3D→2D GPE Reduction

Let us start with fully dimensional 3D GPE eq:

$$i\hbar \psi_t = \Big[-\frac{\hbar^2}{2m}\nabla^2 + \underbrace{\frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)}_{V(x,y,z)} + g|\psi|^2\Big]\psi \quad (1)$$

We suppose a "pancake" trap where $\omega_z \gg \omega_x \approx \omega_y \equiv \omega_r$

$$\Rightarrow \quad V(x,y,z) \longrightarrow \frac{m}{2}(\omega_r^2 r^2 + \omega_z^2 z^2)$$

In this quasi-2D scenario the wave function will not be able to be excited in the z-direction ⇒ in the z-direction the system will be in its "ground state".
[ground state of a harmonic trap → quantum harmonic oscillators → Gaussian]

∴ Let us do: a) separation of variables: $\psi_{3D} = \psi_{2D} \cdot \phi_0(z) \cdot f(t)$
b) method of averaging over z-direction

3D→2D: $\psi_{3D} = \psi_{2D}(x,y)\phi_0(z)e^{-i\frac{\gamma t}{\hbar}}$ (separation of variables)

$$\Rightarrow \phi_0\, i\hbar\frac{\partial}{\partial t}[\psi_2 e^{-i\gamma t}] = -\Big(\frac{\hbar^2}{2m}\nabla_2^2\psi\Big)\phi_0 e - \frac{\hbar^2}{2m}\phi_0'' e\psi + [V + g\phi_0^2|\psi_2|^2]\phi_0\psi e$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}[\psi_2 e] = -\frac{\hbar^2}{2m}\nabla_2^2\psi e - \frac{\hbar^2}{2m}\frac{\phi_0''}{\phi_0}e\psi + [V + g\phi_0^2|\psi_2|^2]\psi e$$

$$\Rightarrow i\hbar\Big(\frac{\partial}{\partial t}\psi_2\Big)\cancel{e} + i\hbar\psi_2\Big(-i\frac{\gamma}{\hbar}\Big)\cancel{e} = -\frac{\hbar^2}{2m}\nabla_2^2\psi\cancel{e} - \frac{\hbar^2}{2m}\frac{\phi_0''}{\phi_0}\cancel{e}\psi + [\quad]\psi\cancel{e}$$

$$\div \psi_2 \Rightarrow i\hbar\frac{\partial_t\psi_2}{\psi_2} + \frac{\hbar\gamma}{\cancel{\hbar}} = -\frac{\hbar^2}{2m}\frac{\nabla_2^2\psi_2}{\psi_2} - \frac{\hbar^2}{2m}\frac{\phi_0''}{\phi_0} + [V + g\phi_0^2|\psi_2|^2]$$

$$i\hbar\frac{\partial_t\psi_2}{\psi_2} + \frac{\hbar^2}{2m}\frac{\nabla_2^2\psi_2}{\psi_2} - \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2) - g\phi_0^2|\psi_2|^2 = -\frac{\hbar\gamma}{\hbar} - \frac{\hbar^2}{2m}\frac{\phi_0''}{\phi_0} + \frac{1}{2}m\omega_z^2 z^2$$

$$\Rightarrow \begin{cases} \frac{\hbar^2}{2m}\phi_0'' - \frac{1}{2}m\omega_z^2 z^2\phi_0 + \frac{\hbar\gamma}{\hbar}\phi_0 = 0 \quad \Rightarrow -\frac{\hbar^2}{2m}\phi_0'' = \left[-\frac{1}{2}m\omega_z^2 z^2 + \gamma\right]\phi_0 \\ i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + \left[\frac{1}{2}V_2 + g_3\phi_0^2|\psi|^2\right]\psi \end{cases}$$

$\boxed{\phi_0}$

$$\phi_0(z) = Ae^{-\frac{z^2}{2\beta^2}} \Rightarrow \cancel{\frac{\hbar^2}{2m}A\left(\frac{1}{4\beta^4}\right)} \quad \phi_0' = A\frac{(-2z)}{2\beta^2}e^{-\frac{z^2}{2\beta^2}}$$

$$\Rightarrow \phi_0'' = A\left[-\frac{1}{\beta^2}e - \frac{z}{\beta^2}\left(\frac{-2z}{2\beta^2}\right)e\right]$$

$$\therefore \frac{\hbar^2}{2m}\left[-\frac{1}{\beta^2} + \frac{z^2}{\beta^4}\right] - \frac{m}{2}\omega_z^2 z^2 + \frac{\hbar\gamma}{\hbar} = 0$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m\beta^2} + \frac{\hbar\gamma}{\hbar}\right] + \left[\frac{\hbar^2}{2m\beta^4} - \frac{m}{2}\omega_z^2\right]z^2 = 0$$

