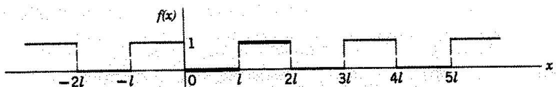


Figure 8.3

First we sketch a graph of $f(x)$ repeated with period $2l$ (Figure 8.3). By equations (8.3), we find

$$\begin{aligned} c_n &= \frac{1}{2l}\int_0^l 0\cdot dx + \frac{1}{2l}\int_l^{2l} 1\cdot e^{-in\pi x/l}\,dx\\ &= \frac{1}{2l}\left.\frac{e^{-in\pi x/l}}{-in\pi/l}\right|_l^{2l} = \frac{1}{-2in\pi}(e^{-2in\pi}-e^{-in\pi})\\ &= \frac{1}{-2in\pi}(1-e^{in\pi}) = \begin{cases} 0, & \text{even } n\neq 0,\\ -\dfrac{1}{in\pi}, & \text{odd } n,\end{cases}\\ c_0 &= \frac{1}{2l}\int_l^{2l} dx = \frac{1}{2}. \end{aligned} \tag{8.4}$$

Then,

$$\begin{aligned} f(x) &= \frac{1}{2} - \frac{1}{i\pi}\left(e^{i\pi x/l} - e^{-i\pi x/l} + \frac{1}{3}e^{3i\pi x/l} - \frac{1}{3}e^{-3i\pi x/l} + \cdots\right)\\ &= \frac{1}{2} - \frac{2}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \cdots\right). \end{aligned} \tag{8.5}$$

▸ PROBLEMS, SECTION 8

1 to 9. In Problems 5.1 to 5.9, define each function by the formulas given but on the interval $(-l,l)$. [That is, replace $\pm\pi$ by $\pm l$ and $\pm\pi/2$ by $\pm l/2$.] Expand each function in a sine-cosine Fourier series and in a complex exponential Fourier series.

10. (a) Sketch several periods of the function $f(x)$ of period 2π which is equal to x on $-\pi<x<\pi$. Expand $f(x)$ in a sine-cosine Fourier series and in a complex exponential Fourier series.

Answer: $f(x)=2(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\cdots)$.

(b) Sketch several periods of the function $f(x)$ of period 2π which is equal to x on $0<x<2\pi$. Expand $f(x)$ in a sine-cosine Fourier series and in a complex exponential Fourier series. Note that this is not the same function or the same series as (a).

Answer: $f(x)=\pi-2\sum_1^\infty \frac{\sin nx}{n}$.

In Problems 11 to 14, parts (a) and (b), you are given in each case one period of a function. Sketch several periods of the function and expand it in a sine-cosine Fourier series, and in a complex exponential Fourier series.

11. (a) $f(x)=x^2$, $-\pi<x<\pi$; (b) $f(x)=x^2$, $0<x<2\pi$.

12. (a) $f(x)=e^x$, $-\pi<x<\pi$; (b) $f(x)=e^x$, $0<x<2\pi$.

13. (a) $f(x)=2-x$, $-2<x<2$; (b) $f(x)=2-x$, $0<x<4$.

14. (a) $f(x)=\sin\pi x$, $-\frac{1}{2}<x<\frac{1}{2}$; (b) $f(x)=\sin\pi x$, $0<x<1$.

15. Sketch (or computer plot) each of the following functions on the interval $(-1,1)$ and expand it in a complex exponential series and in a sine-cosine series.

(a) $f(x)=x$, $-1<x<1$.

Answer: $f(x)=\frac{2}{\pi}\sum_1^\infty(-1)^{n+1}\frac{\sin n\pi x}{n}$.

(b) $f(x)=\begin{cases}1+2x, & -1<x<0,\\ 1-2x, & 0<x<1.\end{cases}$

Answer: $f(x)=\frac{8}{\pi^2}\sum_{\text{odd } n=1}^\infty \frac{\cos n\pi x}{n^2}$.

(c) $f(x)=\begin{cases}x+x^2, & -1<x<0,\\ x-x^2, & 0<x<1.\end{cases}$

Answer: $f(x)=\frac{8}{\pi^3}\sum_{\text{odd } n=1}^\infty \frac{\sin n\pi x}{n^3}$.

Each of the following functions is given over one period. Sketch several periods of the corresponding periodic function and expand it in an appropriate Fourier series.

16. $f(x) = x, \quad 0 < x < 2.$ *Answer:* $f(x) = 1 - \dfrac{2}{\pi}\sum_{1}^{\infty} \dfrac{\sin n\pi x}{n}.$

17. $f(x) = \begin{cases} 0, & -1 < x < 0, \\ 1, & 0 < x < 3. \end{cases}$

18. $f(x) = x^2, \quad 0 < x < 10.$

19. $f(x) = \begin{cases} 0, & -\frac{1}{2} < x < 0, \\ x, & 0 < x < \frac{1}{2}. \end{cases}$

20. $f(x) = \begin{cases} x/2, & 0 < x < 2, \\ 1, & 2 < x < 3. \end{cases}$

21. Write out the details of the derivation of the formulas (8.3).

9. EVEN AND ODD FUNCTIONS

An *even* function is one like x^2 or $\cos x$ (Figure 9.1) whose graph for negative x is just a reflection in the y axis of its graph for positive x. In formulas, the value of $f(x)$ is the same for a given x and its negative; that is

$$(9.1) \qquad f(x) \quad \text{is even if} \quad f(-x) = f(x).$$

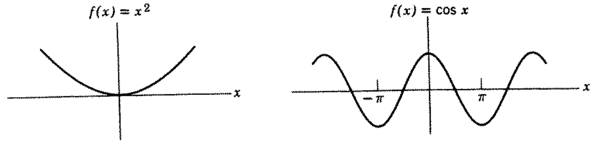


Figure 9.1

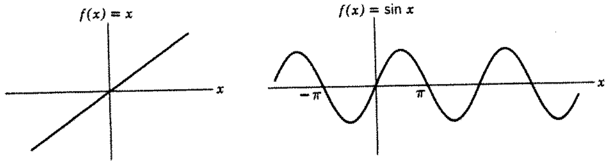


Figure 9.2

An odd function is one like x or $\sin x$ (Figure 9.2) for which the values of $f(x)$ and $f(-x)$ are negatives of each other. By definition

$$(9.2) \qquad f(x) \quad \text{is odd if} \quad f(-x) = -f(x).$$

Notice that even powers of x are even, and odd powers of x are odd; in fact, this

is the reason for the names. You should verify (Problem 14) the following rules for the product of two functions: An even function times an even function, or an odd function times an odd function, gives an even function; an odd function times an even function gives an odd function. Some functions are even, some are odd, and some (for example, e^x) are neither. However, any function can be written as the sum of an even function and an odd function, like this:

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)];$$

the first part is even and the second part is odd. For example,

$$e^x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \cosh x + \sinh x;$$

$\cosh x$ is even and $\sinh x$ is odd (look at the graphs).

Integrals of even functions or of odd functions, over symmetric intervals like $(-\pi, \pi)$ or $(-l, l)$, can be simplified. Look at the graph of $\sin x$ and think about $\int_{-\pi}^{\pi} \sin x\,dx$. The negative area from $-\pi$ to 0 cancels the positive area from 0 to π, so the integral is zero. This integral is still zero for any interval $(-l, l)$ which is symmetric about the origin, as you can see from the graph. The same is true for *any* odd $f(x)$; the areas to the left and to the right cancel. Next look at the cosine graph and the integral $\int_{-\pi/2}^{\pi/2} \cos x\,dx$. You see that the area from $-\pi/2$ to 0 is the same as the area from 0 to $\pi/2$. We could then just as well find the integral from 0 to $\pi/2$ and multiply it by 2. In general, if $f(x)$ is even, the integral of $f(x)$ from $-l$ to l is twice the integral from 0 to l. Then we have

$$(9.3) \qquad \int_{-l}^{l} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd,} \\ 2\displaystyle\int_0^l f(x)\,dx & \text{if } f(x) \text{ is even.} \end{cases}$$

Suppose now that we are given a function on the interval $(0, l)$. If we want to represent it by a Fourier series of period $2l$, we must have $f(x)$ defined on $(-l, 0)$ too. There are several things we could do. We *could* define it to be zero (or, indeed, anything else) on $(-l, 0)$ and go ahead as we have done previously to find either an exponential or a sine-cosine series of period $2l$. However, it often happens in practice that we need (for physical reasons—see Chapter 13) to have an even function (or, in a different problem, an odd function). We first sketch the given function on $(0, l)$ (heavy lines in Figures 9.3 and 9.4). Then we extend the function on $(-l, 0)$ to be even or to be odd as required. To sketch more periods, just repeat the $(-l, l)$ sketch. (If the graph is complicated, it is helpful to trace it with a finger of one

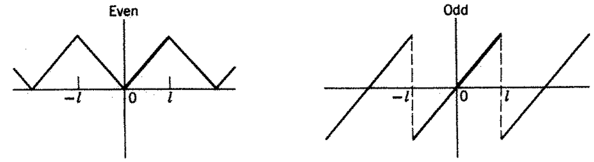


Figure 9.3

hand while you use the other hand to copy exactly what you are tracing. Turn the paper upside down to avoid crossing hands.)

