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A PERTURBATION THEORY FOR SOLITON SYSTEMS

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A simplified perturbational approach appropriate for systems of solitons governed by the perturbed integrable equations is described. Some applications of this method are reviewed. Among them there are soliton structure of the shock waves in dispersive media, double sine-Gordon equation, etc.

I. INTRODUCTION

We consider solitons described by evolution equations of the form

$$u_t = S[u] + \varepsilon R[u] \tag{1.1}$$

where $S[u]$ and $R[u]$ are some operators, ε is a small parameter, and at $\varepsilon = 0$ the system can be solved by the inverse scattering method (ISM). A general form of perturbation theory for such equations has been recently developed in a number of papers [1-9]. By means of this theory a general and very complete description of a single perturbed soliton and a number of applications was considered (see also a review article [10] and references therein).

However, in direct applications of this method to multi-soliton systems one confronts with significant technical difficulties arising from the necessity to use multi-soliton solutions. Fortunately, in many important cases the perturbational effects may be considered without going out of the one-soliton perturbation theory.

If, for instance, soliton velocities are not so close to each other, a time of passing of one soliton through another is small in comparison to the time during which the action of perturbation becomes to be significant. Therefore, in this case the perturbation mainly manifests itself when distance between solitons is large and one can use the single soliton perturbation theory (a more detailed and quantitative analysis of this case is given in [10]; see, also, below, sec. 3).

However, if the soliton velocities are rather close, the soliton interaction time is large and it may be comparable to the "perturbation time". In this case there is a significant interference between the external perturbation and soliton interaction due to their overlapping. Fortunately, it appears that in many of such cases the distances between solitons are large and this gives a possibility to describe a multi-soliton system as linear superposition of single solitons with slowly varying parameters which can be defined by means of perturbational methods.

The present paper is a review of the main results obtained by this approach in [11-15].

In sec. 2 we give some general equations used throughout the paper. In sec. 3, which is based on [11-12], we formulate an essence of our approach and consider multi-soliton system of the perturbed KdV equation (KdVE). Equations describing such system are applied then to the theory of oscillatory shocks. In sec. 4 the same approach is used for the perturbed non-linear Schrödinger equation (NLSE). Here we give a rather simple description of two unbounded and bounded solitons [13, 14]. In sec. 5 we consider two-soliton systems of the perturbed sine-Gordon equation (SGE) and, as a particular case, the double SGE (DSGE) [13, 15].

2. BASIC EQUATIONS

Here we give some general results following from the perturbation theory based on the ISM. Consider, first, the perturbed KdVE

$$u_t - 6uu_x + u_{xxx} = \varepsilon R[u] \tag{2.1}$$

where it is assumed that

$$u \longrightarrow 0\,, \quad R[u] \longrightarrow 0 \qquad (|x| \longrightarrow \infty) \tag{2.2}$$

Evolution of one perturbed soliton is described by the equations [2, 4, 7, 8]

$$\begin{aligned} &u(x,t) = u_s(z, \kappa(t)) + \delta u(x,t), \\ &u_s(z,\kappa) = -2\kappa^2 \operatorname{sech}^2 z\,, \quad z = \kappa[x - \xi(t)] \end{aligned} \tag{2.3}$$

where

$$\frac{d\kappa}{dt} = -\varepsilon A(\kappa)\,, \quad \frac{d\xi}{dt} = 4\kappa^2 - \varepsilon B \tag{2.4}$$

$$A(\kappa) = \frac{1}{4\kappa}\int_{-\infty}^{\infty} \operatorname{sech}^2 z\, R[u_s]\, dz \tag{2.5}$$

$$B(\kappa) = \frac{1}{4\kappa^3}\int_{-\infty}^{\infty} (z \operatorname{sech}^2 z + \tanh z + \tanh^2 z) R[u_s]\, dz \tag{2.6}$$

As for the $\delta u(x,t)$, describing the modification of soliton shape, we discuss here only its tail part [4, 7, 8] (in [8], instead of tail, the term "shelf" is used). It was found that $\delta u(x,t)$ transforms into almost flat tail behind the soliton at a few soliton lengthes κ^{-1}, and if we denote the height of the tail as $\delta u^{(-)}$, one has [7]

$$\delta u^{(-)} = \kappa^2 \varepsilon q(\kappa), \tag{2.7}$$

$$q(\kappa) = \frac{1}{4\kappa^5}\int_{-\infty}^{\infty} \tanh^2 z\, R[u_s]\, dz \tag{2.8}$$

Consider now the perturbed NLS and SG equations

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = \varepsilon R[u], \tag{2.9}$$

$$v_{xt} + \sin v = \varepsilon R[v] \tag{2.10}$$

which are related to the following eigenvalue problem

$$\hat{L}[u]f = \zeta f \tag{2.11}$$

$$\hat{L}[u] = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & u^* \\ -u & 0 \end{pmatrix} \tag{2.12}$$

and $v(x,t)$ in (2.10) is connected with $u(x,t)$ as

$$u(x,t) = \frac{1}{2}\frac{\partial v}{\partial x} \tag{2.13}$$

To the same eigenvalue problem the modified KdVE is related. However, it is not considered here, because for real $u(x,t)$ the results are very similar as those for KdVE, and for complex $u(x,t)$ the physical applications are rather obscure.

Single soliton solutions of the unperturbed NLSE and SGE are described by $u(x,t)$ having the form

$$u_s(z,t) = 2\nu\, \mathrm{sech}\, z\, \exp[i\varphi(z,t)] \tag{2.14}$$

$$z = 2\nu(x - \xi(t)), \quad \varphi(z,t) = \frac{\mu}{\nu}z + \delta(t) \tag{2.15}$$

and $\hat{L}[u_s]$ has only one eigenvalue of discrete spectrum in the upper half-plane

$$\zeta = \mu + i\nu \tag{2.16}$$

For SGE, $u_s(z,t)$ is real, i.e. in one have to assume

$$\mu = 0, \quad \delta = 0, \pi \tag{2.17}$$

Then from (2.14) and (2.13) one has

$$\vartheta(z) = 2\sigma \sin^{-1}(\tanh z) + \gamma\pi \tag{2.18}$$

$$\sigma = e^{i\delta} = \pm 1 , \quad \gamma = \pm 1 \tag{2.19}$$

If $\varepsilon \neq 0$, all soliton parameters, μ , ν , ξ and δ change with time according to [1-3, 7-9]

$$\frac{d\nu}{dt} = \varepsilon N[u] , \quad \frac{d\mu}{dt} = \varepsilon M[u] , \tag{2.20}$$

$$\begin{aligned} \frac{d\xi}{dt} &= -\frac{1}{2\nu} \operatorname{Im} h(\zeta) + \varepsilon\, \Xi[u] \\ \frac{d\delta}{dt} &= 2\mu \frac{d\xi}{dt} + \operatorname{Re} h(\zeta) + \varepsilon\, D[u] \end{aligned} \tag{2.21}$$

