Dynamics of solitons in nearly integrable systems

Yuri S. Kivshar

Institute for Low Temperature Physics and Engineering of the Academy of Sciences of the Ukraine, Kharkov 310164, Union of Soviet Socialist Republics

Boris A. Malomed

P. P. Shirshov Institute for Oceanology of the U.S.S.R. Academy of Sciences, Moscow 117218, Union of Soviet Socialist Republics

A detailed survey of the technique of perturbation theory for nearly integrable systems, based upon the inverse scattering transform, and a minute account of results obtained by means of that technique and alternative methods are given. Attention is focused on four classical nonlinear equations: the Korteweg–de Vries, nonlinear Schrödinger, sine-Gordon, and Landau-Lifshitz equations perturbed by various Hamiltonian and/or dissipative terms; a comprehensive list of physical applications of these perturbed equations is compiled. Systems of weakly coupled equations, which become exactly integrable when decoupled, are also considered in detail. Adiabatic and radiative effects in dynamics of one and several solitons (both simple and compound) are analyzed. Generalizations of the perturbation theory to quasi-one-dimensional and quantum (semiclassical) solitons, as well as to nonsoliton nonlinear wave packets, are also considered.

CONTENTS

I. INTRODUCTION

It is commonly acknowledged that there are several exactly integrable nonlinear equations that play an outstanding role in physical problems: the Korteweg–de Vries (KdV), nonlinear Schrödinger (NS), sine-Gordon (SG), and several other equations. They are so important because they furnish universal mathematical models for some very general physical phenomena. For example, the KdV equation describes, in a general form, competition between weak nonlinearity and weak dispersion, while the NS equation describes the same competition for envelope waves (see, for example, the Introduction in Zakharov *et al.*, 1980). Some other integrable equations are universal as well. The SG equation is the simplest field equation with a potential periodic in a field variable; the so-called three- and four-wave systems are universal models for describing resonant interactions between several simple waves in a nonlinear medium; the Kadomtsev-Petviashvily equation has the same universal sense in two-dimensional problems as the KdV equation in one-dimensional ones, and so on. However, in physical applications these equations usually stem from some asymptotic expansions (e.g., expansions in powers of a wave's amplitude and wave number), so that actually they are approximate equations. Generally, higher terms of those expansions destroy the exact integrability.

Nonintegrable perturbations may also originate in the interaction of separate normal modes of a nonlinear system in the case when nonlinear self-interaction and linear dispersion of each mode is sufficiently strong, while their mutual interaction is weak. In this situation, one deals with coupled systems of nonlinear equations integrable in a decoupled form, while coupling terms play the role of nonintegrable perturbations. Terms that describe effects of external fields, inhomogeneities of a medium, and various dissipative effects are another source of nonintegrable perturbations.

All perturbations can be naturally divided into two classes: Hamiltonian and dissipative. It is important to note that, if a Hamiltonian perturbation does not depend explicitly on time and spatial coordinates, the perturbed equations conserve energy and momentum.

In many cases, the perturbing terms are small. We shall call the corresponding equations nearly integrable. To discuss effects produced by small nonintegrable perturbations added to integrable equations, let us recall that the most remarkable property of exactly integrable equations is the presence of exact solitonic solutions. As we shall see below when discussing physical applications, solitons correspond to localized excitations, which play an important role in diverse physical problems described by the corresponding equations. The existence of a one-soliton solution is not itself a specific property of integrable partial differential equations; many nonintegrable equations also possess simple localized solutions that may be called one-solitonic. However, these are integrable equations only, which possess exact many-soliton solutions. In most cases (including the KdV, NS, and SG equations), many-soliton solutions describe purely elastic interactions between individual solitons. The interacting solitons recover their exact initial shapes and velocities after a collision (interaction). The only result of the interaction is a phase shift, the total phase shift of a soliton induced by collision with any number of other solitons being exactly equal to the sum of partial shifts that would result from separate collisions with each of the solitons involved (this property of exactly integrable systems is commonly referred to as the absence of many-particle effects). [Nonetheless, in an exactly integrable system describing resonant interaction of several simple waves, a soliton may decay into a pair of other solitons (see, for example, Zakharov *et al.*, 1980)]. In general, the concept of a soliton retains its relevance after adding small nonintegrable terms to exactly integrable equations, but the evolution of a soliton may become quite complicated under the action of perturbations. The influence of a small dissipative perturbation is sufficiently obvious: it brakes moving solitons and in addition, it damps oscillating solitons (so-called breathers; see below). Effects generated by conservative perturbations are more subtle. They do not destroy or brake a nonoscillating soliton, but they may, for example, render a collision of solitons inelastic owing to the emission of quasilinear waves (radiation). Perturbation-induced effects are of interest mainly because they represent physical phenomena that cannot be comprised by exactly integrable models. In this connection, nearly integrable systems are of special concern, as perturbation-induced effects in those systems may be treated analytically. In the following section we shall outline briefly the main techniques of analytical perturbation theory for solitons.

Before we proceed further, two remarks are in order concerning the role of solitons in thermodynamics and quantum physics. First of all, it is known that systems described by the NS and SG equations are exactly integrable in the quantum case as well as in the classical one. Perturbation theory for quantum, nearly integrable models is of great interest. At the present time, this perturbation theory exists only in its simplest semiclassical (WKB) version. We shall review it in Sec. X. Second, the thermodynamics of solitons was developed by many workers, and, in fact, it constitutes a separate branch of soliton science, which deserves a special review. In the present survey, we touch lightly on thermodynamic problems in Sec. V when dealing with random perturbations.