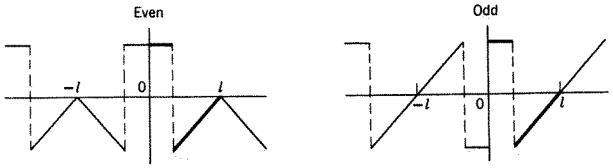


Figure 9.4

For even or odd functions, the coefficient formulas for a_n and b_n simplify. First suppose $f(x)$ is odd. Since sines are odd and cosines are even, $f(x)\sin(n\pi x/l)$ is even and $f(x)\cos(n\pi x/l)$ is odd. Then a_n is the integral, over a symmetric interval $(-l,l)$, of an odd function, namely $f(x)\cos(n\pi x/l)$; a_n is therefore zero. But b_n is the integral of an even function over a symmetric interval and is therefore twice the 0 to l integral. We have:

$$\text{(9.4)} \qquad \text{If } f(x) \text{ is odd, } \begin{cases} b_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\dfrac{n\pi x}{l}\,dx, \\ a_n = 0. \end{cases}$$

We say that we have expanded $f(x)$ in a sine series ($a_n = 0$ so there are no cosine terms). Similarly, if $f(x)$ is even, all the b_n's are zero, and the a_n's are integrals of even functions. We have:

$$\text{(9.5)} \qquad \text{If } f(x) \text{ is even, } \begin{cases} a_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\cos\dfrac{n\pi x}{l}\,dx, \\ b_n = 0. \end{cases}$$

We say that $f(x)$ is expanded in a cosine series. (Remember that the constant term is $a_0/2$.)

You have now learned to find several different kinds of Fourier series that represent a given function $f(x)$ on, let us say, the interval $(0,1)$. How do you know which to use in a given problem? You have to decide this from the physical problem when you are using Fourier series. There are two things to check: (1) the basic period involved in the physical problem; the functions in your series should have this period; and (2) the physical problem may require either an even function or an odd function for its solution; in these cases you must find the appropriate series. Now consider $f(x)$ defined on $(0,1)$. We could find for it a sine-cosine or an exponential series of period 1 (that is, $l = \frac{1}{2}$):

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{2in\pi x} \qquad \text{where} \qquad c_n = \int_0^1 f(x)e^{-2in\pi x}\,dx.$$

(The choice between sine-cosine and exponential series is just one of convenience in evaluating the coefficients—the series are really identical.) But we could also find two other Fourier series representing the same $f(x)$ on $(0,1)$. These series would have period 2 (that is, $l = 1$). One would be a cosine series

$$f(x) = \sum_{n=0}^{\infty} a_n \cos n\pi x, \qquad a_n = 2\int_0^1 f(x)\cos n\pi x\,dx, \qquad b_n = 0,$$

and represent an even function; the other would be a sine series and represent an odd function. In the problems, you may just be told to expand a function in a cosine series, say. You must then see for yourself what the period is when you have sketched an even function, and so choose the proper l in $\cos(n\pi x/l)$ and in the formula for a_n.

▸ **Example.** Represent $f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2}, \\ 0, & \frac{1}{2} < x < 1, \end{cases}$
by (a) a Fourier sine series, (b) a Fourier cosine series, (c) a Fourier series (the last ordinarily means a sine-cosine or exponential series whose period is the interval over which the function is given; in this case the period is 1).

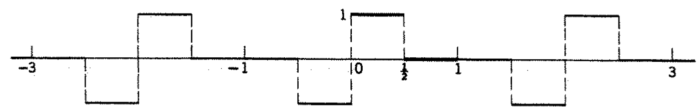


Figure 9.5

(a) Sketch the given function between 0 and 1. Extend it to the interval $(-1,0)$ making it odd. The period is now 2, that is, $l = 1$. Continue the function with period 2 (Figure 9.5). Since we now have an odd function, $a_n = 0$ and

$$\begin{aligned} b_n &= \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx = 2\int_0^{1/2} \sin n\pi x\,dx \\ &= -\frac{2}{n\pi}\cos n\pi x\Big|_0^{1/2} = -\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - 1\right), \\ b_1 &= \frac{2}{\pi}, \quad b_2 = \frac{4}{2\pi}, \quad b_3 = \frac{2}{3\pi}, \quad b_4 = 0, \quad \cdots. \end{aligned}$$

Thus we obtain the *Fourier sine series* for $f(x)$:

$$f(x) = \frac{2}{\pi}\left(\sin\pi x + \frac{2\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \frac{2\sin 6\pi x}{6} + \cdots\right).$$

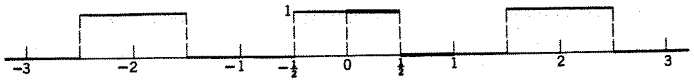


Figure 9.6

(b) Sketch an even function of period 2 (Figure 9.6).
Here $l = 1$, $b_n = 0$, and

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^{1/2} dx = 1,$$

$$a_n = 2\int_0^1 f(x)\cos n\pi x\,dx = \frac{2}{n\pi}\sin n\pi x\bigg|_0^{1/2} = \frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

Then the *Fourier cosine series* for $f(x)$ is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\cos\pi x}{1} - \frac{\cos 3\pi x}{3} + \frac{\cos 5\pi x}{5} - \cdots\right).$$

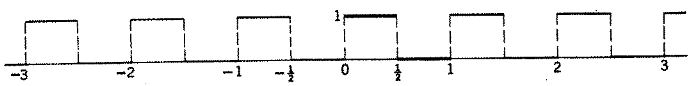


Figure 9.7

(c) Sketch the given function on $(0, 1)$ and continue it with period 1 (Figure 9.7). Here $2l = 1$, and we find c_n as we did in the example of Section 8. As in that example, the exponential series here can then be put in sine-cosine form.

$$c_n = \int_0^1 f(x)e^{-2in\pi x}\,dx = \int_0^{1/2} e^{-2in\pi x}\,dx$$

$$= \frac{1 - e^{-in\pi}}{2in\pi} = \frac{1 - (-1)^n}{2in\pi} = \begin{cases} \dfrac{1}{in\pi}, & n \text{ odd}, \\ 0, & n \text{ even} \neq 0. \end{cases}$$

$$c_0 = \int_0^{1/2} dx = \frac{1}{2}.$$

$$f(x) = \frac{1}{2} + \frac{1}{i\pi}\left(e^{2i\pi x} - e^{-2i\pi x} + \frac{1}{3}e^{6\pi ix} - \frac{1}{3}e^{-6\pi ix} + \cdots\right)$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin 2\pi x + \frac{\sin 6\pi x}{3} + \cdots\right).$$

Alternatively we can find both a_n and b_n directly.

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^{1/2} dx = 1.$$

$$a_n = 2\int_0^{1/2} \cos 2n\pi x\,dx = 0.$$

$$b_n = 2\int_0^{1/2} \sin 2n\pi x\,dx = \frac{1}{n\pi}(1 - \cos n\pi) = \frac{1}{n\pi}[1 - (-1)^n].$$

$$b_1 = \frac{2}{\pi},\quad b_2 = 0,\quad b_3 = \frac{2}{3\pi},\quad b_4 = 0,\quad \cdots.$$

There is one other very useful point to notice about even and odd functions. If you are given a function on $(-l, l)$ to expand in a sine-cosine series (of period $2l$) and happen to notice that it is an even function, you should realize that the b_n's are all going to be zero and you do not have to work them out. Also the a_n's can be written as twice an integral from 0 to l just as in (9.5). Similarly, if the given function is odd, you can use (9.4). Recognizing this may save you a good deal of algebra.

Differentiating Fourier Series Now that we have a supply of Fourier series for reference, let's discuss the question of differentiating a Fourier series term by term. First consider a Fourier series in which a_n and b_n are proportional to $1/n$. Since the derivative of $\frac{1}{n}\sin nx$ is $\cos nx$ (and a similar result for the cosine terms), we see that the differentiated series has no $1/n$ factors to make it converge. Now you might suspect (correctly) that if you can't differentiate the Fourier series, then the function $f(x)$ which it represents can't be differentiated either, at least not at all points. Turn back to examples and problems for which the Fourier series have coefficients proportional to $1/n$ and look at the graphs (or sketch them). Note in every case that $f(x)$ is discontinuous (that is, has jumps) at some points, and so can't be differentiated there. Next consider Fourier series with a_n and b_n proportional to $1/n^2$. If we differentiate such a series once, there are still $1/n$ factors left but we can't differentiate it twice. In that case we would (correctly) expect the function to be continuous with a discontinuous first derivative. (Look for examples.) Continuing, if a_n and b_n are proportional to $1/n^3$, we can find two derivatives, but the second derivative is discontinuous, and so on for Fourier coefficients proportional to higher powers of $1/n$. (See Problems 26 and 27.)