where

$$h(\zeta) = -2\zeta^2 \qquad \text{(NLSE)} \tag{2.22}$$

$$h(\zeta) = (2\zeta)^{-1} \qquad \text{(SG)} \tag{2.23}$$

$$N[u] = \frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} \frac{R[u_s(z)]}{\cosh z}\, e^{-i\varphi(z,t)} dz \tag{2.24}$$

$$M[u] = \frac{1}{2} \operatorname{Im} \int_{-\infty}^{\infty} \frac{\tanh z}{\cosh z} R[u_s(z)]\, e^{-i\varphi(z,t)} dz \tag{2.25}$$

$$\Xi[u] = \frac{1}{4\nu^2} \operatorname{Re} \int_{-\infty}^{\infty} \frac{z\, R[u_s(z)]}{\cosh z}\, e^{-i\varphi(z,t)} dz \tag{2.26}$$

$$D[u] = \frac{1}{2\nu} \operatorname{Im} \int_{-\infty}^{\infty} \frac{1 - z \tanh z}{\cosh z} R[u_s(z)]\, e^{-i\varphi(z,t)} dz \tag{2.27}$$

At $\varepsilon = 0$ one obtains from here the well known equations for the unperturbed solitons [16, 17].

For the perturbed SGE these equations are reduced to the two following

$$\frac{d\nu}{dt} = \frac{\sigma\varepsilon}{4}\int_{-\infty}^{\infty} \operatorname{sech} z\, R[v]\, dz \qquad (2.28)$$

$$\frac{d\xi}{dt} = \frac{1}{4\nu^2} + \frac{\sigma\varepsilon}{8\nu^2}\int_{-\infty}^{\infty} z \operatorname{sech} z\, R[v]\, dz \qquad (2.29)$$

(see also [18]).

For the perturbed equations one should add to $u(x,t)$ a variation of soliton shape $\delta u(x,t)$ which has been investigated in the above mentioned papers [1-10]. We do not consider it here because in all cases under discussion it gives no important for our problem physical effects.

3. MULTI-SOLITON SYSTEM OF THE PERTURBED KdVE. OSCILLATORY SHOCK WAVES

Consider, first, a two-soliton system governed by (2.1). From (2.5) we observe that a characteristic perturbation time scale t_p for a single soliton is defined by

$$t_p^{-1} \sim t_s^{-1}\, \frac{\varepsilon}{4\kappa^5}\int_{-\infty}^{\infty} \operatorname{sech}^2 z\, R[u_{s}(z)]\, dz$$

Here $t_s = \kappa^{-3}$ is the unperturbed soliton time scale. It is a time interval during which an unperturbed soliton passes a distance $\sim \kappa^{-1}$.

If one has two solitons with significantly different amplitudes ($\delta\kappa = \kappa_2 - \kappa_1 \sim \kappa_{1,2}$), the time of passing of the fastest soliton through the slowest one is of the order of t_s. As far as $t_s \ll t_p$ (t_s/t_p is the main parameter of the perturbation theory [4, 7]), soliton interaction during the overlapping process has no important interference with effects of external perturbation. However, such interference may be significant if $\delta\kappa \ll \kappa_{1,2}$. That is why here we consider only this case. The perturbation theory based on the ISM requires calculation of some matrix elements containing the two-soliton solutions. This is very difficult to realize, especially for small $\delta\kappa$, the most important case.

However, in this case there exists another, much simple, way based on the observation by Zabusky and Kruskal [19] and analysis by Lax [20]. They have shown that two-soliton solution of the unperturbed KdVE at $\delta\kappa \ll \kappa_{1,2}$ can be approximately presented as superposition of two single solitons with slowly varying amplitudes. These solitons, first, draw together, up to some minimal distance of the order of $\kappa_1^{-1} \log|\kappa/\kappa_1 - \kappa_2|$, and then slowly diverge. A simplified perturbational approach based on this picture was developed in [21] where some interesting results were obtained. We use here a different method [11] which gives possibility to take into account, in a simple way, effects have not been considered in [21], such as soliton tails and corrections to soliton velocities, which are important for applications considered below.

Consider a chain of solitons, centered at ξ_n ($n = 1, ..., N$), and assume $\xi_n > \xi_{n+1}$,

$$\tau_n = \xi_n - \xi_{n+1} \gg \kappa_n^{-1}, \quad p_n = \kappa_n - \kappa_{n+1} \ll \kappa_n \tag{3.1}$$

and let us look for a solution of eq. (2.1) of the form

$$u(x,t) = \sum_{(n)} \left[u_{sn} + \delta u_n^{(-)} \right] \tag{3.2}$$

where u_n are written in (2.3), $\delta u_n^{(-)}$ describe "tails", and soliton parameters $\kappa = \kappa_n(t)$ and $\xi = \xi_n(t)$ should satisfy eqs. (2.4)-(2.6) where $\varepsilon R[u_s]$ is replaced by

$$\varepsilon R[u_{sn}] + 6\frac{\partial}{\partial x}\left(u_{sn}\, u_{s(n+1)}\right) + 6\frac{\partial}{\partial x}\left(u_{s(n-1)}\, u_{sn}\right) + 6\varepsilon \sum_{i=1}^{n-1} \kappa_i^2 q(\kappa_i) \frac{\partial}{\partial x}(u_{sn})$$

The first term here describes the external perturbation (e.g., dissipation), second and third terms describe interaction of the n-th soliton with its neighbours, and the last term appears due to the influence of tails of $n-1$ first solitons on the n-th soliton. If, in addition,

$$|\kappa_n - \kappa_{n-1}|\tau_n \ll 1, \quad \kappa_n \tau_n \gg 1 \tag{3.3}$$

then equations describing evolution of soliton system take a form [11, 12]

$$d\kappa_m/dt = -64\kappa_{m-1}^4 \exp(-2\kappa_{m-1}\tau_{m-1}) + 64\kappa_m^4 \exp(-2\kappa_m\tau_m) - \varepsilon A_m, \tag{3.4}$$

$$d\kappa_N/dt = -64\kappa_{N-1}^4 \exp(-2\kappa_{N-1}\tau_{N-1}) - \varepsilon A_N, \tag{3.5}$$

$$d\xi_1/dt = 4\kappa_1^2 - 16\kappa_1^2 \exp(-2\kappa_1\tau_1) - \varepsilon B_1, \tag{3.6}$$

$$d\xi_m/dt = 4\kappa_m^2 + 112\kappa_{m-1}^2 \exp(-2\kappa_{m-1}\tau_{m-1}) - 16\kappa_m^2 \exp(-2\kappa_m\tau_m) - \sum_{i=1}^{m-1} 6\varepsilon q_i \kappa_i^2 - \varepsilon B_m \tag{3.7}$$

$$d\xi_N/dt = 4\kappa_N^2 + 112\kappa_{N-1}^2 \exp(-2\kappa_{N-1}\tau_{N-1}) - \\ -\sum_{i=1}^{N-1} 6\varepsilon q_i \kappa_i^2 - \varepsilon B_N \tag{3.8}$$

where $A(\kappa)$, $B(\kappa)$ and $q(\kappa)$ are defined in (2.5), (2.6), and (2.8), and $m = 2, 3, \ldots, N-1$.