A. Perturbation theory for nonlinear waves

Several different approaches to the analytical description of soliton dynamics in nonintegrable systems are known. One's choice of technique should be dictated by the character of the problem under consideration. The simplest are problems concerning the evolution of an amplitude and velocity of one soliton under the action of dissipation. For problems of this sort, a simple approach based on the so-called modified conservation laws is appropriate: assuming, in the first approximation, an unperturbed instantaneous shape of a soliton, one finds dissipation-induced rates of change of quantities that are elementary integrals of motion of the unperturbed system. In general, these are momentum and energy; for the NS equation, one can also employ the so-called plasmon number, discussed below (the KdV equation possesses an additional integral of motion, a so-called mass, but this integral cannot be employed for derivation of the soliton's equations of motion; see Sec. VI.N). Then, expressing the elementary integrals of motion in terms of parameters of the unperturbed soliton (amplitude, velocity, etc.), one obtains evolution equations for these parameters. This approach has been applied successfully to various problems [see, for example, the papers of Ott and Sudan (1969), Pereira and Stenflo (1977), Christiansen and Olsen (1982), Bergman *et al.* (1983), Levring, Samuelsen and Olsen (1984)].

In the presence of Hamiltonian perturbations, the evolution equations for the soliton parameters, which are valid in the lowest approximation, can also be simply obtained: one inserts an unperturbed one-soliton solution into a full Hamiltonian of the system (including a perturbation-induced part) and writes Hamilton's canonical equations of motion. In the framework of the Hamiltonian approach, such parameters as the soliton's velocity and amplitude play the role of canonical momenta, and phase parameters are canonical coordinates. Quite analogously, one may employ the Lagrangian approach (Bondeson, Anderson, and Lisak, 1979); as a matter of fact, these two approaches originate from the well-known variational method of Whitham (1974) in general nonlinear wave theory. Finally, the Hamiltonian approach can be readily modified to incorporate the case when a perturbation contains both conservative and dissipative terms. The Hamiltonian and Lagrangian approaches to dynamics of solitons in nearly integrable systems was employed, for example, by Nozaki (1982) and Malomed (1987d, 1987g, 1988d). In many cases, the simplest techniques based on modified conservative laws or the Hamiltonian/Lagrangian formalism apply to many-soliton problems too. The simplest example is dissipation-induced annihilation of a kink-antikink pair in a perturbed SG equation (Pedersen, Samuelsen, and Welner, 1984; Malomed, 1985).

The methods outlined above for deriving evolution equations in lowest approximation become irrelevant when one wishes to take account of effects that arise in higher orders of perturbation theory. Among these effects, physically interesting are perturbation-induced emission of radiation by solitons and long-range corrections to the soliton's shape. These problems can be solved successfully by means of the so-called direct perturbation theory. In the framework of this approach, original nonlinear equations are linearized on the background of an unperturbed solution to take a perturbation into account directly. A basic technical ingredient of direct perturbation theory is to find eigenfunctions of a linear operator associated with the linearized equation. The perturbation must be decomposed into a series based upon a full set of those eigenfunctions. Then the first-order evolution equations for the soliton's parameters are obtained as a condition of the absence of secular terms. As to emission of radiation from a soliton under the action of perturbations, it can be investigated with the aid of the Green's function (Eilenberger, 1977; McLaughlin and Scott, 1978). It is well known that the Green's function can be constructed as a bilinear combination of eigenfunctions of the linearized equation. Thus direct perturbation theory requires the knowledge of exact unperturbed solutions and eigenfunctions of associated linearized equations, but it does not require unperturbed equations to be exactly integrable. As a rule, direct perturbation theory is applicable to one-soliton problems only. Many-soliton problems can be considered provided the solitons are slightly overlapped, or they move with sufficiently large relative velocities (Sugiyama, 1979). Under these conditions, the overlapping between solitons is also treated as a perturbation.

Direct perturbation theory was first developed by L. A. Ostrovskii and his colleagues; see the survey papers by Gorshkov, Ostrovskii, and Pelinovsky (1974) and Gorshkov and Ostrovskii (1981). A basis for the application of direct perturbation theory to the perturbed SG equation has been elaborated by Fogel *et al.* (1976), Keener and McLaughlin (1977a, 1977b), and McLaughlin and Scott (1978) (see also Mineev and Shmidt, 1980). Important contributions to direct perturbation theory for the NS equations have been made by Keener and McLaughlin (1977a) and Ichikawa (1979). It seems pertinent to mention here as well the recent paper of Flesch and Trullinger (1987), in which a Green's function for a kink of several nonlinear Klein-Gordon equations, including the known ϕ^4 model, is constructed.

The most powerful perturbative technique is based on the inverse scattering transform (IST). This technique requires the unperturbed equation to be exactly solvable by the IST, which restricts the range of applications, but enables one to solve the most sophisticated dynamical problems. Perturbation theory based on the inverse scattering transform was introduced by Kaup (1976, 1977a) and, independently, by Karpman and Maslov (Karpman, 1977; Karpman and Maslov, 1977). An important contribution to the development of IST perturbation theory has been made by Kaup and Newell (1978a). McLaughlin and Scott (1978) noted a link between this theory and the previously developed direct perturbation technique (for the SG equation); they pointed out that, provided an unperturbed equation is exactly integrable, a full set of eigenfunctions of the linearized problem, necessary for the application of direct perturbation theory, can be found in the framework of the inverse scattering transform (Ablowitz *et al.*, 1974). Early results of IST perturbation theory have been reviewed by Karpman (1979a) and Newell (1978a, 1979). Some additional results have been collected in the book by Abdullaev and Khabibullaev (1986).

Equations exactly integrable by the IST possess many remarkable properties, such as Bäcklund transformations, the Painlevé property, a possibility of presentation in the Hirota bilinear form, and so on. Some of these properties may be employed as a basis for constructing alternative versions of the perturbation theory. For example, Yajima (1982) has developed a perturbation theory for the KdV equation based on the Hirota method.