It is interesting to plot (by computer) a given function together with enough terms of its Fourier series to give a reasonable fit. In Section 5 we did this for discontinuous functions and it took many terms of the series. You will find (see Problems 26 and 27) that the more continuous derivatives a function has, the fewer terms of its Fourier series are required to approximate it. We can understand this: The higher order terms oscillate more rapidly (compare $\sin x$, $\sin 2x$, $\sin 10x$), and this rapid oscillation is what is needed to fit a curve which is changing rapidly (for example, a jump). But if $f(x)$ has several continuous derivatives, then it is quite "smooth" and doesn't require so much of the rapid oscillation of the higher order terms. This is reflected in the dependence of the Fourier coefficients on a power of $1/n$.

LEMS, SECTION 9

The functions in Problems 1 to 3 are neither even nor odd. Write each of them as the sum of an even function and an odd function.

1. (a) e^{inx} (b) xe^x

2. (a) $\ln|1-x|$ (b) $(1+x)(\sin x + \cos x)$

3. (a) $x^5 - x^4 + x^3 - 1$ (b) $1+e^x$

4. Using what you know about even and odd functions, prove the first part of (5.2).

Each of the functions in Problems 5 to 12 is given over one period. For each function, sketch several periods and decide whether it is even or odd. Then use (9.4) or (9.5) to expand it in an appropriate Fourier series.

5. $f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$

6. $f(x) = \begin{cases} -1, & -l < x < 0, \\ 1, & 0 < x < l. \end{cases}$

Answer: $f(x) = \dfrac{4}{\pi}\left(\sin\dfrac{\pi x}{l} + \dfrac{1}{3}\sin\dfrac{3\pi x}{l} + \dfrac{1}{5}\sin\dfrac{5\pi x}{l} + \cdots\right).$

7. $f(x) = \begin{cases} 1, & -1 < x < 1, \\ 0, & -2 < x < -1 \text{ and } 1 < x < 2. \end{cases}$

8. $f(x) = x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}.$

9. $f(x) = x^2, \quad -\frac{1}{2} < x < \frac{1}{2}.$

Answer: $f(x) = \dfrac{1}{12} - \dfrac{1}{\pi^2}\left(\cos 2\pi x - \dfrac{1}{2^2}\cos 4\pi x + \dfrac{1}{3^2}\cos 6\pi x - \cdots\right).$

10. $f(x) = |x|, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}.$

11. $f(x) = \cosh x, \quad -\pi < x < \pi.$

Answer: $f(x) = \dfrac{2\sinh\pi}{\pi}\left(\dfrac{1}{2} - \dfrac{1}{2}\cos x + \dfrac{1}{5}\cos 2x - \dfrac{1}{10}\cos 3x + \dfrac{1}{17}\cos 4x - \cdots\right).$

12. $f(x) = \begin{cases} x+1, & -1 < x < 0, \\ x-1, & 0 < x < 1. \end{cases}$

13. Give algebraic proofs of (9.3). *Hint:* Write $\int_{-l}^{l} = \int_{-l}^{0} + \int_0^l$, make the change of variable $x = -t$ in $\int_{-l}^{0}$, and use the definition of even or odd function.

14. Give algebraic proofs that for even and odd functions:

(a) even times even = even; odd times odd = even; even times odd = odd;

(b) the derivative of an even function is odd; the derivative of an odd function is even.

15. Given $f(x) = x$ for $0 < x < 1$, sketch the even function f_c of period 2 and the odd function f_s of period 2, each of which equals $f(x)$ on $0 < x < 1$. Expand f_c in a cosine series and f_s in a sine series.

Answer: $f_c(x) = \dfrac{1}{2} - \dfrac{4}{\pi^2}\left(\cos\pi x + \dfrac{1}{3^2}\cos 3\pi x + \cdots\right),$

$f_s(x) = \dfrac{2}{\pi}\left(\sin\pi x - \dfrac{1}{2}\sin 2\pi x + \dfrac{1}{3}\sin 3\pi x - \cdots\right).$

16. Let $f(x) = \sin^2 x$, $0 < x < \pi$. Sketch (or computer plot) the even function f_c of period 2π, the odd function f_s of period 2π, and the function f_p of period π, each of which is equal to $f(x)$ on $(0,\pi)$. Expand each of these functions in an appropriate Fourier series.

In Problems 17 to 22 you are given $f(x)$ on an interval, say $0 < x < b$. Sketch several periods of the even function f_c of period $2b$, the odd function f_s of period $2b$, and the function f_p of period b, each of which equals $f(x)$ on $0 < x < b$. Expand each of the three functions in an appropriate Fourier series.

17. $f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x < 1. \end{cases}$

18. $f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & 1 < x < 3. \end{cases}$

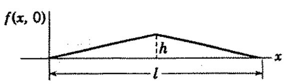
19. $f(x) = |\cos x|, \quad 0 < x < \pi.$

20. $f(x) = x^2, \quad 0 < x < 1.$

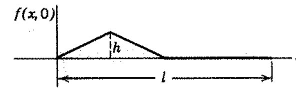
21. $f(x) = \begin{cases} x, & 0 < x < 1, \\ 2-x, & 1 < x < 2. \end{cases}$

22. $f(x) = \begin{cases} 10, & 0 < x < 10, \\ 20, & 10 < x < 20. \end{cases}$

23. If a violin string is plucked (pulled aside and let go), it is possible to find a formula $f(x,t)$ for the displacement at time t of any point x of the vibrating string from its equilibrium position. It turns out that in solving this problem we need to expand the function $f(x,0)$, whose graph is the initial shape of the string, in a Fourier sine series. Find this series if a string of length l is pulled aside a small distance h at its center, as shown.



24. If, in Problem 23, the string is stopped at the center and half of it is plucked, then the function to be expanded in a sine series is shown here. Find the series. *Caution:* Note that $f(x,0) = 0$ for $l/2 < x < l$.



25. Suppose that $f(x)$ and its derivative $f'(x)$ are both expanded in Fourier series on $(-\pi,\pi)$. Call the coefficients in the $f(x)$ series a_n and b_n and the coefficients in the $f'(x)$ series a_n' and b_n'. Write the integral for a_n [equation (5.9)] and integrate it by parts to get an integral of $f'(x)\sin nx$. Recognize this integral in terms of b_n' [equation (5.10) for $f'(x)$] and so show that $b_n' = -na_n$. (In the integration by parts, the integrated term is zero because $f(\pi) = f(-\pi)$ since f is continuous—sketch several periods.). Find a similar relation for a_n' and b_n. Now show that this is the result you get by differentiating the $f(x)$ series term by term. Thus you have shown that the Fourier series for $f'(x)$ is correctly given by differentiating the $f(x)$ series term by term (assuming that $f'(x)$ is expandable in a Fourier series).

In Problems 26 and 27, find the indicated Fourier series. Then differentiate your result repeatedly (both the function and the series) until you get a discontinuous function. Use a computer to plot $f(x)$ and the derivative functions. For each graph, plot on the same axes one or more terms of the corresponding Fourier series. Note the number of terms needed for a good fit (see comment at the end of the section).