$$\Rightarrow \gamma = \frac{\hbar^2}{2m\beta^2}, \quad m\omega_z^2 = \frac{\hbar^2}{m\beta^4}, \quad \gamma = \frac{\hbar^2}{2m}\frac{1}{\beta^2} = \frac{\hbar^2}{2m\,a_z^2} = \frac{\hbar^2}{2m}\frac{m\omega_z}{\hbar}$$

$$\boxed{\gamma = \frac{\hbar\omega_z}{2}}$$

$$\Rightarrow \boxed{\beta^2 = \frac{\hbar}{m\omega_z} = a_z^2} \qquad \left[a_z^2 \equiv \frac{\hbar}{m\omega_z}\right]$$

$$\therefore \phi_0(z) = A\,e^{-\frac{z^2}{2a_z^2}} \qquad \int_{-\infty}^{\infty} e^{-\frac{z^2}{2a^2}}dz = \sqrt{2\pi}\,a$$

$$\therefore \text{If } \underbrace{\int|\phi_0|^2 = 1}_{\text{Normalize}} \Rightarrow A^2\int e^{-\frac{z^2}{a_z^2}} = 1 \Rightarrow A^2 a_z\sqrt{\pi} = 1$$

$$\Rightarrow \boxed{A = \pi^{-1/4}a_z^{-1/2}}$$

$$\therefore \boxed{\phi_0(z) = \pi^{-1/4}a_z^{-1/2}e^{-\frac{z^2}{2a_z^2}}} \Rightarrow \phi_0'' = \pi^{-1/4}a_z^{-1/2}\left[-\frac{1}{a_z^2} + \frac{z^2}{a_z^4}\right]e$$

$$= \left[-\frac{1}{a_z^2} + \frac{z^2}{a_z^4}\right]\phi_0$$

$\boxed{\psi_2}$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{1}{2}V_2 +$$

GPE : $i\hbar \psi_t = \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}V_3 + g_3|\psi|^2\right]\psi$

$$\psi_3 = \psi_2 \phi_0 e^{-\frac{i\gamma t}{\hbar}}$$

Average over z: $\boxed{\int GPE \cdot \phi_0^* \, dz}$ $\Rightarrow$ $i\hbar \frac{\partial \psi_2}{\partial t}\phi_0 + \gamma \psi_2 \phi_0 = -\frac{\hbar^2}{2m}\nabla^2\psi_2\phi_0 - \frac{\hbar^2}{2m}\phi_0''\psi_2 + [V + g\phi_0^2|\psi_2|^2]\psi_2\phi_0$

$$\cancel{\Rightarrow \quad i\hbar\frac{\partial \psi_2}{\partial t} + \gamma\psi_2 = -\frac{\hbar^2}{2m}\nabla^2\psi_2 - \frac{\hbar^2}{2m}\left[-\frac{1}{a_z^2} + \frac{z^2}{a_z^4}\right]\psi_2 + [V+g}$$

$\times \phi_0^*$ $(= \times\phi_0)$ $\Rightarrow$ $i\hbar\frac{\partial\psi_2}{\partial t}\phi_0^2 + \gamma\psi_2\phi_0^2 = -\frac{\hbar^2}{2m}\nabla^2\psi_2\phi_0^2 - \frac{\hbar^2}{2m}\phi_0''\phi_0\psi_2 + [\quad]\psi_2\phi_0^2$

$\int dz$ $\Rightarrow$ $i\hbar\frac{\partial\psi_2}{\partial t}\phi_0^2 + \gamma\psi_2\phi_0^2 = -\frac{\hbar^2}{2m}\nabla^2\psi_2\phi_0^2 - \left[-\frac{1}{2}m\omega_z^2 z^2 + \gamma\right]\phi_0^2\psi_2 + [\quad]\psi_2\phi_0^2$

$\int dz$ $\Rightarrow$ $i\hbar\frac{\partial\psi_2}{\partial t} + \cancel{\gamma\psi_2} = -\frac{\hbar^2}{2m}\nabla^2\psi_2 - \frac{1}{2}m\omega_z^2 z^2\psi_2 + \cancel{\gamma\psi_2} + [\cancel{V_1} + V_2]\psi_2 + g\int\phi_0^4\psi_2$

$[\int\phi_0^2 = 1]$

$$\Rightarrow \quad i\hbar\frac{\partial\psi_2}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_2 + V_2\psi_2 + g_3\int\phi_0^4\,|\psi_2|^2\psi_2$$

$$\therefore \quad g_2 \equiv g_3\int\phi_0^4 = g_3\,\frac{1}{2}\sqrt{\frac{2}{\pi}}\,\frac{1}{a_z} = g_3\,\frac{1}{\sqrt{2\pi}\,a_z}$$

$$\therefore \quad \boxed{g_2 = g_3\,\frac{1}{\sqrt{2\pi}\,a_z}}$$

$$\Rightarrow \quad \boxed{i\hbar\frac{\partial\psi_2}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_2 + \left[\frac{1}{2}m\left(\omega_x^2x^2 + \omega_y^2y^2\right) + g_2|\psi_2|^2\right]\psi_2} \quad (2)$$