At $\varepsilon = 0$ and $N = 2$ we come, in particular, to equations for two interacting solitons which leads to the same results as two-soliton solution of the unperturbed KdVE (details are given in [11]).

Consider now under which conditions the soliton system might be stationary. For that one should require

$$\frac{d\kappa_n}{dt} = 0, \qquad \frac{d\xi_n}{dt} = V \tag{3.9}$$

where V is a common velocity of the system. Applying (3.9) to $n = 1, 2, \ldots, N-1$ and using (3.4)-(3.8), we obtain

$$\tau_m = -\frac{1}{2\kappa_m}\log\left(\frac{\varepsilon\sum_{i=1}^{m} A_i}{64\,\kappa_m^4}\right) \tag{3.10}$$

$$p_m \approx \frac{3\varepsilon A_m}{16\,\kappa_m^3} - \frac{3\varepsilon q_m \kappa_m}{4} \tag{3.11}$$

However, one sees that

$$d\kappa_N/dt \neq 0 \;, \quad d\xi_N/dt \neq V$$

for any $\varepsilon R[u]$. Therefore there does not exist such external perturbation (with small ε) which could provide a stationary state of KdV solitons for $N \geqslant 2$. However, at large N , conditions (3.10) and (3.11) define a quasistationary state of soliton system with the time-life increasing with N . Indeed, let conditions (3.10) and (3.11) to be hold for $n = 1, 2, \ldots, N-1$, at some $t = t_0$. Then for $t > t_0$ the system would decay because they are not satisfied for $n = N$. However, the decay is the slower, the larger N .

The system might be stationary if, apart of external perturbation $\varepsilon R[u]$, there exists some other external force which compensates a tendency to the decay. Such situation is realized in a stationary shock wave, where a piston moving with constant velocity plays a role of the external force. Thus, one comes to conclusion that soliton system satisfying the conditions (3.9)-(3.11) may form a front part of a stationary shock described by (2.1). Let us elucidate some relations describing a structure of such shock [12].

From eq. $d\xi_1/dt = V$, with V as a shock velocity, one has for the front soliton in the shock

$$K_1 = \frac{1}{2} V^{1/2}\left(1 + \frac{\varepsilon A_1}{2V^2} + \frac{\varepsilon B_1}{2V}\right) \tag{3.12}$$

The amplitudes of subsequent solitons and distances between them are defined by eqs. (3.10) and (3.11). If the external perturbation such that

$$\varepsilon \int_{-\infty}^{\infty} \operatorname{sech}^2 z \, R[u_s(z)]\, dz > 0, \qquad \varepsilon q \le 0 \tag{3.13}$$

(this is assumed throughout this section), then all $\rho_m > 0$ (i.e. $K_m - K_{m+1} > 0$).

Now define the shock profile as $v(x,t) = -u(x,t)$. Then soliton peaks correspond to the maxima v_m, and

$$v_m - v_{m-1} \approx -\frac{3\varepsilon A_{m-1}}{4K_{m-1}^2} + 2\varepsilon q_{m-1} K_{m-1}^2 < 0 \tag{3.14}$$

Eqs. (3.10)-(3.14) give a complete description of the front part of the shock which may be considered as a sequence of solitons (Fig. 1).

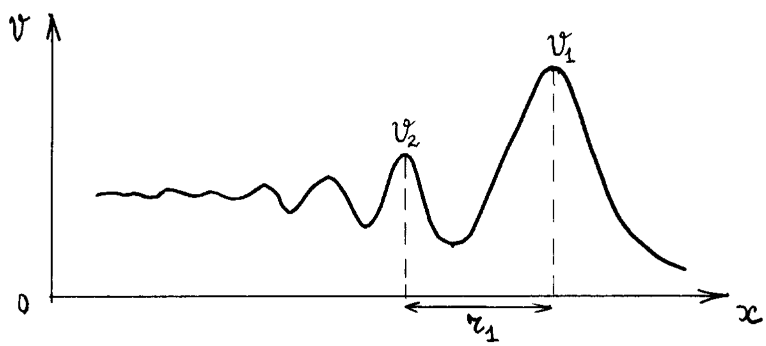

Figure 1
A Profile of the Oscillatory Shock Wave

Conditions of applicability of obtained relations are

$$2K_j \tau_j \gg 1, \quad |\varepsilon q_j| \ll 1 \tag{3.15}$$

which are necessary for applicability of our approach. They are violated in the back part of the shock where pulsation amplitudes are small. This part can be investigated from different point of view. By putting into (2.1)

$$u = -v(x - Vt)$$

we have

$$v_{xx}(x) + 3v^2(x) - Vv(x) = \varepsilon \int_{x+Vt}^{\infty} R[-v(x')]\,dx'$$

At $x \to -\infty$ one obtains

$$v(-\infty) \approx \frac{1}{3}V + \frac{\varepsilon}{V}\int_{-\infty}^{\infty} R[u(x)]\,dx \qquad (3.16)$$

In the back part of the shock, the difference $v(x) - v(-\infty)$ describes small oscillations which should damp at $x \to -\infty$. A condition of the damping, and a convergence of the integral in (3.16) put some restrictions on the external perturbation $\varepsilon R[u]$, necessary for the existence of the shock. A more detailed analysis of conditions of the existence of the shock is given in [11] (see also references therein).

As a simplest and important example, consider the KdV-B equation (i.e. $\varepsilon R = \varepsilon\, \partial^2/\partial x^2$, $\varepsilon > 0$). Then from the above written we have

$$\tau_n \approx -(2\kappa_n)^{-1} \log\left[(\kappa_1/\kappa_{n+1})^4 - 1/240\right] \qquad (3.17)$$

$$p_n \approx \varepsilon/2 \,, \qquad \kappa_{n+1} \approx \kappa_1 - n\varepsilon/2 \qquad (3.18)$$

$$v_{n+1} \approx v_1 - \frac{22\varepsilon n}{15}\left(\kappa_1 - \frac{\varepsilon n}{4}\right) \qquad (3.19)$$

From (3.18) and (3.19) one sees that a number of solitons in the shock is of the order of κ_1/ε. In particular, at $m = \kappa_1/2\varepsilon$ we have

$$\kappa_m/\kappa_1 \approx 0.75, \qquad \tau_m = 2.36\,\kappa_m^{-1} \,, \qquad v_m = 1.36\,\kappa_1^2 \qquad (3.20)$$

It is interesting to note that v_m in (3.20) is very close to the limit value $v(-\infty)$ from (3.16), which is

$$v(-\infty) = 1.33\,\kappa_1^2$$

4. A TWO-SOLITON SYSTEM OF THE NLSE [13, 14]

Following the method outlined in the previous section we look for the solution of eq. (2.9) in the form

$$u(x,t) = u_1(x,t) + u_2(x,t) \qquad (4.1)$$

where

$$u_n(x,t) = 2\nu_n \operatorname{sech}[2\nu_n(x-\xi_n)]\exp[i2\mu_n(x-\xi_n)+i\delta_n] \quad (4.2)$$

and it is assumed that $\nu_n = \nu_n(t)$ and $\mu_n = \mu_n(t)$, together with $\xi_n(t)$ and $\delta_n(t)$ are slow functions of time (n = 1, 2).