26. $f(x) = \begin{cases} 3x^2 + 2x^3, & -1 < x < 0, \\ 3x^2 - 2x^3, & 0 < x < 1. \end{cases}$

27. $f(x) = (x^2 - \pi^2)^2, \quad -\pi < x < \pi.$

▸ 10. AN APPLICATION TO SOUND

We have said that when a sound wave passes through the air and we hear it, the air pressure where we are varies with time. Suppose the excess pressure above (and below) atmospheric pressure in a sound wave is given by the graph in Figure 10.1. (We shall not be concerned here with the units of p; however, reasonable units in Figure 10.1 would be p in 10^{-6} atmospheres.) Let us ask what frequencies we hear when we listen to this sound. To find out, we expand $p(t)$ in a Fourier series. The period of $p(t)$ is $\frac{1}{262}$; that is, the sound wave repeats itself 262 times per second. We have called the period $2l$ in our formulas, so here $l = \frac{1}{524}$. The functions we have called $\sin(n\pi x/l)$ here become $\sin 524n\pi t$. We can save some work by observing

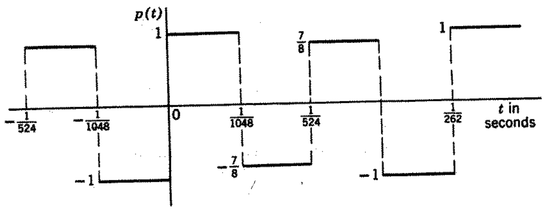


Figure 10.1

that $p(t)$ is an odd function; there are then only sine terms in its Fourier series and we need to compute only b_n. Using (9.4), we have

$$(10.1)\qquad b_n = 2(524)\int_0^{1/524} p(t)\sin 524n\pi t\,dt$$

$$= 1048\int_0^{1/1048}\sin 524n\pi t\,dt - \frac{7}{8}(1048)\int_{1/1048}^{1/524}\sin 524n\pi t\,dt$$

$$= 1048\left(-\frac{\cos\frac{n\pi}{2}-1}{524n\pi} + \frac{7}{8}\,\frac{\cos n\pi - \cos\frac{n\pi}{2}}{524n\pi}\right)$$

$$= \frac{2}{n\pi}\left(-\frac{15}{8}\cos\frac{n\pi}{2} + 1 + \frac{7}{8}\cos n\pi\right).$$

From this we can compute the values of b_n for the first few values of n:

$$(10.2)\qquad \begin{aligned} b_1 &= \frac{2}{\pi}\left(1-\frac{7}{8}\right) = \frac{2}{\pi}\left(\frac{1}{8}\right) = \frac{1}{\pi}\cdot\frac{1}{4} & b_5 &= \frac{1}{5\pi}\cdot\frac{1}{4}\\ b_2 &= \frac{2}{2\pi}\left(\frac{15}{8}+1+\frac{7}{8}\right) = \frac{1}{2\pi}\left(\frac{15}{2}\right) & b_6 &= \frac{1}{6\pi}\left(\frac{15}{2}\right)\\ b_3 &= \frac{2}{3\pi}\left(1-\frac{7}{8}\right) = \frac{1}{3\pi}\cdot\frac{1}{4} & b_7 &= \frac{1}{7\pi}\cdot\frac{1}{4}\\ b_4 &= \frac{2}{4\pi}\left(-\frac{15}{8}+1+\frac{7}{8}\right) = 0 & b_8 &= 0, \quad \text{etc.} \end{aligned}$$

Then we have

$$(10.3)\qquad p(t) = \frac{1}{4\pi}\left(\frac{\sin 524\pi t}{1} + \frac{30\sin(524\cdot 2\pi t)}{2} + \frac{\sin(524\cdot 3\pi t)}{3} + \frac{\sin(524\cdot 5\pi t)}{5} + \frac{30\sin(524\cdot 6\pi t)}{6} + \frac{\sin(524\cdot 7\pi t)}{7} + \cdots\right).$$

We can see just by looking at the coefficients that the most important term is the second one. The first term corresponds to the fundamental with frequency 262 vibrations per second (this is approximately middle C on a piano). But it is much weaker in this case than the first overtone (second harmonic) corresponding to the second term; this tone has frequency 524 vibrations per second (approximately high C). (You might like to use a computer to play one or several terms of the series.) The sixth harmonic (corresponding to $n = 6$) and also the harmonics for $n = 10$, 14, 18, 22, and 26 are all more prominent (that is, have larger coefficients) than the fundamental. We can be even more specific about the relative importance of the various frequencies. Recall that in discussing a simple harmonic oscillator, we showed that its average energy was proportional to the square of its velocity amplitude. It can be proved that the intensity of a sound wave (average energy striking unit area of your ear per second) is proportional to the average of the square of the excess pressure. Thus for a sinusoidal pressure variation $A\sin 2\pi ft$, the intensity is proportional to A^2. In the Fourier series for $p(t)$, the intensities of the various harmonics are then proportional to the squares of the corresponding Fourier coefficients. (The intensity corresponds roughly to the loudness of the tone—not exactly because the ear is not uniformly sensitive to all frequencies.) The relative intensities of the harmonics in our example are then:

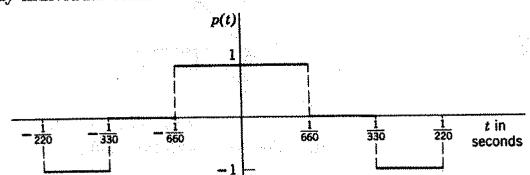
n	= 1	2	3	4	5	6	7	8	9	10	⋯
Relative intensity	= 1	225	$\frac{1}{9}$	0	$\frac{1}{25}$	25	$\frac{1}{49}$	0	$\frac{1}{81}$	9	⋯

From this we see even more clearly that we would hear principally the second harmonic with frequency 524 (high C).

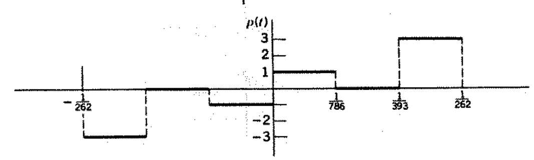
▸ PROBLEMS, SECTION 10

In Problems 1 to 3, the graphs sketched represent one period of the excess pressure $p(t)$ in a sound wave. Find the important harmonics and their relative intensities. Use a computer to play individual terms or a sum of several terms of the series.

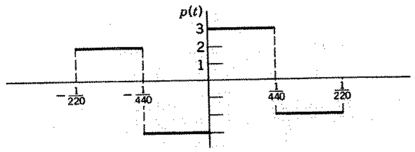
1.



2.

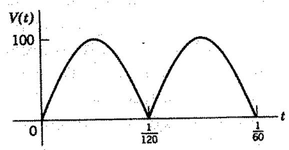


3.

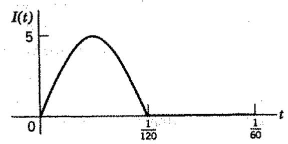


In Problems 4 to 10, the sketches show several practical examples of electrical signals (voltages or currents). In each case we want to know the harmonic content of the signal, that is, what frequencies it contains and in what proportions. To find this, expand each function in an appropriate Fourier series. Assume in each case that the part of the graph shown is repeated sixty times per second.

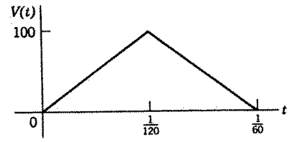
4. Output of a simple d-c generator; the shape of the curve is the absolute value of a sine function. Let the maximum voltage be 100 v.



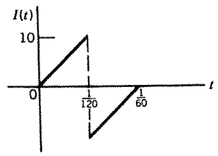
5. Rectified half-wave; the curve is a sine function for half the cycle and zero for the other half. Let the maximum current be 5 amp. *Hint:* Be careful! The value of l here is $1/60$, but $I(t) = \sin t$ only from $t = 0$ to $t = 1/120$.



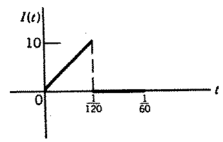
6. Triangular wave; the graph consists of two straight lines whose equations you must write! The maximum voltage of 100 v occurs at the middle of the cycle.



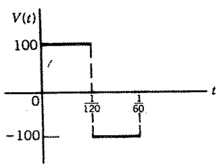
7. Sawtooth



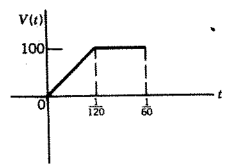
8. Rectified sawtooth



9. Square wave



10. Periodic ramp function



▶ 11. PARSEVAL'S THEOREM

We shall now find a relation between the average of the square (or absolute square) of $f(x)$ and the coefficients in the Fourier series for $f(x)$, assuming that $\int_{-\pi}^{\pi} |f(x)|^2\,dx$ is finite. The result is known as *Parseval's theorem* or the *completeness relation.* You should understand that the point of the theorem is *not* to get the average of the square of a given $f(x)$ by using its Fourier series. [Given $f(x)$, it is easy to get its average square just by doing the integration in (11.2) below.] The point of the theorem is to show the *relation* between the average of the square of $f(x)$ and the Fourier coefficients. We can derive a form of Parseval's theorem from any of the various Fourier expansions we have made; let us use (5.1).