There also exists another approach to nearly integrable systems with conservative (Hamiltonian time-independent) perturbations which originates from the well-known approach of classical mechanics: A perturbed system can be formally cast in unperturbed form by an infinite array of canonical transformations. These transformations constitute a series in powers of the small perturbation parameter ϵ. However, the series generally diverges due to the appearance of small denominators in higher orders. An analogous approach to nearly integrable partial differential equations was developed by Kodama (1985a, 1985b, 1988c, 1987) and Menyuk (1986a, 1986b; see also Menyuk and Chen, 1985; Menyuk, Chen, and Lee, 1985). In another context, this theme was touched upon by Benilov and Malomed (1988). All these authors cancel a perturbative term $\sim\epsilon$ by means of a first canonical transformation to arrive at a new effective perturbation $\sim\epsilon^2$ (it is important that this transformation encompasses both radiative and solitonic modes). However, the possibility of such a transformation does not render meaningless the results obtained in first-order perturbation theory. Indeed, the transformation mentioned changes the form of the solitons involved by quantities $\sim\epsilon$ (for instance, it may add a radiative component to a soliton), while in physical problems we are interested in interactions of solitons without initial radiative "tails."

The perturbative techniques discussed above suggest that there are two different levels of problems concerning dynamics of solitons in nearly integrable systems. Problems that can be formulated and solved in the adiabatic approximation, which disregards emission of radiation and perturbation-induced distortion of the shape of solitons, may be classified as belonging to the lower level. As a matter of fact, these problems are rather primitive since, in the adiabatic approximation, solitons may be regarded as structureless particles. Problems dealing with emission and related topics require more refined techniques, and they may be classified as belonging to a higher level.

B. Physical problems

In this section we enumerate physical applications of the KdV, modified KdV, NS, and SG equations and describe factors that give rise to various perturbation terms in those equations. Some additional examples will be given below in Secs. III–XI in the context of particular problems of IST perturbation theory. Magnetic systems described by the perturbed Landau-Lifshitz equation are discussed in Sec. XII, and physical applications of other nearly integrable equations are mentioned very briefly in the concluding section (Sec. XIII). We have tried to make the list of physical systems comprehensive, although we cannot be sure that no important item has been overlooked.

From the discussion below of a large number of physical problems, it will be seen that perturbation theory is fairly universal: One can distinguish few unperturbed exactly integrable equations and several important perturbating terms to them that occur in various problems.

1. Korteweg–de Vries and modified Korteweg–de Vries equations

Among the three basic equations that we shall deal with in the present paper, the KdV equation,

$$u_t - 6uu_x + u_{xxx} = \epsilon P(u) , \tag{1.1}$$

where $P(u)$ is a perturbation and ϵ is a small parameter, was historically the first to appear in physical applications. Almost a century ago, this equation (with $\epsilon=0$) was derived in the classic paper of Korteweg and de Vries (1895) as a fundamental equation governing propagation of waves in shallow water. More recently, the same equation appeared in the theory of nonlinear ion acoustic waves in cold plasmas [see, for example, Zabusky and Kruskal (1965), Washimi and Taniuti (1966)].

As the universal nature of this equation became clear, the KdV equation was detected in a number of other physical problems. As interesting examples, we can mention quasi-one-dimensional solid-state physics [the theory of chains of interacting atoms; e.g., Flytzanis, Pnevmatikos, and Remoissenet (1985)] and nonlinear transmission lines (e.g., Yoshinaga and Kakutani, 1984). There is even a problem from nuclear physics, in which a perturbed KdV equation plays a central role (Hefter, Raha, and Weiner, 1985).

In all these applications, the KdV equation arises as an approximate equation valid in a certain asymptotic sense (see the preceding section). Taking account of additional physical factors, one can obtain two different kinds of small perturbations for the KdV equation. First, the higher nonlinear dispersion and the higher spatial dispersion add the terms $\epsilon_1 u^2 u_x$ and $\epsilon_2 u_{xxxxx}$, respectively [see, for example, Yoshinaga and Kakutani (1984); note that the perturbation $\sim\epsilon_1$ does not break the exact integrability according to Zakharov *et al.* (1980)]. Second, dissipation may give rise to the following two perturbing terms:

$$\epsilon P(u) = -\alpha u \ , \tag{1.2a}$$

$$\epsilon P(u) = \gamma u_{xx} \ , \tag{1.2b}$$

where α and γ are small dissipative coefficients (see, for example, Kawahara and Toh, 1985a). In particular, the KdV equation perturbed only by the term (1.2b) ($\alpha=0$, $\gamma>0$) is commonly called the Korteweg–de Vries–Burgers equation. This equation plays an especially important role in applications. In certain cases, the coefficient γ in Eq. (1.2b) may be negative. In those cases, the perturbed KdV equation describes the development of an instability [Kawahara and Toh, 1985b; in such a situation, the higher stabilizing term $\sim -u_{xxxx}$ must be taken into account (Toh and Kawahara, 1985)]. The term (1.2a) with a time-dependent $\alpha(t)$ of either sign is physically meaningful too. According to Kakutani (1971), Johnson (1973), Newell (1980), Knickerbocker and Newell (1985), and others, this perturbation describes a variable depth in the shallow water theory; $\alpha=$const corresponds to a constant depth gradient. The KdV equation with the dissipative term (1.2a) finds a similar application in the description of nonlinear ion acoustic waves in an inhomogeneous plasma [see, for example, the paper by Chang *et al.* (1986)].

Landau damping acting upon nonlinear ion acoustic waves in a plasma introduces a nonlocal dissipative term of the form (Ott and Sudan, 1969; VanDam and Taniuti, 1973)

$$P[u(x)] = -⨍_{-\infty}^{+\infty} u(x')(x-x')^{-1}dx' \ , \tag{1.2c}$$

where $⨍$ is the symbol for the principle value of an integral. The same perturbation occurs in hydrodynamics of stratified liquids (Ostrovskii, Stepanyants, and Tsimring, 1984a).

