Summarized Notes on Bäcklund transf for KdV

A
32 2/5 A

KdV : $u_t - 6uu_x + u_{xxx} = 0$ ①

• gral Miura transf : $u = \lambda + v^2 + v_x$ ② ($\lambda = 0$ standard M.T.)

KdV $\Longrightarrow$ mKdV : $v_t - 6(v^2+\lambda)v_x + v_{xxx} = 0$ ③

• v sol $\Rightarrow$ $-v$ sol $\Longrightarrow$ $\begin{cases} u_+ = \lambda + v^2 \oplus v_x \\ u_- = \lambda + v^2 \ominus v_x \end{cases}$

• Another transf: $u_\pm = (w_\mp)_x$ ③.⑤

$\Longrightarrow$

$$\boxed{\begin{aligned} &(w_+ + w_-)_x = 2\lambda + \tfrac{1}{2}(w_x - w_-)^2 \quad ④ \\ &(w_+ - w_-)_t = 3(w_{+x}^2 - w_{-x}^2) + (w_+ - w_-)_{xxx} = 0 \quad ⑤ \end{aligned}}$$

• mKdV + ③.⑤ ④ $\Longrightarrow$

Auto-Bäcklund of mKdV

<u>Find sols of KdV</u>: start with trivial $w_- = 0$ ($\rightarrow u = 0$)

∴ $w_- \xrightarrow[⑤]{④} w_+$

• $\xrightarrow{④}$ $w_{+x} = 2\lambda + \frac{1}{2}(w_+)^2$

const. of $\int$

$\xrightarrow{\text{sep var in } x}$ $\boxed{w_+ = -2k \tanh[kx + f(t)]}$

• $\xrightarrow{⑤}$ $\boxed{f(t) = -4k^3 t - kx_0 \quad (k \equiv \sqrt{-\lambda})}$

$\xrightarrow[\text{[} \tanh' = \operatorname{sech}^2 \text{]}]{③.⑤}$ $\xrightarrow{\int dx}$ $u_+ = -2k^2 \operatorname{sech}^2[k(x - 4k^2 t - x_0)]$

$2k^2 = c/2 \Rightarrow u_+ = -\dfrac{c}{2}\operatorname{sech}^2\left[\dfrac{\sqrt{c}}{2}(x - ct - x_0)\right]$ ✓

sech² sol !!

<u>Repeating process to obtain more solutions</u> $\longrightarrow$ Messy !!

<u>More elegant</u> (and easier)

- Start with w_0 and def. $w_0 \xrightarrow{\lambda_1} w_1$, $w_0 \xrightarrow{\lambda_2} w_2$

$$\stackrel{(4)}{\Longrightarrow} \begin{cases} (w_1+w_0)_x = 2\lambda_1 + \frac{1}{2}(w_1-w_0)^2 & (6a) \\ (w_2+w_0)_x = 2\lambda_2 + \frac{1}{2}(w_2-w_0)^2 & (6b) \end{cases} \quad (6)$$

- Now do same process starting with $w_1 \xrightarrow{\lambda_2} w_{12}$, $w_2 \xrightarrow{\lambda_1} w_{21}$

$$\stackrel{(4)}{\Longrightarrow} \begin{cases} (w_{12}+w_1)_x = 2\lambda_2 + \frac{1}{2}(w_{12}-w_1)^2 & (7a) \\ (w_{21}+w_2)_x = 2\lambda_1 + \frac{1}{2}(w_{21}-w_2)^2 & (7b) \end{cases} \quad (7)$$

- From Bianchi's theo. of permutability $\Longrightarrow$ $w_{12} = w_{21}$!!!

i.e.

$$\begin{tikzcd} & w_1 \arrow[dr, "\lambda_2"] & \\ w_0 \arrow[ur, "\lambda_1"] \arrow[dr, "\lambda_2"'] & & w_{12} = w_{21} \\ & w_2 \arrow[ur, "\lambda_1"'] & \end{tikzcd}$$

- Do $[(7a)-(7b)] - [(6a)-(6b)] \Longrightarrow$ $\boxed{w_{12} = w_0 - \dfrac{4(\lambda_1-\lambda_2)}{w_1-w_2}}$ (8)

to solve $w_{12} = w_{12}(w_0, w_1, w_2)$

So now
- start with trivial w_0
- construct $w_0 \xrightarrow[(6a)]{\lambda_1} w_1$ & $w_0 \xrightarrow[(6b)]{\lambda_2} w_2$ — for any desired λ_1 & λ_2!
- $\stackrel{(8)}{\Longrightarrow}$ w_{12} = NEW SOL!!!

<u>Ex</u>: Do this precisely for the velocities in our collision example:

- $v_1 = 4$, $v_2 = 16$ $\xrightarrow{k=\sqrt{c}/2}$ $k^2 = \frac{c}{4}$; $k_1 = 1$, $k_2 = 2$ $\xrightarrow{-\lambda = k^2}$ $-\lambda = c/4$; $\lambda_1 = -1$, $\lambda_2 = -4$
- $w_0 = 0$ $\xrightarrow{\lambda_1=-1}$ $w_1 = \ldots\ldots \tanh$; $\xrightarrow{\lambda_2=-4}$ $w_2 = \coth$
- $\stackrel{(8)}{\Longrightarrow}$ $w_{12} = 0 - \dfrac{4(-1+4)}{(\ldots\tanh\ldots \cancel{+} \coth)} = \ldots\ldots$] exactly the sol. that we wrote before for the collision of 2 solitons!