Suppose, first, that ε = 0. Then, to define the unknown functions, we have to solve eqs. (2.20)-(2.22) and (2.24)-(2.27), where, instead of $\varepsilon R[u]$, the following expression should be taken

$$\varepsilon_m R_m[u_n] = i\left(u_m^* u_n^2 + 2u_m u_n u_n^*\right) \quad (4.3)$$

m, n = 1, 2, $m \neq n$. (4.3) describes action of the m-th soliton on the n-th one due to their overlapping. After calculating the corresponding integrals, one has

$$\frac{d\mu_n}{dt} = (-1)^n 16\nu^3 e^{-2\nu\tau}\cos(2\mu\tau+\psi), \quad (4.4)$$

$$\frac{d\nu_n}{dt} = (-1)^n 16\nu^3 e^{-2\nu\tau}\sin(2\mu\tau+\psi), \quad (4.5)$$

$$\frac{d\xi_n}{dt} = 2\mu_n + 4\nu e^{-2\nu\tau}\sin(2\mu\tau+\psi), \quad (4.6)$$

$$\frac{d\delta_n}{dt} = 2(\nu_n^2+\mu_n^2) + 8\mu\nu e^{-2\nu\tau}\sin(2\mu\tau+\psi) + 24\nu^2 e^{-2\nu\tau}\cos(2\mu\tau+\psi) \quad (4.7)$$

with the notations

$$\tau = \xi_1 - \xi_2, \quad \psi = \delta_2 - \delta_1, \quad (4.8)$$

$$\nu = \frac{1}{2}(\nu_1+\nu_2), \quad \mu = \frac{1}{2}(\mu_1+\mu_2) \quad (4.9)$$

and it is assumed that $\tau > 0$, and, also,

$$|\mu_1 - \mu_2| \ll \mu, \quad |\nu_1-\nu_2| \ll \nu, \quad \nu\tau \gg 1, \quad (4.10)$$

$$|\nu_1 - \nu_2|\tau \ll 1. \quad (4.11)$$

As it will be seen from the solution obtained below, the conditions (4.10) are necessary in order to soliton parameters change slowly. Condition (4.11) is introduced to simplify computations. It is not an essential restriction, because terms describing soliton interaction may be neglected at the distances $\tau \gtrsim |\nu_1-\nu_2|^{-1}$.

Let us, also, define

$$p = \nu_2 - \nu_1 \,,\quad q = \mu_2 - \mu_1 \,,\quad Y = q + ip \,. \tag{4.12}$$

Then it is possible to show that eqs. (4.4)-(4.7) have the following constants of motion

$$\mu = \mathrm{const} \,,\qquad \nu = \mathrm{const} \tag{4.13}$$

$$Y^2 - 16\nu^2 e^{-2\nu\tau} \exp[i(2\mu\tau + \psi)] = \Lambda^2 , \tag{4.14}$$

where Λ is a complex constant. Using this relations one can deduce from (4.4)-(4.7) the equation

$$\frac{dY}{dt} - 2\nu\left(Y^2 - \Lambda^2\right) = 0 \tag{4.15}$$

which has the solution

$$Y = -\Lambda \tanh\left(2\nu\Lambda t - \alpha_1 - i\alpha_2\right) \tag{4.16}$$

where α_1 and α_2 are real constants. Introducing the notation

$$\Lambda = m + in \tag{4.17}$$

and calculating real and imaginary parts of (4.16), one has

$$q(t) = -\frac{m \sinh(4\nu m t - 2\alpha_1) - n \sin(4\nu n t - 2\alpha_2)}{\cosh(4\nu m t - 2\alpha_1) + \cos(4\nu n t - 2\alpha_2)} \,, \tag{4.18}$$

$$p(t) = -\frac{n \sinh(4\nu m t - 2\alpha_1) + m \sin(4\nu n t - 2\alpha_2)}{\cosh(4\nu m t - 2\alpha_1) + \cos(4\nu n t - 2\alpha_2)} \tag{4.19}$$

From eqs. (4.4)-(4.7) one can deduce, also,

$$\frac{d\tau}{dt} = -2q \,,\qquad \frac{d\psi}{dt} = 4(\nu p + \mu q) \tag{4.20}$$

Integration gives

$$\tau(t) - \tau(0) = \frac{1}{2\nu} \log \frac{\cosh(4\nu m t - 2\alpha_1) + \cos(4\nu n t - 2\alpha_2)}{\cosh 2\alpha_1 + \cos 2\alpha_2} \tag{4.21}$$

$$\psi(t)-\psi(0) = -2\tan^{-1}\left[\tanh(2\nu m t-\alpha_1)\tan(2\nu n t-\alpha_2)\right] + $$
$$+2\tan^{-1}(\tanh\alpha_1\tan\alpha_2) - 2\mu(\tau-\tau(0)). \tag{4.22}$$

Before an examination of these equations we point out that, according to direct calculations performed in [14] under conditions (4.10) and (4.11), the eigenvalues of $L[u]$, corresponding to (4.1) and (4.2), have the form

$$\zeta_{1,2} = \mu \mp \frac{m}{2} + i\left(\nu \mp \frac{n}{2}\right) \tag{4.23}$$

We stress that this expression has been obtained without use the equations of motion and, therefore, it is valid for $\varepsilon \neq 0$, as well as for $\varepsilon = 0$. However, in the last case it depends only on constans of motion, in accordance with ISM. Consider now three cases.