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx. \tag{11.1}$$

We square $f(x)$ and then average the square over $(-\pi, \pi)$:

$$\text{The average of } [f(x)]^2 \text{ is } \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx. \tag{11.2}$$

When we square the Fourier series in (11.1) we get many terms. To avoid writing out a large number of them, consider instead what types of terms there are and what the averages of the different kinds of terms are. First, there are the squares of the individual terms. Using the fact that the average of the square of a sine or cosine over a period is $\frac{1}{2}$, we have:

$$\begin{aligned} &\text{The average of } (\tfrac{1}{2}a_0)^2 && \text{is} && (\tfrac{1}{2}a_0)^2. \\ &\text{The average of } (a_n \cos nx)^2 && \text{is} && a_n^2 \cdot \tfrac{1}{2}. \\ &\text{The average of } (b_n \sin nx)^2 && \text{is} && b_n^2 \cdot \tfrac{1}{2}. \end{aligned} \tag{11.3}$$

Then there are cross-product terms of the forms $2 \cdot \frac{1}{2}a_0 a_n \cos nx$, $2 \cdot \frac{1}{2}a_0 b_n \sin nx$, and $2a_n b_m \cos nx \sin mx$ with $m \neq n$ (we write n in the cosine factor and m in the sine factor since every sine term must be multiplied times every cosine term). By (5.2), the average values of terms of all these types are zero. Then we have

$$\text{The average of } [f(x)]^2 \text{ (over a period)} = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_1^\infty a_n^2 + \frac{1}{2}\sum_1^\infty b_n^2. \tag{11.4}$$

This is one form of Parseval's theorem. You can easily verify (Problem 1) that the theorem is unchanged if $f(x)$ has period $2l$ instead of 2π and its square is averaged over any period of length $2l$. You can also verify (Problem 3) that if $f(x)$ is written as a complex exponential Fourier series, and if in addition we include the possibility that $f(x)$ itself may be complex, then we find:

$$\text{The average of } |f(x)|^2 \text{ (over a period)} = \sum_{-\infty}^{\infty} |c_n|^2. \tag{11.5}$$

Parseval's theorem is also called the *completeness relation.* In the problem of representing a given sound wave as a sum of harmonics, suppose we had left one of the harmonics out of the series. It seems plausible physically, and it can be proved mathematically, that with one or more harmonics left out, we would not be able to represent sound waves containing the omitted harmonics. We say that the set of functions $\sin nx$, $\cos nx$ is a *complete set* of functions on any interval of length 2π; that is, any function (satisfying Dirichlet conditions) can be expanded in a Fourier series whose terms are constants times $\sin nx$ and $\cos nx$. If we left out some values of n, we would have an incomplete set of basis functions (see basis, page 357) and could not use it to expand some given functions. For example, suppose that you made a mistake in finding the period (that is, the value of l) of your given function and tried to use the set of functions $\sin 2nx$, $\cos 2nx$ in expanding a given function of period 2π. You would get a wrong answer because you used an incomplete set of basis functions (with the $\sin x$, $\cos x$, $\sin 3x$, $\cos 3x$, $\cdots$, terms missing). If your Fourier series is wrong because the set of basis functions you use is incomplete, then the results you get from Parseval's theorem (11.4) or (11.5) will be wrong too. In fact, if we use an incomplete basis set in, say, (11.5), then there are missing (non-negative) terms on the right-hand side, so the equation becomes the inequality: Average of $|f(x)|^2 \geq \sum_{-\infty}^{\infty} |c_n|^2$. This is known as Bessel's inequality. Conversely, if (11.4) and (11.5) are correct for *all* $f(x)$, then the set of basis functions used is a complete set. This is why Parseval's theorem is often called the completeness relation. (Also see page 377 and Chapter 12, Section 6.)

Let us look at some examples of the physical meaning and the use of Parseval's theorem.

▸ **Example 1.** In Section 10 we said that the intensity (energy per square centimeter per second) of a sound wave is proportional to the average value of the square of the excess pressure. If for simplicity we write (10.3) with letters instead of numerical values, we have

$$p(t) = \sum_{1}^{\infty} b_n \sin 2\pi n f t. \tag{11.6}$$

For this case, Parseval's theorem (11.4) says that:

$$\text{The average of } [p(t)]^2 = \sum_{1}^{\infty} b_n^2 \cdot \frac{1}{2} = \sum_{1}^{\infty} \text{the average of } b_n^2 \sin^2 2n\pi f t\,. \tag{11.7}$$

Now the intensity or energy (per square centimeter per second) of the sound wave is proportional to the average of $[p(t)]^2$, and the energy associated with the nth harmonic is proportional to the average of $b_n^2 \sin^2 2n\pi f t$. Thus Parseval's theorem says that the total energy of the sound wave is equal to the sum of the energies associated with the various harmonics.

▸ **Example 2.** Let us use Parseval's theorem to find the sum of an infinite series. From Problem 8.15(a) written in complex exponential form we get:

The function $f(x)$ of period 2 which is equal to x on $(-1, 1)$

$$= -\frac{i}{\pi}\left(e^{i\pi x} - e^{-i\pi x} - \frac{1}{2}e^{2i\pi x} + \frac{1}{2}e^{-2i\pi x} + \frac{1}{3}e^{3i\pi x} - \frac{1}{3}e^{-3i\pi x} + \cdots\right).$$

Let us find the average of $[f(x)]^2$ on $(-1, 1)$.

$$\text{The average of } [f(x)]^2 = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\bigg|_{-1}^{1}\right] = \frac{1}{3}.$$

By Parseval's theorem (11.5), this is equal to $\sum_{-\infty}^{\infty} |c_n|^2$, so we have

$$\frac{1}{3} = \sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{\pi^2}\left(1 + 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{9} + \frac{1}{9} + \cdots\right) = \frac{2}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}.$$

Then we get the sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{2}\cdot\frac{1}{3} = \frac{\pi^2}{6}.$$

We have seen that a function given on $(0, l)$ can be expanded in a sine series by defining it on $(-l, 0)$ to make it odd, or in a cosine series by defining it on $(-l, 0)$ to make it even. Here is another useful example of defining a function to suit our purposes. (We will need this in Chapter 13.) Suppose we want to expand a function defined on $(0, l)$ in terms of the basis functions $\sin(n + \frac{1}{2})\frac{\pi x}{l} = \sin\frac{(2n+1)\pi x}{2l}$. Can we do it, that is, do these functions make up a complete set for this problem? Note that our proposed basis functions have period $4l$, say $(-2l, 2l)$ (observe the $2l$ in the denominator where you are used to l). So given $f(x)$ on $(0, l)$, we can define it as we like on $(l, 2l)$ and on $(-2l, 0)$. We know (by the Dirichlet theorem) that the functions $\sin\frac{n\pi x}{2l}$ and $\cos\frac{n\pi x}{2l}$, *all* n, make up a complete set on $(-2l, 2l)$. We need to see how, on $(0, l)$ we can use just the sines (that's easy—make the function odd) and only the odd values of n. It turns out (see Problem 11) that if we define $f(x)$ on $(l, 2l)$ to make it symmetric around $x = l$, then all the b_n's for even n are equal to zero. So our desired basis set is indeed a complete set on $(0, l)$. Similarly we can show (Problem 11) that the functions $\cos\frac{(2n+1)\pi x}{2l}$ make up a complete set on $(0, l)$.

▸ PROBLEMS, SECTION 11

1. Prove (11.4) for a function of period $2l$ expanded in a sine-cosine series.
2. Prove that if $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$, then the average value of $[f(x)]^2$ is $\sum_{-\infty}^{\infty} c_n c_{-n}$. Show by Problem 7.12 that for real $f(x)$ this becomes (11.5).
3. If $f(x)$ is complex, we usually want the average of the square of the absolute value of $f(x)$. Recall that $|f(x)|^2 = f(x) \cdot \bar{f}(x)$, where $\bar{f}(x)$ means the complex conjugate of $f(x)$. Show that if a complex $f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}$, then (11.5) holds.
4. When a current I flows through a resistance R, the heat energy dissipated per second is the average value of RI^2. Let a periodic (not sinusoidal) current $I(t)$ be expanded in a Fourier series $I(t) = \sum_{-\infty}^{\infty} c_n e^{120in\pi t}$. Give a physical meaning to Parseval's theorem for this problem.

Use Parseval's theorem and the results of the indicated problems to find the sum of the series in Problems 5 to 9.

5. The series $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$, using Problem 9.6.
6. The series $\sum_{n=1}^{\infty}\frac{1}{n^4}$, using Problem 9.9.

7. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, using Problem 5.8.

8. The series $\sum_{\text{odd } n} \frac{1}{n^4}$, using Problem 9.10.

9. The series $\frac{1}{3^2} + \frac{1}{15^2} + \frac{1}{35^2} + \cdots$, using Problem 5.11.

10. A general form of Parseval's theorem says that if two functions are expanded in Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx,$$

$$g(x) = \frac{1}{2}a_0' + \sum_1^\infty a_n' \cos nx + \sum_1^\infty b_n' \sin nx,$$

then the average value of $f(x)g(x)$ is $\frac{1}{4}a_0a_0' + \frac{1}{2}\sum_1^\infty a_na_n' + \frac{1}{2}\sum_1^\infty b_nb_n'$. Prove this.