In hydrodynamic problems, there occurs a coupled system of two KdV equations, the most general form of which is

$$u_{1t} - 6u_1u_{1x} + u_{1xxx} + \epsilon_1 u_2 u_{2x} + \epsilon_2 (u_1u_2)_x + \epsilon_3 u_{2xxx} = 0 \ , \tag{1.3}$$

$$u_{2t} - 6\beta u_2 u_{2x} + \beta u_{2xxx} - V_0 u_{2x} + \alpha[\epsilon_1 (u_1u_2)_x + \epsilon_2 u_1 u_{1x} + \epsilon_3 u_{1xxx}] = 0 \ , \tag{1.4}$$

where ϵ_1, ϵ_2, and ϵ_3 are small parameters, while α, β, and V_0 may be arbitrary. This system describes, for example, a resonant interaction of two wave modes in a shallow stratified liquid (Gear and Grimshaw, 1984; Gear, 1985). The conservative system of equations (1.3) and (1.4) is an interesting object for application of the perturbation theory.

To conclude our discussion of physical applications of the perturbed KdV equation, it is pertinent to mention the modified KdV equation

$$u_t - 6u^2 u_x + u_{xxx} = \epsilon P(u) \ , \tag{1.5}$$

which occurs, for instance, in the theory of quasi-one-dimensional solids (atomic chains; see, for example, Flytzanis, Pnevmatikos, and Remoissenet, 1985) and in liquid-crystal hydrodynamics (e.g., Kamenskii and Rozhkov, 1985). In addition, Eq. (1.5) arises in the special situation when a coefficient in front of the basic nonlinear term in Eq. (1.1) vanishes. Examples are waves in shallow two-layer liquids (Kakutani and Yamasaki, 1978; Helfrich, Melville, and Miles, 1984) and ion acoustic waves in a plasma with negative ions (Watanabe, 1984) or in a two-electron-temperature plasma (Tajiri and Nishihara, 1985). In such a situation, a perturbation from the KdV equation plays the role of that in the resultant modified KdV equation, for example, a modified KdV equation with Landau damping [Eq. (1.2c); see Tajiri and Nishihara, 1985]. As is well known, the unperturbed modified KdV equation can be transformed into the unperturbed KdV one by means of the Miura transformation (Miura, 1968).

By analogy with the KdV equation, natural perturbing terms to Eq. (1.5) are the higher dispersions $u^4 u_x$ or u_{xxxxx} and the dissipative terms (1.2).

2. Nonlinear Schrödinger equation

As we mentioned above, the NS equation

$$iu_t + u_{xx} + 2|u|^2 u = \epsilon P(u) \tag{1.6}$$

is a universal equation describing the evolution of wave envelopes in a dispersive weakly nonlinear medium (see, for example, Taniuti, 1974). Equation (1.6) finds an important application in plasma physics, where it describes electron (Langmuir) waves (Asano, Taniuti, and Yajima, 1968; Ichikawa, Inamura, and Taniuti, 1972).

Another application is known in nonlinear optics, where the function $u(x,t)$ has the sense of a complex envelope of electromagnetic field, and Eq. (1.6) describes self-modulation and self-focusing of light in a Kerr-type nonlinear medium. The first papers devoted to the NS equation in nonlinear optics appeared as early as 1964 (Chiao, Garmire, and Townes) and 1965 (Kelley). The great current interest in this application was initiated by the paper of Hasegawa and Tappert (1973), who predicted solitons in nonlinear optical fibers with the aid of the NS equation. [In the NS equation (1.6) describing an optical fiber, the variables x and t have the sense of time and spatial coordinates, in contrast with the usual notation.] Another strong stimulus to research in this field was provided by the concept of the soliton laser (Mollenauer and Stolen, 1984; Haus and Islam, 1985).

Further, the NS equation occurs in dynamics of quasi-one-dimensional ferromagnets with easy-axis anisotropy (Corones, 1977; Lakshmanan, 1977). Other applications are related to Davydov's solitons in molecular chains (Davydov, 1979), to slightly nonideal Bose gas with attraction (Lieb, 1963; Lieb and Leninger, 1963), to hydrodynamic description of nuclear matter (Heftner, 1985), and so on. Dynamics of strong phonon beams in a solid-state medium may also be described by the NS equation (Tappert and Varma, 1970), as well as gravity waves on deep water (Zakharov, 1968; see also Yuen and Lake, 1975; Hogan, 1985; Shivamoggi and Debnath, 1986).

Proceeding to a description of the perturbations, it is natural to start with the higher nonlinear-dispersion term

$$\epsilon P(u)=\epsilon|u|^4u \tag{1.7a}$$

with real ϵ. This perturbation is known in nonlinear optics (Pushkarov *et al.*, 1979; Kumar, Sarkar, and Ghatak, 1986). From the general viewpoint, it was considered by Kivshar and Malomed (1986c) and Nozaki (1986). We shall also consider the more general perturbation term

$$\epsilon P(u)=\epsilon|u|^{2N}u \tag{1.7b}$$

with an arbitrary integer $N\geq 2$.

In applications to nonlinear optical fibers, an important role is played by the perturbative terms $i\epsilon_1 u_{xxx}$ and $i\epsilon_2(|u|^2u)_x$ [or a more general combination $i\epsilon_2^{(1)}(|u|^2)_x u+i\epsilon_2^{(2)}u^2u_x^*$] with real $\epsilon_{1,2}$. According to Marcuse (1980) and Hasegawa and Kodama (1981), the former term accounts for the higher spatial dispersion, while the latter may be regarded as taking account of the nonlinear dispersion of group velocity (DeMartini *et al.*, 1967; Anderson and Lisak, 1983; Yang and Shen, 1984; and Kodama, 1985d).