(i) $m \neq 0$. Without loss of generality one can assume that $\alpha_1 = 0$. Then from (4.14) it follows

$$\tau(0) = \frac{1}{2\nu}\log\left(\frac{16\nu^2}{n^2+m^2}\cos^2\alpha_2\right) \tag{4.24}$$

$$\psi(0) = 2\tan^{-1}(n/m) - 2\mu\tau(0) \tag{4.25}$$

We see, also, that for $t \to \pm\infty$

$$q \to \pm m, \quad p \to \pm n, \quad \tau \to \infty \tag{4.26}$$

Thus, in this case solitons are brought together from infinity up to some minimal distance $\tau_{min} \sim \tau(0)$ and after they diverge, i.e. the case under consideration corresponds to collision of two unbounded solitons, and for $t = \pm\infty$ one has

$$\begin{aligned} \nu_1(\mp\infty) &= \nu \mp n/2, \quad \mu_1(\mp\infty) = \mu \mp m/2 \\ \nu_2(\mp\infty) &= \nu \pm n/2, \quad \mu_2(\mp\infty) = \mu \pm m/2 \end{aligned} \tag{4.27}$$

These relations, together with (4.23), are in complete agreement with the exact two-soliton solution of NLSE. Defining asymptotic values at $t \to \pm\infty$

$$\tau(t) \to \tau_\pm(t), \quad \psi(t) \to \psi_\pm(t),$$

one has

$$\tau_+(t) + \tau_-(t) = \frac{1}{\nu}\log\frac{4\nu^2}{n^2+m^2} \tag{4.28}$$

$$\psi_+(t) + \psi_-(t) = 4\tan^{-1}(n/m) - 2\frac{\mu}{\nu}\log\frac{4\nu^2}{n^2+m^2} \tag{4.29}$$

These relations define position and phase shifts of solitons at $t \to \pm\infty$ caused by their interaction. They coincide with those following from the ISM [16] if

$$m, n \ll \nu, \quad |\alpha_2| < \frac{\pi}{2} \tag{4.30}$$

It is easy to see that conditions (4.30) provide fulfilment of (4.10) and (4.11). Therefore we conclude that, within the scope of our approximations, our results are in full agreement with the ISM. Plots of relative soliton velocity at $m \neq 0$ are shown in the Fig. 2. One can see that soliton approach and divergence have an oscillatory character.

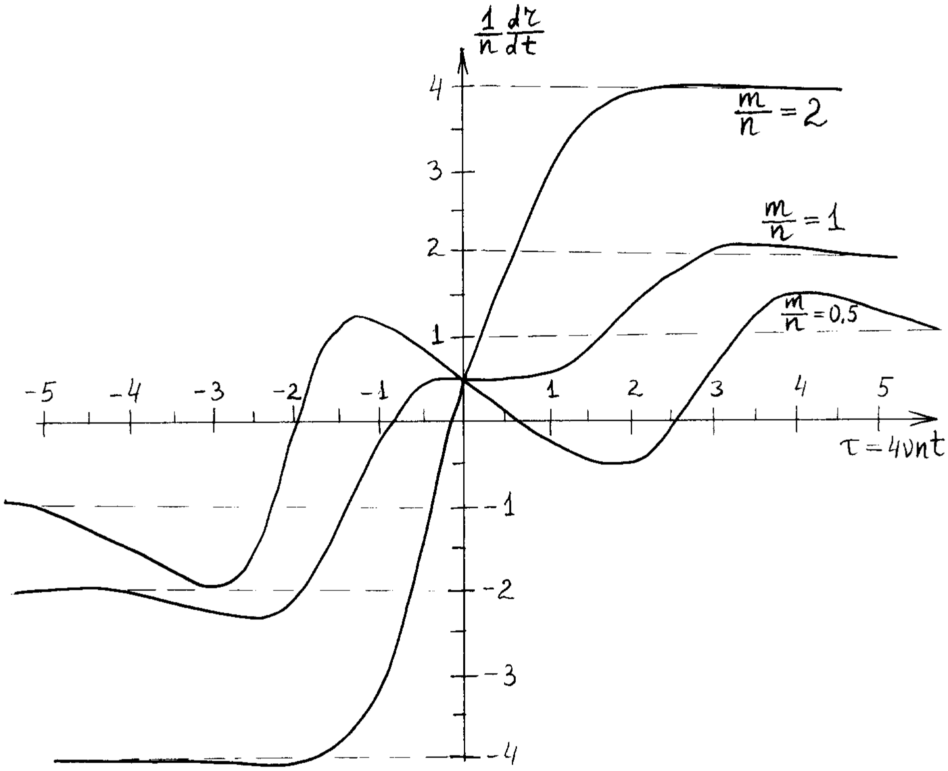

Figure 2
Plots of $(1/n)\dot{\tau}$ for $m \neq 0$ ($\alpha_1 = 0$, $\alpha_2 = 0.25$)

(ii) $m = 0$, $n \neq 0$. In this case, without loss of generality, one can put $\alpha_2 = 0$. Then eqs. (4.18), (4.19), (4.21) and (4.22) give

$$q(t) = n\frac{\sin(4\nu n t)}{\cosh 2\alpha_1 + \cos(4\nu n t)} \tag{4.31}$$

$$p(t) = n\frac{\sinh 2\alpha_1}{\cosh 2\alpha_1 + \cos(4\nu n t)} \tag{4.32}$$

$$\tau(t) = \frac{1}{2\nu}\log\left\{\frac{8\nu^2}{n^2}\left[\cosh 2\alpha_1 + \cos(4\nu n t)\right]\right\} \tag{4.33}$$

$$\Psi(t) + 2\mu\tau(t) = 2\tan^{-1}\left[\tan(2\nu n t)\tanh\alpha_1\right] \tag{4.34}$$

i.e. relative soliton velocities, their phases and distances between them, oscillate with the period

$$T = \frac{\pi}{2\nu|n|} \tag{4.35}$$

and

$$\tau_{max} = \frac{1}{\nu}\log\left|\frac{4\nu\cosh\alpha_1}{n}\right|, \quad \tau_{min} = \frac{1}{\nu}\log\left|\frac{4\nu\sinh\alpha_1}{n}\right|. \tag{4.36}$$

Therefore at $m = 0$, i.e. $\mathrm{Re}\,\zeta_1 = \mathrm{Re}\,\zeta_2$, we obtain a bound soliton system, again in accordance with the ISM. However, the case considered here is different from breathers which are usually extracted from the general two-soliton solution of the NLSE, because we have not a one "breathing" pulse but two oscillating solitons.

(iii) $m = 0$, $n = 0$ (i.e. $\Lambda = 0$). This case corresponds to degenerate eigenvalue: $\zeta_1 = \zeta_2$. To analyse it by our approach we solve eq. (4.15) at $\Lambda = 0$ to obtain

$$Y = -\left[2\nu(t - i\alpha)\right]^{-1}$$

where α is an integration constant which may be considered as real, but $\alpha \neq 0$. Calculating p and q we have

$$q(t) = -t\left[2\nu(t^2+\alpha^2)\right]^{-1}, \quad p(t) = -\alpha\left[2\nu(t^2+\alpha^2)\right]^{-1} \tag{4.37}$$

From that it follows

$$\tau(t) = \frac{1}{\nu} \log\left[8\nu^2 (t^2+\alpha^2)^{1/2}\right] \tag{4.38}$$

$$\tau_{min} = \frac{1}{\nu} \log(8\nu^2\alpha) \tag{4.39}$$

$$\psi(t) = -2\tan^{-1}(t/\alpha) - (\mu/\nu)\log\left[64\nu^4(t^2+\alpha^2)\right] \tag{4.40}$$

Therefore, in this case solitons move monotonically from $\tau = \infty$ to τ_{min} and then diverge, and the distance between them varies as $\log|t|$

In a similar way one can consider a two-soliton system under external perturbation. For that one have to add to (4.3) the term $\varepsilon R[u]$ describing an external perturbation. This results in adding to the r.h. sides of eqs. (4.4)-(4.7) additional terms: $\varepsilon M[u_n]$, $\varepsilon N[u_n]$, $\varepsilon \Xi[u_n]$, and $\varepsilon D[u_n]$ which are defined by (2.24)-(2.27). The equations obtained in such a way are investigated in [13, 14] and will be published elsewhere. Here we only mention the following consequences. Quantities μ, ν, m, and n are no more constants of motion. Therefore, the eigenvalues of $\hat{L}$, which are still expressed by (4.23), also change in time. If an external perturbation is such that $dm/dt \neq 0$, the bound soliton state may be destroyed.