11. (a) Let $f(x)$ on $(0, 2l)$ satisfy $f(2l - x) = f(x)$, that is, $f(x)$ is symmetric about $x = l$. If you expand $f(x)$ on $(0, 2l)$ in a sine series $\sum b_n \sin \frac{n\pi x}{2l}$, show that for even n, $b_n = 0$. *Hint:* Note that the period of the sines is $4l$. Sketch an $f(x)$ which is symmetric about $x = l$, and on the same axes sketch a few sines to see that the even ones are antisymmetric about $x = l$. Alternatively, write the integral for b_n as an integral from 0 to l plus an integral from l to $2l$, and replace x by $2l - x$ in the second integral.

(b) Similarly, show that if we define $f(2l-x) = -f(x)$, the cosine series has $a_n = 0$ for even n.

12. FOURIER TRANSFORMS

We have been expanding *periodic* functions in series of sines, cosines, and complex exponentials. Physically, we could think of the terms of these Fourier series as representing a set of harmonics. In music these would be an infinite set of frequencies nf, $n = 1, 2, 3, \cdots$; notice that this set, although infinite, does not by any means include all possible frequencies. In electricity, a Fourier series could represent a periodic voltage; again we could think of this as made up of an infinite but discrete (that is, not continuous) set of a-c voltages of frequencies $n\omega$. Similarly, in discussing light, a Fourier series could represent light consisting of a discrete set of wavelengths λ/n, $n = 1, 2, \cdots$, that is, a discrete set of colors. Two related questions might occur to us here. First, is it possible to represent a function which is *not* periodic by something analogous to a Fourier series? Second, can we somehow extend or modify Fourier series to cover the case of a continuous spectrum of wavelengths of light, or a sound wave containing a continuous set of frequencies?

If you recall that an integral is a limit of a sum, it may not surprise you very much to learn that the Fourier *series* (that is, a *sum* of terms) is replaced by a Fourier *integral* in the above cases. The Fourier integral can be used to represent nonperiodic functions, for example a single voltage pulse not repeated, or a flash of light, or a sound which is not repeated. The Fourier integral also represents a continuous set (spectrum) of frequencies, for example a whole range of musical tones or colors of light rather than a discrete set.

Recall from equations (8.2) and (8.3), these complex Fourier series formulas:

(12.1)
$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/l},$$
$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx.$$

The period of $f(x)$ is $2l$ and the frequencies of the terms in the series are $n/(2l)$. We now want to consider the case of continuous frequencies.

Definition of Fourier Transforms We state without proof (see plausibility arguments below) the formulas corresponding to (12.1) for a continuous range of frequencies.

(12.2)
$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,d\alpha,$$
$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx.$$

Compare (12.2) and (12.1); $g(\alpha)$ corresponds to c_n, α corresponds to n, and $\int_{-\infty}^{\infty}$ corresponds to $\sum_{-\infty}^{\infty}$. This agrees with our discussion of the physical meaning and use of Fourier integrals. The quantity α is a continuous analog of the integral-valued variable n, and so the set of coefficients c_n has become a function $g(\alpha)$; the sum over n has become an integral over α. The two functions $f(x)$ and $g(\alpha)$ are called a pair of *Fourier transforms*. Usually, $g(\alpha)$ is called the Fourier transform of $f(x)$, and $f(x)$ is called the inverse Fourier transform of $g(\alpha)$, but since the two integrals differ in form only in the sign in the exponent, it is rather common simply to call either a Fourier transform of the other. You should check the notation of any book or computer program you are using. Another point on which various references differ is the position of the factor $1/(2\pi)$ in (12.2); it is possible to have it multiply the $f(x)$ integral instead of the $g(\alpha)$ integral, or to have the factor $1/\sqrt{2\pi}$ multiply each of the integrals.

The *Fourier integral theorem* says that, if a function $f(x)$ satisfies the Dirichlet conditions (Section 6) on every finite interval, and if $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite, then (12.2) is correct. That is, if $g(\alpha)$ is computed and substituted into the integral for $f(x)$ [compare the procedure of computing the c_n's for a Fourier series and substituting them into the series for $f(x)$], then the integral gives the value of $f(x)$ anywhere that $f(x)$ is continuous; at jumps of $f(x)$, the integral gives the midpoint of the jump (again compare Fourier series, Section 6). The following discussion is not a mathematical proof of this theorem but is intended to help you see more clearly how Fourier integrals are related to Fourier series.

It might seem reasonable to think of trying to represent a function which is not periodic by letting the period $(-l, l)$ increase to $(-\infty, \infty)$. Let us try to do this, starting with (12.1). If we call $n\pi/l = \alpha_n$ and $\alpha_{n+1} - \alpha_n = \pi/l = \Delta\alpha$, then $1/(2l) = \Delta\alpha/(2\pi)$ and (12.1) can be rewritten as

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\alpha_n x}, \tag{12.3}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-i\alpha_n x}\,dx = \frac{\Delta\alpha}{2\pi}\int_{-l}^{l} f(u)e^{-i\alpha_n u}\,du. \tag{12.4}$$

(We have changed the dummy integration variable in c_n from x to u to avoid later confusion.) Substituting (12.4) into (12.3), we have

$$\begin{aligned} f(x) &= \sum_{-\infty}^{\infty}\left[\frac{\Delta\alpha}{2\pi}\int_{-l}^{l} f(u)e^{-i\alpha_n u}\,du\right]e^{i\alpha_n x} \\ &= \sum_{-\infty}^{\infty}\frac{\Delta\alpha}{2\pi}\int_{-l}^{l} f(u)e^{i\alpha_n(x-u)}\,du = \frac{1}{2\pi}\sum_{-\infty}^{\infty} F(\alpha_n)\,\Delta\alpha, \end{aligned} \tag{12.5}$$

where

$$F(\alpha_n) = \int_{-l}^{l} f(u)e^{i\alpha_n(x-u)}\,du. \tag{12.6}$$

Now $\sum_{-\infty}^{\infty} F(\alpha_n)\Delta\alpha$ looks rather like the formula in calculus for the sum whose limit, as $\Delta\alpha$ tends to zero, is an integral. If we let l tend to infinity [that is, let the period of $f(x)$ tend to infinity], then $\Delta\alpha = \pi/l \to 0$, and the sum $\sum_{-\infty}^{\infty} F(\alpha_n)\Delta\alpha$ goes over formally to $\int_{-\infty}^{\infty} F(\alpha)d\alpha$; we have dropped the subscript n on α now that it is a continuous variable. We also let l tend to infinity and $\alpha_n = \alpha$ in (12.6) to get

$$F(\alpha) = \int_{-\infty}^{\infty} f(u)e^{i\alpha(x-u)}\,du. \tag{12.7}$$

Replacing $\sum_{-\infty}^{\infty} F(\alpha_n)\Delta\alpha$ in (12.5) by $\int_{-\infty}^{\infty} F(\alpha)\,d\alpha$ and substituting from (12.7) for $F(\alpha)$ gives

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\,d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)e^{i\alpha(x-u)}\,du\,d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\,d\alpha\int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\,du. \end{aligned} \tag{12.8}$$

If we define $g(\alpha)$ by

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\,du, \tag{12.9}$$

then (12.8) gives

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,d\alpha. \tag{12.10}$$

These equations are the same as (12.2). Notice that the actual requirement for the factor $1/(2\pi)$ is that the *product* of the constants multiplying the two integrals for $g(\alpha)$ and $f(x)$ should be $1/(2\pi)$; this accounts for the various notations we have discussed before.

Just as we have sine series representing odd functions and cosine series representing even functions (Section 9), so we have sine and cosine Fourier integrals which represent odd or even functions respectively. Let us prove that if $f(x)$ is odd, then $g(\alpha)$ is odd too, and show that in this case (12.2) reduces to a pair of sine transforms. The corresponding proof for even $f(x)$ is similar (Problem 1). We substitute

$$e^{-i\alpha x} = \cos\alpha x - i\sin\alpha x$$

into (12.9) to get

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)(\cos\alpha x - i\sin\alpha x)\,dx. \tag{12.11}$$

Since $\cos\alpha x$ is even and we are assuming that $f(x)$ is odd, the product $f(x)\cos\alpha x$ is odd. Recall that the integral of an odd function over a symmetric interval about the origin (here, $-\infty$ to $+\infty$) is zero, so the term $\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$ in (12.11) is zero. The product $f(x)\sin\alpha x$ is even (product of two odd functions); recall that the integral of an even function over a symmetric interval is twice the integral over positive x. Substituting these results into (12.11), we have

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)(-i\sin\alpha x)\,dx = -\frac{i}{\pi}\int_{0}^{\infty} f(x)\sin\alpha x\,dx. \tag{12.12}$$

From (12.12), we can see that replacing α by $-\alpha$ changes the sign of $\sin\alpha x$ and so changes the sign of $g(\alpha)$. That is, $g(-\alpha) = -g(\alpha)$, so $g(\alpha)$ is an odd function as we claimed. Then expanding the exponential in (12.10) and arguing as we did to obtain (12.12), we find

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,dx = 2i\int_{0}^{\infty} g(\alpha)\sin\alpha x\,d\alpha. \tag{12.13}$$

If we substitute $g(\alpha)$ from (12.12) into (12.13) to obtain an equation like (12.8), the numerical factor is $(-i/\pi)(2i) = 2/\pi$; thus the imaginary factors are not needed. The factor $2/\pi$ may multiply either of the two integrals or each integral may be multiplied by $\sqrt{2/\pi}$. Let us make the latter choice in giving the following definition.