The NS equation perturbed by the dissipative terms

$$\epsilon P(u)=-i\alpha_1 u\ , \tag{1.8a}$$

$$\epsilon P(u)=i\alpha_2 u_{xx}-i\alpha_3|u|^2u\ , \tag{1.8b}$$

with α_1, α_2, and α_3 real, occurs in hydrodynamics as a generalized Ginzburg-Landau equation describing the development of instability of the Poiseuille flow (Stewartson and Stuart, 1971), Couette-Taylor flow (DiPrima, 1970; DiPrima, Eckhaus, and Segel, 1971), and plane-parallel flow (DiPrima, Eckhaus, and Segel, 1971). It also arises in plasma physics when one studies the interaction between Langmuir and ion acoustic waves, provided the velocity V of the Langmuir waves is small in comparison with the sound velocity c (Gorev *et al.*, 1976; Fabrikant, 1984; for this case, Gibbons (1978) has derived a nondissipative perturbing term $\epsilon[|u|^2u_{xx}+(|u^2|)_xu_x-|u_x|^2u]$, $\epsilon\sim(V/c)^2$, which may be regarded as combining nonlinear and spatial dispersion).

The Landau damping of Langmuir waves in plasmas is an origin of other dissipative terms to Eq. (1.6). The linear Landau damping gives rise to a linear nonlocal dissipative term [cf. Eq. (1.2c)] with a kernel of a rather complicated form (see, for example, Nickolson and Goldman, 1977). More interesting, for our purposes, is the term

$$\epsilon P[u(x)]=-\epsilon u(x)\int_{-\infty}^{\infty}|u(x')|^2(x-x')^{-1}dx'$$

with a real small ϵ, which describes the nonlinear Landau damping (Ichikawa and Taniuti, 1973; Dysthe and Pécseli, 1977; Ichikawa, 1979). The important difference between this term and those of Eqs. (1.8) is the fact that the nonlinear Landau damping conserves the so-called plasmon number $N\equiv\int_{-\infty}^{\infty}|u(x)|^2dx$.

In the theory of nonlinear optical fibers, an important role is played by the nonlinear dissipative perturbation $\epsilon P(u)=\epsilon u(|u|^2)_x$ with positive real ϵ, which describes dissipation due to induced Raman scattering (Kodama and Hasegawa, 1986; Kodama and Nozaki, 1987).

According to Stenflo (1988), a general envelope equation of the NS type may contain the additional term $P(u)=(u_{xx}+u^{-1}u_x^2)$. This term, with a small coefficient in front of it, may be regarded as a new perturbation to the NS equation. Stenflo (1988) has found an exact solution to the NS equation with this term which resembles a quiescent soliton.

In certain problems a perturbation arises that combines a dissipative term with an external time-periodic drive:

$$\epsilon P(u)=-i\alpha_1 u+\epsilon e^{i\Omega t} \tag{1.9}$$

[see also a more general perturbation in Nozaki and Bekki (1984)]. For instance, this perturbation occurs in the theory of charge-density waves when one considers a small-amplitude localized dipolar excitation driven by an external ac electric field (Kaup and Newell, 1978b). An allied perturbation $P=|u|^2\exp(i\Omega t-ikx)$ describes the action of an external electromagnetic wave on Langmuir waves in plasmas according to Galeev *et al.* (1975).

In a system with pumping (external driving force) of the drift type and dissipation, nonlinear envelope waves are described by the perturbed NS equation

$$iu_t+u_{xx}+2|u|^2u=i\gamma_0u+i\gamma_1u_x+i\gamma_2u_{xx}\ ,$$

where γ_0, γ_1, and γ_2 are real coefficients of the pumping and dissipation. This equation was derived by Vigdorchik and Ioffe (1988) to describe magnetostatic solitons propagating at a ferromagnet-semiconductor surface, pumped by a drift flow of charge carriers in the semiconductor.

In many physical problems, there occur interactions of high-frequency and low-frequency waves. Examples are the interaction of Langmuir and ion acoustic waves in plasmas (Zakharov, 1972), of intermolecular vibrations with sound in molecular chains (Davydov, 1979), of high- and low-frequency acoustic gravity disturbances in an atmosphere (Stenflo, 1986), of surface and internal waves in the ocean (Petrov, 1978), etc. In the simplest case, such an interaction is described by the well-known system of Zakharov,

$$iu_t + u_{xx} = nu \ , \tag{1.10a}$$

$$n_{tt} - n_{xx} = 2(|u|^2)_{xx} \ , \tag{1.10b}$$

where $n(x,t)$ is the low-frequency field and $u(x,t)$ is an envelope of the high-frequency field, the sound velocity being normalized to unity. In the subsonic case, when group velocities of the high-frequency waves are much smaller than the sound velocity, Eqs. (1.10) may be regarded as a variant of a perturbed NS equation.

Another interesting application of perturbation theory for the NS equation is a system of coupled NS equations:

$$iu_t + u_{xx} + 2|u|^2 u = \epsilon P(u,v) \ , \tag{1.11}$$

$$iv_t + v_{xx} + 2|v|^2 v = \epsilon P(v,u) \ , \tag{1.12}$$

where ϵ is a small parameter. In particular, for two interacting molecular chains (see, for example, Davydov, 1979) and for two weakly coupled wave modes in a nonlinear waveguide (Blow, Doran, and Wood, 1987; Trillo *et al.*, 1988),

$$P(u,v) = \epsilon_1 v + \epsilon_2 |v|^2 u + \cdots \ , \tag{1.13}$$

where other nonlinear terms may be added. A similar system describes interacting Langmuir and dispersive ion acoustic waves in plasmas (Spatschek, 1978; Som, Gupta, and Dasgupta, 1979). The same system with the additional coupling term $\epsilon P(u,v) = i\epsilon v^*$ (ϵ is real, and the asterisk stands for the complex conjugation) has been derived by Akylas and Knopping (1986) in hydrodynamics to describe generation of beach waves by an incident wave.

Another coupling term,

$$P(u,v) = v_{xx} \tag{1.14}$$

(with a real ϵ), arises when one studies two weakly coupled long Josephson junctions (Mineev, Mkrtchyan, and Shmidt, 1981); in Josephson-junction theory, the NS equation describes a small-amplitude breather (bion); see below.