If perturbation is so that the period (4.35) is much less than perturbation time t_p, one can average over the oscillation period of bound system, and if

$$\overline{dm}/dt = 0 \tag{4.41}$$

then the bound state may be considered as conserving untill the averaging procedure is justified. In many cases, however (examples are given in [13, 14]), $\overline{dn}/dt < 0$, along with (4.41). Due to that $\overline{n}$ decreases and, at appropriately small $\overline{n}$, the period (4.35) becomes to be comparable to t_p, and, so, the adiabaticity condition breaks, and averaging has no more meaning. In general, this is the end of the existence of bound soliton system. Phenomena of such type have been observed in numerical simulations [22].

5. TWO-SOLITON SYSTEM OF THE PERTURBED SGE

As in previous cases we look for the solutions of the perturbed SGE in the form

$$\vartheta(x,t) = \vartheta_1(x,t) + \vartheta_2(x,t) \tag{5.1}$$

where

$$\vartheta_n(x,t) = 2\sigma_n \sin^{-1}\tanh z_n + \gamma_n \pi \tag{5.2}$$

$\sigma_n = \pm 1$, $\gamma_n = \pm 1$, $z_n = 2\nu_n(t)[x - \xi(t)]$ ($n = 1, 2$), and $\nu_n(t)$, $\xi(t)$ are slow functions of t. At $\sigma_1\sigma_2 = 1$ we have kink-kink system, and at $\sigma_1\sigma_2 = -1$ it is kink-antikink. We can also consider them as two-soliton systems, because to each $\vartheta(x,t)$ corresponds quantity $u(x,t)$, defined by (2.13), which has a soliton form. It is important, however, that often perturbation terms depend on ϑ and, therefore, additional constants $\gamma_n \pi$ are significant.

To obtain equations for $\nu_n(t)$ and $\xi_n(t)$ we use (2.28) and (2.29) with the following change

$$\varepsilon R[u] \longrightarrow \varepsilon_{12} R_{12} + \varepsilon R[u] \tag{5.3}$$

Here $\varepsilon_{12} R_{12}$ describes soliton interaction due to overlapping and $\varepsilon R[u]$ in the r.h.s. of (5.3) corresponds to external perturbation. It is easy to find

$$\begin{aligned}\varepsilon_{12} R_{12} = &-4\sigma_1 \operatorname{sech} z_1 \tanh z_1 \operatorname{sech}^2 z_2 - \\ &-4\sigma_2 \operatorname{sech} z_2 \tanh z_2 \operatorname{sech}^2 z_1\end{aligned} \tag{5.4}$$

As before, we introduce the notations

$$\tau = \xi_1 - \xi_2, \quad \nu = \frac{1}{2}(\nu_1 + \nu_2), \quad p = \nu_2 - \nu_1 \tag{5.5}$$

and assume

$$|p| << \nu, \quad \nu\tau >> 1, \quad |p|\tau << 1 \tag{5.6}$$

Then the basic equations take the form

$$\frac{d\nu_n}{dt} = 4\sigma_1\sigma_2(-1)^n e^{-2\nu\tau} + \frac{\varepsilon}{4}\sigma_n \int_{-\infty}^{\infty} \operatorname{sech} z_n R[\vartheta]\, dz_n, \tag{5.7}$$

$$\frac{d\xi_n}{dt} = \frac{1}{4\nu_n^2} + \frac{\sigma_1\sigma_2}{\nu^2} e^{-2\nu\tau} + \frac{\varepsilon}{8\nu_n^2}\sigma_n \int_{-\infty}^{\infty} z_n \operatorname{sech} z_n R[\vartheta]\, dz_n \tag{5.8}$$

The eigenvalues of $\hat{L}[u]$, corresponding to (5.1) and (5.2), are [15]

$$\zeta_{1,2} = i\nu \mp \frac{i}{2}\sqrt{p^2 + 16\sigma_{12}\nu^2 \exp(-2\nu\tau)}, \tag{5.9}$$

where

$$\sigma_{12} = \sigma_1\sigma_2 \tag{5.10}$$

From (5.7) and (5.8) it follows that at $\varepsilon = 0$

$$\nu = (\nu_1 + \nu_2)/2 = \mathrm{const} \tag{5.11}$$

$$d\tau/dt = p/2\nu^3 \tag{5.12}$$

$$dp/dt = 8\sigma_{12} e^{-2\nu\tau} \tag{5.13}$$

These equations have an integral of motion

$$p^2 + 16\sigma_{12}\nu^2 e^{-2\nu\tau} = \mathrm{const} \tag{5.14}$$

Exactly this quantity appears under the square root in (5.9) and this ensures a constancy of eigenvalues, in agreement with the ISM. From (5.14) one sees that solitons are repulsed (attracted) at $\sigma_{12} = 1$ (-1). A character of soliton motion is clear from the Fig. 3.

At $\sigma_{12} = -1$, const < 0, soliton distance oscillates with amplitude

$$\tau_0 = -\frac{1}{2\nu} \log \frac{|\mathrm{const}|}{16\nu^2} \tag{5.15}$$

However, in this case the solution (5.1) is valid only in the vicinity of the turning point (and under the condition $\nu\tau_0 \gg 1$), because our approximation breaks at $\nu\tau \lesssim 1$. From (5.9) one has

$$\zeta_{1,2} = i\nu \pm 2\nu e^{-\nu\tau_0} \qquad (\sigma_{12} = -1) \tag{5.16}$$

$$\zeta_{1,2} = i\nu(1 \mp 2e^{-\nu\tau_0}) \qquad (\sigma_{12} = 1) \tag{5.17}$$

where $p(\tau_0) = 0$ and τ_0 is maximum (minimum) distance between solitons at $\sigma_{12} = -1$ ($\sigma_{12} = 1$).

