

Summarized notes on Bäcklund transf for KdV

(A)
32 2/5 A

KdV: $u_t - 6uu_x + u_{xxx} = 0$ (1)

• gal Miura transf: $u = \lambda + v^2 + v_x$ (2) ($\lambda = 0$ standard M.T.)

$\text{KdV} \Rightarrow \text{mKdV}$: $U_t - 6(v^2 + \lambda)U_x + U_{xxx} = 0$ (3)

• v sol $\Rightarrow -v$ sol $\Rightarrow \begin{cases} u_+ = \lambda + v^2 + v_x \\ u_- = \lambda + v^2 - v_x \end{cases}$

• Another transf: $u_{\pm} = (w_{\pm})_x$ (3.5)

$\Rightarrow$

$$(u_+ + w_-)_x = 2\lambda + \frac{1}{2}(w_+ - w_-)^2$$
 (4)

• mKdV $\Rightarrow$

$$(w_+ - w_-)_t = 3(w_+^2 - w_-^2) + (w_+ - w_-)_{xxx} = 0$$
 (5)

Auto-Bäcklund of mKdV

Find sols of KdV: start with trivial $w_- = 0 \Rightarrow u = 0$

$\therefore w_- \xrightarrow{(4)} w_+$

• $\xrightarrow{(4)}$ $w_{+x} = 2\lambda + \frac{1}{2}(w_+)^2$ const. of $\int$

$\xrightarrow{\text{sep. var.}} \int \frac{dw_+}{w_+^2} = \int (2\lambda + \frac{1}{2}w_+^2) dx$

• $\xrightarrow{(5)}$ $f(t) = -4k^3t - kx_0$ ($k = \sqrt{\lambda}$)

$\xrightarrow{(3.5)} \int dx \Rightarrow$ $u_+ = -2k^2 \text{sech}^2 [k(x - 4k^2t - x_0)]$

[$\tanh' = \text{sech}^2$]

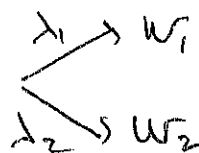
$2k^2 = c/2 \Rightarrow u_+ = -\frac{c}{2} \text{sech}^2 \left[\frac{\sqrt{c}}{2} (x - ct - x_0) \right]$ ✓

sech² sol !!

Repeating process to obtain more solutions $\rightarrow$ Messy !!

More elegant (and easier)

Start with w_0 and def.



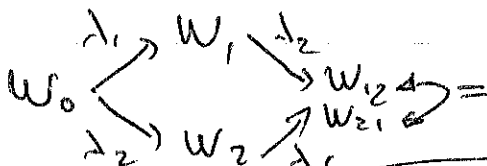
(B)
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$$\Rightarrow \begin{cases} (w_1 + w_0)_x = 2\lambda_1 + \frac{1}{2}(w_1 - w_0)^2 & (6a) \\ (w_2 + w_0)_x = 2\lambda_2 + \frac{1}{2}(w_2 - w_0)^2 & (6b) \end{cases} \quad (6)$$

Now do same process starting with $w_1 \xrightarrow{\lambda_2} w_{12}$
 $w_2 \xrightarrow{\lambda_1} w_{21}$

$$\Rightarrow \begin{cases} (w_{12} + w_1)_x = 2\lambda_2 + \frac{1}{2}(w_{12} - w_1)^2 & (7a) \\ (w_{21} + w_1)_x = 2\lambda_1 + \frac{1}{2}(w_{21} - w_2)^2 & (7b) \end{cases} \quad (7)$$

From Branchi's theo. of permutability $\Rightarrow w_{12} = w_{21} !!!$
i.e.



Do $[(7a)-(7b)] - [(6a)-(6b)] \Rightarrow$ $w_{12} = w_0 - \frac{4(\lambda_1 - \lambda_2)}{w_1 - w_2}$ (8)

to solve $w_{12} = w_{12}(w_0, w_1, w_2)$

So now start with trivial w_0

- construct $w_0 \xrightarrow{\lambda_1} w_1$ & $w_0 \xrightarrow{\lambda_2} w_2$ for any desired λ_1 & λ_2 !
- $\Rightarrow w_{12} = \text{NEW SOL} !!!$

Ex: Do this precisely for the velocities in our collision example:

$$\begin{array}{l} v_1 = 4 \\ v_2 = 16 \end{array} \xrightarrow{k=c/2} \begin{array}{l} k_1 = 1 \\ k_2 = 2 \end{array} \xrightarrow{-\lambda = k^2} \begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = -4 \end{array}$$

$$w_0 = 0 \xrightarrow{\lambda_1 = -1} w_1 = \dots \tanh \quad \xrightarrow{\lambda_2 = -4} w_2 = \dots \coth$$

$$\Rightarrow w_{12} = 0 - \frac{4(-1+4)}{(-\tanh - \coth)} = \dots$$

exactly the sol. that we wrote before for the collision of 2 solitons!