3. Sine-Gordon equation

a. A single equation

The SG equation

$$u_{tt} - u_{xx} + \sin u = \epsilon P(u) \tag{1.15}$$

covers a vast area of physical applications. An early review of these applications was that of Barone *et al.* (1971; see also Scott, 1970). The earliest example is the model of dislocations in solids put forward by Frenkel and Kontorova (1938). Later, a similar model was proposed by Frank and van der Merwe (1949) to describe a chain of adsorbed atoms on a metallic surface.

A very important application of the SG equation takes its origin in the theory of long Josephson junctions (Josephson, 1962; see also Barone and Paterno, 1982; Likharev, 1985; Lomdahl, 1985). In this case, $u(x,t)$ has the sense of a phase jump of the wave function of superconducting electrons across the junction, or, equivalently, $u(x,t)$ gives the magnetic flux confined in the junction. Topological solitons (kinks) in the form of so-called fluxons, i.e., quanta of magnetic flux, have become a central concept in the dynamical theory of long Josephson junctions since the appearance of the paper by Fulton and Dynes (1973).

The SG equation finds another important application in the dynamics of quasi-one-dimensional ferromagnets with easy-plane anisotropy. [Enz (1964), Mikeska (1978), and Kjems and Steiner (1978) were the first to employ the SG model for description of domain walls as topological solitons in ferromagnets.] The dynamics of weak ferromagnets (antiferromagnets) is also described by the SG equation; see the papers of Zvezdin (1979) and Bar'yakhtar, Ivanov, and Sukstanskii (1980), as well as the review of Bar'yakhtar, Ivanov, and Chetkin (1985). In "magnetic" applications, the dynamical variable $u(x,t)$ is proportional to an angle that determines local orientation of the magnetization vector lying in a certain plane. Quite analogously, the SG equation may be used to describe the dynamics of quasi-one-dimensional ferroelectric systems (Pouget and Maugin, 1984, 1985a, 1985b).

Another physical application of the SG equation, of great current interest, dates back to 1974, when Lee, Rice, and Anderson demonstrated that the SG Hamiltonian arises naturally in a charge-density-wave (CDW) system when the Peierls wave number is commensurate with the inverse spacing of an underlying ionic lattice (see also Fleming *et al.*, 1980). In this case, the variable $u(x,t)$ has the sense of a phase misfit between the electronic CDW and the lattice. In 1976, Rice *et al.* introduced the concept of solitons into the phase dynamics of CDW's in commensurate systems. Fukuyama (1978), Grüner, Zawadowski, and Chaikin (1981), and Hansen and Carneiro (1984) all noted that an effective commensurability, which again leads to the SG model, might be induced in an incommensurate system by an ionic superlattice with an expedient spacing.

At the same time, it is necessary to emphasize that, while the existence of kinklike (topological) solitons in real Josephson junctions, ferromagnets and ferroelectrics, liquid crystals, and two-level resonant optical media (references pertaining to the two latter subjects are given below) is universally recognized, the existence of phase solitons in the CDW systems (one-dimensional metals) is, so far, a hypothesis (Krive, Rozhavskii, and Kulik, 1986).

Other applications of the SG equation have been made to liquid crystals (Lin, Shu, Shen, Lam, and Yun, 1982; Kamenskii, 1984; a short review of SG-type solitons in liquid crystals was given by Lin, Shu, and Xu, 1985), spin waves in liquid helium (Maki, 1975; Maki and Kumar, 1976; Bullough and Caudrey, 1978), and self-induced transparency of a two-level medium in nonlinear optics (McCall and Hahn, 1969; Lamb, 1971). Finally, the SG equation has some applications in hydrodynamics (Gibbon, James, and Moroz, 1979; Moroz and Brindley, 1981; Coullet and Huerre, 1986) and even as a model of hadrons (Uchiyama, 1976). To conclude the list of applications, it is pertinent to mention the ingenious mechanical analog of the SG equation constructed by Scott (1969) in the form of a chain of pendula suspended on a horizontal elastic thread.

For our purposes, it is important that, in most applications mentioned above, formulation of physically meaningful problems necessarily includes small perturbing terms. Diverse problems give rise to various perturbations $P(u)$ in Eq. (1.15). First of all, dissipation is usually described by the terms

$$\epsilon P(u) = -\gamma u_t \quad (\gamma > 0) , \tag{1.16a}$$

$$\epsilon P(u) = \beta u_{xxt} \quad (\beta > 0) . \tag{1.16b}$$

For instance, in Josephson-junction theory, the term (1.16a) accounts for the dissipative losses due to tunneling of normal electrons across the dielectric barrier, while (1.16b) accounts for the losses due to a current along the barrier (see, for example, Barone and Paterno, 1982). Dissipative terms in other problems have the same form (1.16); however, one should remember that, strictly speaking, this form is phenomenological, and the genuine form of dissipative terms may be fairly complicated (see, for example, McCumber, 1968, Stewart, 1968, and Olsen and Samuelsen, 1984).

Another general type of dissipative perturbation is that localized in space, $\epsilon P(u) = -\beta\delta(x)u_t$ ($\beta > 0$). This perturbation may describe, for example, a narrow region in a long Josephson junction with locally enhanced dissipative losses. It can be created by the action of a focused laser beam on the junction (Chang, 1985), by imposing a shorter resistor onto it (Akoh *et al.*, 1985), or by implanting a microshort made of a normal metal (Kivshar and Malomed, 1987c, 1988e).

Proceeding to Hamiltonian perturbations, let us first consider the perturbed SG equation that takes its origin in Josephson-junction theory. The most important nondissipative perturbation describes a uniform density of the bias current (generally speaking, time-variable):

$$P = f(t) \tag{1.17}$$

(Barone and Paterno, 1982). The same term (1.17) describes an external drive in many other applications, for instance, the external dc or ac electric field in a commensurate CDW system (Maki, 1978), an external field of electric or mechanical nature in models of the Frenkel-Kontorova type (Braun *et al.*, 1988), a shear flow in nematic liquid crystals (Lin, Shu, and Xu, 1985), etc.