From (5.12)-(5.14) one has ($\sigma_{12} = 1$)

$$\tau(t) = \frac{1}{\nu} \log\left\{\frac{4\nu}{|\zeta_1 - \zeta_2|} \cosh\left(\frac{|\zeta_1 - \zeta_2|}{2\nu^2} t\right)\right\} \tag{5.18}$$

$$\tau_0 = \frac{1}{\nu} \log\left|\frac{4\nu}{\zeta_1 - \zeta_2}\right| \tag{5.19}$$

$$p(t) = |\zeta_1 - \zeta_2| \tanh\left(\frac{|\zeta_1 - \zeta_2|}{2\nu^2} t\right) \tag{5.20}$$

Conditions $\nu\tau_0 \gg 1$ and $p/\nu \ll 1$ are realized at

$$|\zeta_1 - \zeta_2|/\nu \ll 1 \tag{5.21}$$

If one defines the asymptotics $\tau_\pm(t)$ as

$$\tau(t) \longrightarrow \tau_\pm(t) \qquad (t \to \pm\infty)$$

then

$$\nu[\tau_+(t) + \tau_-(t)] = 2\log\frac{2\nu}{|\zeta_1 - \zeta_2|} \tag{5.22}$$

It can be shown that this equation is completely equivalent to the relation between shifts of soliton positions due to their collision, which follows from the ISM.

Returning to the perturbed SGE we consider the DSGE

$$\vartheta_{xt} + \sin\vartheta = -\frac{\lambda}{2}\sin\frac{\vartheta}{2} \tag{5.23}$$

which have been already investigated in a number of papers (e.g., [23-26] and references therein). Eq. (5.23) is usually treated by perturbation methods with λ as a small parameter. Taking in (5.7) and (5.8)

$$\varepsilon R[\vartheta] = -\frac{\lambda}{2}\sin\frac{\vartheta}{2}\ , \tag{5.24}$$

one has (n = 1, 2)

$$\frac{d\nu_n}{dt} = (-1)^n\left[4\sigma_{12}e^{-2\nu\tau} - \frac{\lambda}{4}\sigma_{12}\gamma_1\gamma_2\right] \tag{5.25}$$

$$\frac{d\xi_n}{dt} = \frac{1}{4\nu_n^2} + \frac{1}{\nu^2}\sigma_{12}e^{-2\nu\tau} \tag{5.26}$$

From (5.25) and (5.26) it follows that eqs. (5.11) and (5.12) are still valid (as for λ = 0). Instead of (5.13) one has

$$\frac{dp}{dt} = 8\sigma_{12}e^{-2\nu\tau} - \frac{\lambda}{2}\sigma_{12}\gamma_1\gamma_2 \tag{5.27}$$

From (5.27) and (5.12) an "energy" integral follows

$$\frac{1}{2}p^2 + U(\tau) = E \tag{5.28}$$

with "potential energy"

$$U(\tau) = \nu^2\sigma_{12}\left(8e^{-2\nu\tau} + \lambda\gamma_1\gamma_2\nu\tau\right) \tag{5.29}$$

We see that in the presence of perturbation $U(\tau)$ has an extremum in the point τ_m

$$\tau_m = \frac{1}{2\nu} \log \frac{16}{\lambda \gamma_1 \gamma_2} \tag{5.30}$$

$$U(\tau_m) = \frac{\sigma_{12}\lambda}{2} \nu^2 \left(1 + \log \frac{16}{\lambda}\right) \tag{5.31}$$

(Here and below is assumed $\gamma_1\gamma_2 = 1$, $\lambda > 0$). A plot of $U(\tau)$ is shown in Fig. 3. One sees that, at $\sigma_{12} = 1$, perturbation leads to bound states of solitons which are impossible at $\lambda = 0$. At $\sigma_{12} = -1$, the perturbation only results in some modification of breathers and in diminishing of a region of their existence.

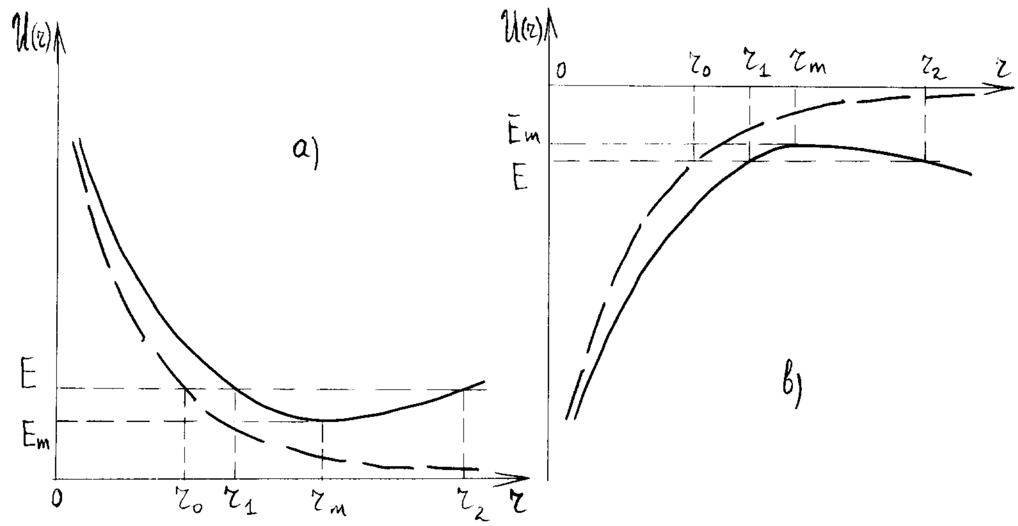

Figure 3
Potential Energy of Solitons $U(\tau) = \nu^2\sigma_{12}(8e^{-2\nu\tau} + \lambda\nu\tau)$
a) $\sigma_{12} = 1$, b) $\sigma_{12} = -1$. Dotted Lines Correspond to $U(\tau)$ at $\lambda = 0$

A difference between these cases is also in eigenvalues of (2.11). (It is naturally, that at $\lambda \neq 0$, the eigenvalues depend on $\tau = \tau(t)$). In particular, at the turning points $\tau = \bar{\tau}$, where. $p(\bar{\tau}) = 0$, $U(\bar{\tau}) = E$, ($\bar{\tau} = \tau_1, \tau_2$),

$$\zeta_{1,2} = i\nu \mp 2i\nu\sqrt{\sigma_{12}}\, e^{-\nu\bar{\tau}} \tag{5.32}$$

We see that at $\sigma_{12} = 1$ eigenvalues are purely imaginary in the all region between the turning points $\tau_1 \leq \tau \leq \tau_2$. At $\sigma_{12} = -1$ one should distinguish two regions. For $\tau \leq \tau_1$ the eigenvalues are complex with opposite signs of the real part. At some $\tau > \tau_m$ the real part of $\zeta_{1,2}$ vanishes, i.e. the system decays into two diverging solitons (see also the end of the paper).