Fourier Sine Transforms We define $f_s(x)$ and $g_s(\alpha)$, a pair of *Fourier sine transforms* representing *odd functions*, by the equations

$$\begin{aligned} f_s(x) &= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g_s(\alpha)\sin\alpha x\,d\alpha, \\ g_s(x) &= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f_s(x)\sin\alpha x\,dx. \end{aligned} \tag{12.14}$$

We discuss even functions in a similar way (Problem 1).

Fourier Cosine Transforms We define $f_c(x)$ and $g_c(\alpha)$, a pair of *Fourier cosine transforms* representing *even functions*, by the equations

$$(12.15)\qquad f_c(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty g_c(\alpha)\cos\alpha x\,d\alpha,$$
$$g_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f_c(x)\cos\alpha x\,dx.$$

▶ **Example 1.** Let us represent a nonperiodic function as a Fourier integral. The function

$$f(x) = \begin{cases} 1, & -1 < x < 1, \\ 0, & |x| > 1, \end{cases}$$

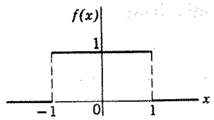


Figure 12.1

shown in Figure 12.1 might represent an impulse in mechanics (that is, a force applied only over a short time such as a bat hitting a baseball), or a sudden short surge of current in electricity, or a short pulse of sound or light which is not repeated. Since the given function is not periodic, it cannot be expanded in a Fourier *series*, since a Fourier series always represents a *periodic* function. Instead, we write $f(x)$ as a Fourier integral as follows. Using (12.9), we calculate $g(\alpha)$; this process is like finding the c_n's for a Fourier series. We find

$$(12.16)\qquad g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx = \frac{1}{2\pi}\int_{-1}^{1} e^{-i\alpha x}\,dx$$
$$= \frac{1}{2\pi}\left.\frac{e^{-i\alpha x}}{-i\alpha}\right|_{-1}^{1} = \frac{1}{\pi\alpha}\frac{e^{-i\alpha}-e^{i\alpha}}{-2i} = \frac{\sin\alpha}{\pi\alpha}.$$

We substitute $g(\alpha)$ from (12.16) into the formula (12.10) for $f(x)$ (this is like substituting the evaluated coefficients into a Fourier series). We get

$$(12.17)\qquad f(x) = \int_{-\infty}^{\infty}\frac{\sin\alpha}{\pi\alpha}e^{i\alpha x}\,dx$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\alpha(\cos\alpha x + i\sin\alpha x)}{\alpha}d\alpha = \frac{2}{\pi}\int_0^\infty\frac{\sin\alpha\cos\alpha x}{\alpha}\,d\alpha$$

since $(\sin\alpha)/\alpha$ is an even function. We thus have an integral representing the function $f(x)$ shown in Figure 12.1.

▶ **Example 2.** We can use (12.17) to evaluate a definite integral. Using $f(x)$ in Figure 12.1, we find

$$(12.18)\qquad \int_0^\infty \frac{\sin\alpha\cos\alpha x}{\alpha}\,d\alpha = \frac{\pi}{2}f(x) = \begin{cases} \frac{\pi}{2} & \text{for } |x| < 1, \quad \frac{\pi}{4} \text{ for } |x| = 1, \\ 0 & \text{for } |x| > 1. \end{cases}$$

Notice that we have used the fact that the Fourier integral represents the midpoint of the jump in $f(x)$ at $|x| = 1$. If we let $x = 0$, we get

$$(12.19)\qquad \int_0^\infty \frac{\sin\alpha}{\alpha}\,d\alpha = \frac{\pi}{2}.$$

We could have done this problem by observing that $f(x)$ is an even function and so can be represented by a cosine transform. The final results (12.17) to (12.19) would be just the same (Problem 2).

In Section 9, we sometimes started with a function defined only for $x > 0$ and extended it to be even or odd so that we could represent it by a cosine series or by a sine series. Similarly, for Fourier transforms, we can represent a function defined for $x > 0$ by either a Fourier cosine integral (by defining it for $x < 0$ so that it is even), or by a Fourier sine integral (by defining it for $x < 0$ so that it is odd). (See Problem 2 and Problems 27 to 30.)

Parseval's Theorem for Fourier Integrals Recall (Section 11) that Parseval's theorem for a Fourier series $f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}$ relates $\int_{-l}^{l}|f|^2\,dx$ and $\sum_{-\infty}^{\infty}|c_n|^2$. In physical applications (see Section 11), Parseval's theorem says that the total energy (say in a sound wave, or in an electrical signal) is equal to the sum of the energies associated with the various harmonics. Remember that a Fourier integral represents a continuous spectrum of frequencies and that $g(\alpha)$ corresponds to c_n. Then we might expect that $\sum_{-\infty}^{\infty}|c_n|^2$ would be replaced by $\int_{-\infty}^{\infty}|g(\alpha)|^2 d\alpha$ (that is, a "sum" over a continuous rather than a discrete spectrum) and that Parseval's theorem would relate $\int_{-\infty}^{\infty}|f|^2 dx$ and $\int_{-\infty}^{\infty}|g|^2 d\alpha$. Let us try to find the relation.

We will first find a generalized form of Parseval's theorem involving two functions $f_1(x)$, $f_2(x)$ and their Fourier transforms $g_1(\alpha)$, $g_2(\alpha)$. Let $\bar{g}_1(\alpha)$ be the complex conjugate of $g_1(\alpha)$; from (12.1), we have

$$(12.20)\qquad \bar{g}_1(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}_1(x)e^{i\alpha x}dx.$$

We now multiply (12.20) by $g_2(\alpha)$ and integrate with respect to α:

$$(12.21)\qquad \int_{-\infty}^{\infty}\bar{g}_1(\alpha)g_2(\alpha)\,d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\bar{f}_1(x)e^{i\alpha x}\,dx\right]g_2(\alpha)d\alpha.$$

Let us rearrange (12.21) so that we integrate first with respect to α. [This is justified assuming that the absolute values of the functions f_1 and f_2 are integrable on $(-\infty,\infty)$.]

$$(12.22)\qquad \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}_1(x)\,dx\left[\int_{-\infty}^{\infty}g_2(\alpha)e^{i\alpha x}\,d\alpha\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}_1(x)f_2(x)\,dx$$

by (12.2). Thus

$$(12.23)\qquad \int_{-\infty}^{\infty}\bar{g}_1(\alpha)g_2(\alpha)\,d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}_1(x)f_2(x)\,dx.$$

(Compare this with the corresponding Fourier series theorem in Problem 11.10.) If we set $f_1 = f_2 = f$ and $g_1 = g_2 = g$, we get Parseval's theorem:

$$(12.24)\qquad \int_{-\infty}^{\infty}|g(\alpha)|^2\,d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty}|f(x)|^2\,dx.$$

PROBLEMS, SECTION 12

1. Following a method similar to that used in obtaining equations (12.11) to (12.14), show that if $f(x)$ is even, then $g(\alpha)$ is even too. Show that in this case $f(x)$ and $g(\alpha)$ can be written as Fourier cosine transforms and obtain (12.15).

2. Do Example 1 above by using a cosine transform (12.15). Obtain (12.17); for $x > 0$, the 0 to ∞ integral represents the function

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$$

Represent this function also by a Fourier sine integral (see the paragraph just before Parseval's theorem).

In Problems 3 to 12, find the exponential Fourier transform of the given $f(x)$ and write $f(x)$ as a Fourier integral [that is, find $g(\alpha)$ in equation (12.2) and substitute your result into the first integral in equation (12.2)].

3. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & |x| > \pi \end{cases}$

4. $f(x) = \begin{cases} 1, & \pi/2 < |x| < \pi \\ 0, & \text{otherwise} \end{cases}$

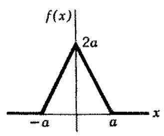
5. $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

6. $f(x) = \begin{cases} x, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

7. $f(x) = \begin{cases} |x|, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

8. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

9.