Inhomogeneities of long Josephson junctions are another abundant source of perturbations for the SG equation. A weak inhomogeneity of the maximum Josephson current density is described by the perturbing term (Mkrtchyan and Shmidt, 1979)

$$P(u) = g(x)\sin u . \tag{1.18}$$

In particular,

$$P(u) = \pm\delta(x)\sin u \tag{1.19}$$

corresponds to the so-called microresistor (+) and microshort (microshunt) (−) (McLaughlin and Scott, 1978). A microshort can be realized as a narrow bridge connecting two bulk superconductors; a microresistor is a local bulge of the dielectric layer that separates the superconductors. A periodic lattice of the pointlike inhomogeneities (1.19), described by the perturbation (1.18) with $g(x) = \sum_{n=0}^{\infty} \delta(x - an)$, is of great physical interest too. A lattice of pointlike microresistors ($\epsilon > 0$) is the only example of a long periodically inhomogeneous Josephson junction realized so far in an experiment (Serpuchenko and Ustinov, 1987; see also Golubov, Serpuchenko, and Ustinov, 1988, and Malomed *et al.*, 1988).

The same perturbation (1.19) describes a local magnetic impurity in magnetic systems comprised by the SG model (Bar'yakhtar, Ivanov, and Chetkin, 1985). As has been demonstrated by Malomed (1988e) and Malomed and Nepomnyashchy (1989b), a charged impurity in a commensurate CDW system may give rise to a more general perturbation $P(u) = \delta(x)\sin(u/M + \theta)$, where $M \geq 1$ is an integer, and θ is an arbitrary parameter.

A nonuniformity of the dielectric barrier in a long Josephson junction renders inhomogeneous two other basic local characteristics of the junction, viz., its local inductance and capacity. According to Sakai, Samuelsen, and Olsen (1987), a perturbing term generated by a small inhomogeneous part $[\sim g(x)]$ of the local inductance can be brought into the form $P(u) = g'(x)u_x$. A capacity inhomogeneity is described by $P(u) = g(x)u_{tt}$ (see, e.g., Malomed and Ustinov, 1989b).

Another local nondissipative perturbing term $P = \delta'(x)$ in Josephson-junction theory can be generated by an Abrikosov vortex lying in the junction's plane perpendicular to its local dimension (Aslamazov and Gurovich, 1984); the same term describes a local deformation of a CDW system (Brazovsky and Bak, 1978). However, a more realistic configuration is that of an Abrikosov vortex lying in the junction perpendicular to its plane; in this case an effective perturbation reduces to the form of Eq. (1.19) (Golubov and Ustinov, 1988).

In many cases, it is necessary to consider the dynamics of solitons (fluxons) in long Josephson junctions of a finite length. Boundary conditions at a junction's edge may be regarded as effective perturbations (Olsen *et al.*, 1986). For example, a semi-infinite Josephson junction is described by the SG equation on the half-axis $x>0$ with the boundary condition $u_x|_{x=0}=\epsilon$, where ϵ is a sum of two terms accounting for current injection through the edge $x=0$ and external magnetic field (Owen and Scalapino, 1967). Evidently, the corresponding dynamical problem is equivalent to that on the whole axis $-\infty<x<+\infty$ with the effective perturbation

$$P(u)=2\delta(x)\ , \tag{1.20}$$

if $u(x)$ is defined on the negative half-axis according to the rule $u(-x)=u(x)$. In this regard, it is pertinent to mention the experimental work of Davidson *et al.* (1985), in which a ringlike long Josephson junction with trapped solitons (fluxons) has been constructed. In the theoretical model of the ringlike Josephson junction (e.g., If, Soerensen, and Christiansen, 1984; If *et al.*, 1985; Fordsmand, Christiansen, and If, 1986; Kivshar and Malomed, 1986d, 1986e; Marchesoni, 1986a), the problem of boundary conditions is absent.

In models of the Frenkel-Kontorova type, a weak long-scale substrate potential $\epsilon\phi(x)$ introduces the perturbative term $-\epsilon\phi'_x(x)$ into a corresponding SG equation (Fogel *et al.*, 1976; Malomed, 1987d). In particular, $\phi(x)=-\mathrm{sgn}(x)$ gives rise to the perturbation (1.20). The same term describes application of a pointlike electrode carrying bias current to a long Josephson junction. In an experiment, this was done by Akoh *et al.* (1985).

In many other physical problems, an external constant or time-variable drive is accounted for by the perturbative term

$$P(u)=f(t)\sin(u/2)\ . \tag{1.21}$$

In the case $f=\mathrm{const}$, Eq. (1.15) with the perturbation (1.21) is commonly called the double SG equation. It is interesting to note that its mechanical analog, similar to that of the usual SG equation mentioned above (Scott, 1969), has been constructed by Salerno (1985). In weak ferromagnets $f(t)$ is proportional to an external magnetic field directed perpendicular to the x axis and aligned with the magnetization vector at $x=\pm\infty$ (Zvezdin, 1979; Bar'yakhtar, Ivanov, and Sukstanskii, 1980). An external magnetic field perpendicular both to the x axis and to the magnetization vector is described by the perturbation

$$P(u)=f(t)\cos(u/2)\ , \tag{1.22}$$

and an external field directed along the x axis corresponds to the perturbative term [Maugin and Miled, 1986; cf. Eq. (1.17)]

$$P(u)=\frac{df(t)}{dt}\ . \tag{1.23}$$

As to the easy-plane ferromagnets, in that system an external ac magnetic field orthogonal to the easy plane is described by the term (Kosevich *et al.*, 1983)

$$P(u)=f(t)\sin u\ . \tag{1.24}$$

Another important "magnetic" perturbation is (Eleonskii *et al.*, 1978).

$$P(u)=\sin(2u)\ . \tag{1.25}$$

It describes weak higher anisotropy in an easy-plane ferromagnet. The SG equation with the same effective perturbation appears if one considers a ferromagnet subject to the action of a very strong dc magnetic field. In this case, the former perturbing term (1.21) [with $f(t)\equiv 1$] becomes, on evident rescaling, the basic term $\sin u$ of the unperturbed SG equation, while the former basic term becomes the perturbation (1.25). The same perturbation (1.25) also occurs in Josephson-junction theory when one deals with the Josephson effect in layered superconductors (Gvozdikov, 1988). Finally, in some problems of CDW theory (Krive, Rozhavskii, and Kulik, 1986) and field theory (Casher, Kogut, and Susskind, 1974) there occurs an "exotic" perturbation $P(u)=u$.