In the equilibrium points, $\tau = \tau_m$, the system is described by stationary solutions of eq. (5.23) which have the form

$$\vartheta_m = 2\sigma_1 \sin^{-1} \tanh\left\{2\nu\left[x - \left(\xi + \frac{\tau_m}{2}\right)\right]\right\} + $$
$$+ 2\sigma_2 \sin^{-1} \tanh\left\{2\nu\left[x - \left(\xi - \frac{\tau_m}{2}\right)\right]\right\} + \pi(\gamma_1 + \gamma_2), \quad (5.33)$$

where, according to (5.26) and (5.30)

$$\dot{\xi} = \frac{1}{2}\left(\dot{\xi}_1 + \dot{\xi}_2\right) = \frac{1}{4\nu^2} + \frac{\sigma_{12}}{\nu^2} e^{-2\nu\tau_m} = \frac{1}{4\nu^2}\left(1 + \frac{\lambda}{4}\sigma_{12}\right). \quad (5.34)$$

(5.33) may be considered as soliton-like solution of the eq.(5.23) satisfying the boundary conditions

$$\frac{\partial \vartheta_m}{\partial x} \longrightarrow 0 \qquad (|x| \to \infty),$$
$$\vartheta_m(\infty) - \vartheta_m(-\infty) = 4\pi\sigma_1 \qquad (\sigma_{12} = 1),$$
$$\vartheta_m(\infty) - \vartheta_m(-\infty) = 0 \qquad (\sigma_{12} = -1).$$

However, at $\sigma_{12} = 1$ the soliton (5.33) is stable and at $\sigma_{12} = -1$, it is unstable.

It is interesting to note that if one looks for an exact stationary solution of eq. (5.23) in the form (5.33), with $\dot{\xi}$ = const, τ_m = const, then for τ_m one obtains

$$\cosh(2\nu\tau_m) = \frac{\delta}{\lambda\gamma_1\gamma_2} + \sigma_{12} \quad (5.35)$$

In the first order of λ/δ , its solution gives (5.30). As for $\dot{\xi}$ one obtains for it exactly (5.34). It is important that even for λ = 1, it is a good agreement of our approximate expressions with the exact ones.

The case σ_{12} = 1 corresponds to conditions considered in [25] and σ_{12} = -1 to [23, 24, 26], where similar results have been obtained by different approach.

And, finally, we present solutions of DSGE which are close to stationary ones (their derivations are given in [15]).

At σ_{12} = 1 (kink-kink), the solution corresponding to harmonic oscillations of solitons near the equilibrium point $\tau = \tau_m$ can be written as

$$\vartheta \approx 4\sigma_1 \tan^{-1}\left\{\frac{\sqrt{\lambda}}{2} e^{-\delta(x,t)} \sinh[2\nu(x - \xi)]\right\} + 2\pi\gamma_1, \quad (5.36)$$

where

$$\delta(x,t) = \sqrt{2\lambda}\,\nu^2 a \cos\Omega t\,[x - \xi(t)] + \nu a \sin\Omega t \tag{5.37}$$

$$\xi(t) \approx \frac{1}{4\nu^2}\left(1 + \frac{\lambda}{4} + \frac{3\lambda\nu^2 a^2}{4}\right)t - \left(\log\frac{16}{\lambda} - 1\right)\sqrt{\frac{\lambda}{2}}\,\frac{a}{4}\cos\Omega t + \xi_0 \tag{5.38}$$

Here a and Ω are amplitude and frequency of soliton oscillations

$$\tau - \tau_m = a \sin\Omega t\,, \qquad \Omega = \frac{1}{\nu}\sqrt{\frac{\lambda}{2}}\,, \tag{5.39}$$

and it is supposed $a\nu \ll 1$. At $a = 0$ formula (5.36) reduces to the stationary solution discussed above.

At $\sigma_{12} = -1$ (kink-antikink), we have a solution describing a stable breather

$$\vartheta = -4\sigma_1 \tan^{-1}\left[\frac{\left(1 - \frac{\lambda\nu^2}{4\eta^2}\right)G(x,t) + \frac{\lambda\nu^2}{4\eta^2}}{F(x,t)}\right] + 2\pi\gamma_1\,, \tag{5.40}$$

where

$$\begin{aligned} G(x,t) &= \cos[2\eta(x + Vt)] \\ F(x,t) &= \frac{\mu}{\nu}\cosh[2\nu(x - Vt)] \\ V &= \frac{1}{4(\nu^2 + \eta^2)} \end{aligned} \tag{5.41}$$

This solution corresponds to solitons oscillating in the potential well (Fig. 3b), and parameter η is expressed through the turning point τ_1 by

$$\eta = \frac{\nu\sqrt{\lambda}}{2}\exp[\nu(\tau_m - \tau_1)] = 2\nu e^{-2\nu\tau_1} \tag{5.42}$$

In derivation of (5.40) it was assumed $\lambda \ll 1$. From (5.42) we see that at $\tau_1 = \tau_m$

$$\eta_{min} = \frac{\nu\sqrt{\lambda}}{2} \tag{5.43}$$

In this case $\lambda\nu^2/4\eta^2 = 1$ and (5.40) takes a form

$$\vartheta = -4\sigma_1 \tan^{-1}\left\{\frac{2}{\sqrt{\lambda}}\,\mathrm{sech}[2\nu(x - \xi)]\right\} + 2\pi\gamma_1 \tag{5.44}$$

It is easy to check that (5.44) is exactly equivalent to the stationary solution (5.33) at $\sigma_{12} = -1$, as it should be.

In the opposite case

$$\frac{\lambda\nu^2}{4\eta^2} << 1, \tag{5.45}$$

the solution (5.40) is very close to the unperturbed breather of SGE.

Consider now solutions corresponding to the case when distance between the kink and antikink $\tau > \tau_2$. Then from (5.9) one has

$$\zeta_{1,2}(\tau) = i\nu \pm 2\nu\left[e^{-2\nu\tau_2} - \frac{\lambda}{8}\nu(\tau-\tau_2)\right]^{1/2} \tag{5.46}$$

If one introduces "a critical distance" τ_{cr}

$$\tau_{cr} = \tau_2 + \frac{8}{\lambda\nu} e^{-2\nu\tau_2}, \tag{5.47}$$

then

$$\mathrm{Re}\,\zeta_{1,2} \neq 0 \qquad (\tau_2 \leq \tau < \tau_{cr})$$

$$\mathrm{Re}\,\zeta_{1,2} = 0 \qquad (\tau \geqslant \tau_{cr})$$

In the point $\tau = \tau_{cr}$ the system decays into two independent diverging solitons. At $\tau \gg \tau_{cr}$, evolution of them may be described by eqs. (5.25) and (5.26) where terms with $exp(-2\nu\tau)$ are neglected, i.e.

$$\frac{d\nu_n}{dt} \approx (-1)^n \frac{\lambda}{4}, \quad \frac{d\xi_n}{dt} \approx \frac{1}{4\nu_n^2}. \tag{5.48}$$

These results are in good agreement with the numerical analysis given in [24, 26].

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