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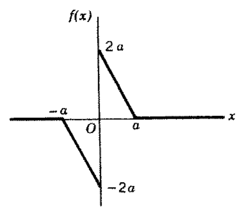

11. $f(x) = \begin{cases} \cos x, & -\pi/2 < x < \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$

12. $f(x) = \begin{cases} \sin x, & |x| < \pi/2 \\ 0, & |x| > \pi/2 \end{cases}$

Hint: In Problems 11 and 12, use complex exponentials.

In Problems 13 to 16, find the Fourier cosine transform of the function in the indicated problem, and write $f(x)$ as a Fourier integral [use equation (12.15)]. Verify that the cosine integral for $f(x)$ is the same as the exponential integral found previously.

13. Problem 4.

14. Problem 7.

15. Problem 9.

16. Problem 11.

In Problems 17 to 20, find the Fourier sine transform of the function in the indicated problem, and write $f(x)$ as a Fourier integral [use equation (12.14)]. Verify that the sine integral for $f(x)$ is the same as the exponential integral found previously.

17. Problem 3.

18. Problem 6.

19. Problem 10.

20. Problem 12.

21. Find the Fourier transform of $f(x) = e^{-x^2/(2\sigma^2)}$. *Hint:* Complete the square in the x terms in the exponent and make the change of variable $y = x + \sigma^2 i\alpha$. Use tables or computer to evaluate the definite integral.

22. The function $j_1(\alpha) = (\alpha\cos\alpha - \sin\alpha)/\alpha$ is of interest in quantum mechanics. [It is called a spherical Bessel function; see Chapter 12, equation (17.4).] Using Problem 18, show that

$$\int_0^\infty j_1(\alpha)\sin\alpha x\,d\alpha = \begin{cases} \pi x/2, & -1 < x < 1, \\ 0, & |x| > 1. \end{cases}$$

23. Using Problem 17, show that

$$\int_0^\infty \frac{1-\cos\pi\alpha}{\alpha}\sin\alpha\,d\alpha = \frac{\pi}{2},$$
$$\int_0^\infty \frac{1-\cos\pi\alpha}{\alpha}\sin\pi\alpha\,d\alpha = \frac{\pi}{4}.$$

24. (a) Find the exponential Fourier transform of $f(x) = e^{-|x|}$ and write the inverse transform. You should find

$$\int_0^\infty \frac{\cos\alpha x}{\alpha^2+1}\,d\alpha = \frac{\pi}{2}e^{-|x|}.$$

(b) Obtain the result in (a) by using the Fourier cosine transform equations (12.15).

(c) Find the Fourier cosine transform of $f(x) = 1/(1+x^2)$. *Hint:* Write your result in (b) with x and α interchanged.

25. (a) Represent as an exponential Fourier transform the function

$$f(x) = \begin{cases} \sin x, & 0 < x < \pi, \\ 0, & \text{otherwise}. \end{cases}$$

Hint: Write $\sin x$ in complex exponential form.

(b) Show that your result can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos\alpha x + \cos\alpha(x-\pi)}{1-\alpha^2}\,d\alpha.$$

26. Using Problem 15, show that

$$\int_0^\infty \frac{1-\cos\alpha}{\alpha^2}\,d\alpha = \frac{\pi}{2}.$$

Represent each of the following functions (a) by a Fourier cosine integral; (b) by a Fourier sine integral. *Hint:* See the discussion just before Parseval's theorem.

27. $f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ 0, & x > \pi/2 \end{cases}$

28. $f(x) = \begin{cases} 1, & 2 < x < 4 \\ 0, & 0 < x < 2,\ x > 4 \end{cases}$

10. (a) Sketch at least three periods of the graph of the function represented by the cosine series for $f(x)$ in Problem 9.

(b) Sketch at least three periods of the graph of the exponential Fourier series of period 2 for $f(x)$ in Problem 9.

(c) To what value does the cosine series in (a) converge at $x = 0$? At $x = 1$? At $x = 2$? At $x = -2$?

(d) To what value does the exponential series in (b) converge at $x = 0$? At $x = 1$? At $x = \frac{3}{2}$? At $x = -2$.

11. Find the three Fourier series in Problems 9 and 10.

12. What would be the apparent frequency of a sound wave represented by

$$p(t) = \sum_{n=1}^{\infty} \frac{\cos 60n\pi t}{100(n-3)^2 + 1}?$$

13. (a) Given $f(x) = (\pi - x)/2$ on $(0, \pi)$, find the sine series of period 2π for $f(x)$.

(b) Use your result in (a) to evaluate $\sum 1/n^2$.

14. (a) Find the Fourier series of period 2 for $f(x) = (x-1)^2$ on $(0, 2)$.

(b) Use your result in (a) to evaluate $\sum 1/n^4$.

15. Given

$$f(x) = \begin{cases} 1, & -2 < x < 0, \\ -1, & 0 < x < 2, \end{cases}$$

find the exponential Fourier transform $g(\alpha)$ and the sine transform $g_s(\alpha)$. Write $f(x)$ as an integral and use your result to evaluate

$$\int_0^\infty \frac{(\cos 2\alpha - 1)\sin 2\alpha}{\alpha}\, d\alpha.$$

16. Given

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 \le x \le 2, \\ 0, & x \ge 2, \end{cases}$$

find the cosine transform of $f(x)$ and use it to write $f(x)$ as an integral. Use your result to evaluate

$$\int_0^\infty \frac{\cos^2 \alpha \sin^2 \alpha/2}{\alpha^2}\, d\alpha.$$

17. Show that the Fourier sine transform of $x^{-1/2}$ is $\alpha^{-1/2}$. *Hint:* Make the change of variable $z = \alpha x$. The integral $\int_0^\infty z^{-1/2} \sin z\, dz$ can be found by computer or in tables.

18. Let $f(x)$ and $g(\alpha)$ be a pair of Fourier transforms. Show that df/dx and $i\alpha g(\alpha)$ are a pair of Fourier transforms. *Hint:* Differentiate the first integral in (12.2) under the integral sign with respect to x. Use (12.23) to show that

$$\int_{-\infty}^{\infty} \alpha |g(\alpha)|^2\, d\alpha = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \bar{f}(x) \frac{d}{dx} f(x)\, dx.$$

Comment: This result is of interest in quantum mechanics where it would read, in the notation of Problem 12.35:

$$\int_{-\infty}^{\infty} p|\phi(p)|^2\, dp = \int_{-\infty}^{\infty} \psi^*(x) \left(\frac{-ih}{2\pi} \frac{d}{dx}\right) \psi(x)\, dx.$$

19. Find the form of Parseval's theorem (12.24) for sine transforms (12.14) and for cosine transforms (12.15).

20. Find the exponential Fourier transform of

$$f(x) = \begin{cases} 2a - |x|, & |x| < 2a, \\ 0, & |x| > 2a, \end{cases}$$

and use your result with Parseval's theorem to evaluate

$$\int_0^\infty \frac{\sin^4 a\alpha}{\alpha^4}\, d\alpha.$$

21. Define a function $h(x) = \sum_{k=-\infty}^{\infty} f(x + 2k\pi)$, assuming that the series converges to a function satisfying Dirichlet conditions (Section 6). Verify that $h(x)$ does have period 2π.

(a) Expand $h(x)$ in an exponential Fourier series $h(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$; show that $c_n = g(n)$ where $g(\alpha)$ is the Fourier transform of $f(x)$. *Hint:* Write c_n as an integral from 0 to 2π and make the change of variable $u = x + 2k\pi$. Note that $e^{-2ink\pi} = 1$, and the sum on k gives a single integral from $-\infty$ to ∞.

(b) Let $x = 0$ in (a) to get *Poisson's summation formula* $\sum_{-\infty}^{\infty} f(2k\pi) = \sum_{-\infty}^{\infty} g(n)$. This result has many applications; for example: statistical mechanics, communication theory, theory of optical instruments, scattering of light in a liquid, and so on. (See Problem 22.)

22. Use Poisson's formula (Problem 21b) and Problem 20 to show that

$$\sum_{-\infty}^{\infty} \frac{\sin^2 n\theta}{n^2} = \pi\theta, \quad 0 < \theta < \pi.$$

(This sum is needed in the theory of scattering of light in a liquid.) *Hint:* Consider $f(x)$ and $g(\alpha)$ as in Problem 20. Note that $f(2k\pi) = 0$ except for $k = 0$ if $a < \pi$. Put $\alpha = n$, $a = \theta$.

23. Use Parseval's theorem and Problem 12.11 to evaluate

$$\int_0^\infty \frac{\cos^2(\alpha\pi/2)}{(1-\alpha^2)^2}\, d\alpha.$$