$\psi_3 = \psi_2\phi_0 e^{-\frac{i\gamma t}{\hbar}}$, $\phi_0(z) = \pi^{-1/4}a_z^{-1/2}e^{-\frac{z^2}{2a_z^2}}$,

$a_z^2 \equiv \frac{\hbar}{m\omega_z}$, $\gamma \equiv \frac{\hbar\omega_z}{2}$ $\Rightarrow$ $\frac{\gamma}{\hbar} = \frac{\omega_z}{2} \Rightarrow [\;] = 1/t$ ✓

$g_2 \equiv g_3\frac{1}{\sqrt{2\pi}\,a_z}$, $g_3 = \frac{4\pi\hbar^2 a}{m}$ $\Rightarrow$ $\sqrt{2\pi}\,a_z g_2 = \frac{4\pi\hbar^2 a}{m}$

Adim :

$$\omega_r \equiv \omega_x = \omega_y$$

$$\tau = \omega_z t \ , \ R = r/\zeta \ , \ \zeta^2 = \frac{\hbar}{m\omega_z}$$

$$\xrightarrow[t \to \tau,\ r \to R]{(2)} \quad i\hbar\omega_z \frac{\partial \psi}{\partial \tau} = -\frac{\hbar^2}{2m}\frac{m\omega_z}{\hbar}\nabla_R^2\psi + \left[\frac{1}{2}m\omega_r^2 \frac{\hbar}{m\omega_z} R^2 + g_2|\psi|^2\right]\psi$$

$$\div \hbar\omega_z \quad i\psi_\tau = -\frac{1}{2}\nabla_R^2\psi + \left[\frac{1}{2}\Omega^2 R^2 + \frac{g_2}{\hbar\omega_z}|\psi|^2\right]\psi$$

$$\boxed{\Omega \equiv \frac{\omega_r}{\omega_z}}$$

$$\bar{\psi} = (4\pi\zeta^2 a)^{1/2}\psi = \left(\cancel{4\pi}\frac{\cancel{\hbar}}{\cancel{m}\omega_z}\frac{g_3 \cancel{m}}{\cancel{4\pi\hbar}}\right)^{1/2}\psi = \left(\frac{g_3}{\omega_z}\right)^{1/2}\psi$$

$$\Rightarrow \quad i\bar{\psi}_\tau = -\frac{1}{2}\nabla_R^2\bar{\psi} + \left[\frac{1}{2}\Omega^2 R^2 + \frac{g_3}{\sqrt{2\pi}a_z}\frac{1}{\hbar\omega_z}\frac{\omega_z}{g_3}|\bar{\psi}|^2\right]\bar{\psi}$$

Adim ψ : $\quad b\bar{\psi} = \psi \Rightarrow \quad i\bar{\psi}_\tau = -\frac{1}{2}\nabla^2\bar{\psi} + \left[\frac{1}{2}\Omega^2R^2 + \frac{g_2}{\hbar\omega_z} b^2 |\bar{\psi}|^2\right]\bar{\psi}$

$$\therefore \quad b^2 = \frac{\hbar\omega_z}{g_2}$$

$$\left[b^2 = \frac{\hbar\omega_z}{g_3} = \frac{\cancel{\hbar}\omega_z m}{4\pi\cancel{\hbar}^2 a} = \frac{m\omega_z}{4\pi a\hbar} \right.$$

$$\left. 4\pi\zeta^2 a = 4\pi\frac{\hbar}{m\omega_z}a \right]$$

$$\Rightarrow \boxed{\psi_2 = \sqrt{\frac{\hbar\omega_z}{g_2}}\,\bar{\psi}_2}$$

$$\Rightarrow \boxed{i\frac{\partial\bar{\psi}_2}{\partial\tau} = -\frac{1}{2}\nabla^2\bar{\psi}_2 + \left[\frac{1}{2}\Omega^2R^2 + |\bar{\psi}_2|^2\right]\bar{\psi}_2}$$

$$\psi_3 = \psi_2\phi_0 e^{-\frac{i\gamma t}{\hbar}} \ , \ \phi_0(z) = \pi^{-1/4}a_z^{-1/2}e^{-\frac{z^2}{2a_z^2}}$$

$$\zeta^2 = a_z^2 \equiv \frac{\hbar}{m\omega_z} \ , \ \gamma \equiv \frac{\hbar\omega_z}{2} \ , \ \Omega = \omega_r/\omega_z$$

$$\psi_2 = \sqrt{\frac{\hbar\omega_z}{g_2}}\,\bar{\psi}_2 \ , \ g_2 \equiv g_3\frac{1}{\sqrt{2\pi}a_z} \ , \ g_3 = \frac{4\pi\hbar^2 a}{m}$$

$$\tau = \omega_z t \ , \ R = r/\zeta \ , \ \zeta^2 = \frac{\hbar}{m\omega_z} = a_z^2$$

Adimensionalized of reduced 3D-2D GPE.