In many problems, the usual SG model appears as a continuum limit of an original discrete model (the Frenkel-Kontorova dislocation model, adsorbed atomic chains, CDW systems, etc.). The continuum approximation is relevant provided characteristic sizes of solitons are much greater than the spacing of the underlying lattice. In this case, "residual" discreteness can be approximately accounted for by a small perturbation in the continuum model; typical resultant perturbative terms are Eq. (1.18) with $g(x)=\cos(2\pi x/a)$ and with $a\ll 1$ the lattice spacing, and the higher spatial-dispersion term $\sim u_{xxxx}$ (Ishimori and Munakata, 1982; see also Peyrard and Kruskal, 1984; Homma and Takeno, 1985; Willis, El-Batanouny, and Stancioff, 1985; Stancioff *et al.*, 1986; Takeno and Homma, 1986a, 1986b; and De Lillo, 1987).

b. Systems of coupled equations

The perturbation theory for the SG equation is applicable to systems of weakly coupled equations of the SG type. A simple and physically interesting example[1] is

$$u_{tt}-u_{xx}+\sin u=\epsilon v_{xx}\ , \tag{1.26}$$

$$v_{tt}-v_{xx}+\sin v=\epsilon u_{xx}\ , \tag{1.27}$$

which describes two weakly coupled parallel long Josephson junctions [Mineev, Mkrtchyan, and Shmidt, 1981; for

[1]Here and below [see Eqs. (1.28), (1.29); (1.28′), (1.29′); (1.30)–(1.32)] we write the systems of coupled equations in the most symmetric form. In Secs. IV and VI we deal with nonsymmetric generalizations of the systems (1.28′),(1.29′) and (1.28),(1.29).

a more rigorous derivation of Eqs. (1.26) and (1.27), see Volkov (1987)].

Two weakly interacting arrays of adsorbed atoms are described by the system

$$u_{tt}-u_{xx}+\sin u=\epsilon\sin(u-v)\ , \tag{1.28}$$

$$v_{tt}-v_{xx}+\sin v=\epsilon\sin(v-u) \tag{1.29}$$

[Braun *et al.*, 1988; in general, the linear-derivative coupling terms from the right-hand side of Eqs. (1.26) and (1.27) (with another coupling constant) should be added to Eqs. (1.28) and (1.29)]. Earlier, this same system (1.28) and (1.29) appeared in the papers of Yomosa (1984, 1985) and Homma and Takeno (1984) as a base-rotator dynamical model of a double helix of DNA. Later, an alternative model of the double helix was put forward by Zhang (1987):

$$u_{tt}-u_{xx}+\sin u=-\epsilon\sin(u/2)\cos(v/2)\ , \tag{1.28$'$}$$

$$v_{tt}-v_{xx}+\sin v=-\epsilon\sin(v/2)\cos(u/2)\ . \tag{1.29$'$}$$

The system (1.28′) and (1.29′) can be naturally called a system of coupled double SG equations. Setting $u=0$ or $v=0$ in Eqs. (1.28′) and (1.29′), one arrives at the SG equation perturbed by the term (1.21) with $f=1$, which is a double SG equation. It is interesting that this system gives rise to solitons with richer dynamics than the system (1.28) and (1.29) (Malomed, 1987h). At the same time, it is necessary to note that, as far as we know, solitons have not yet been detected in experiments with DNA molecules.

Our last example of a physically interesting subject for the application of perturbation theory is the SG equation coupled to two linear wave equations:

$$u_{tt}-u_{xx}+\sin u=\epsilon(V_x\cos u+U_x\sin u)\ , \tag{1.30}$$

$$V_{tt}-V_{xx}=\epsilon(\sin u)_x\ , \tag{1.31}$$

$$U_{tt}-U_{xx}=\epsilon(\cos u)_x\ . \tag{1.32}$$

According to Pouget and Maugin (1984) and Maugin and Miled (1986), the system (1.30)–(1.32) describes the coupling between polarization or magnetization waves and sound in elastic ferroelectric or ferromagnetic systems, $U(x,t)$ and $V(x,t)$ being, respectively, transverse and longitudinal acoustic modes.

C. Experimental observations of solitons

To conclude the list of perturbed equations, it seems relevant to mention several pioneering experiments in which solitons described by the perturbed KdV, NS, and SG equations have been observed.

Hydrodynamics. It is a commonplace to cite the observation of a large-amplitude soliton on a water surface by Scott Russell in 1834 (Scott Russell, 1844). This soliton is described by the KdV equation.

As we discuss in Secs. III.A, VIII.C, and VIII.D, scattering of a shallow-water KdV soliton by a depth inhomogeneity gives rise to a number of interesting dynamical effects. Some of these effects have been observed in the experiments of Seabra-Santos, Renouard, and Temporville (1987). In Secs. IV.C.1 and VII.F.1 we consider the interactions of solitons described by a system of two coupled KdV equations. Weidman and Johnson (1982) observed similar effects in an experiment with a stratified liquid.

Plasma physics. Different kinds of solitons are known in plasma physics [see, for example, the reviews of Pécseli (1985) and Petviashvili and Yan'kov (1985)]. From our standpoint, the most interesting are the Langmuir and ion acoustic solitons, described by the NS and KdV equations, respectively. Langmuir solitons in the form of cavitons, i.e., local regions from which the plasma is ousted by the electromagnetic field, were observed by Ikezi, Nishikawa, and Mima (1974), Kim, Stenzel, and Wong (1974), and Wong and Quon (1975). In the presence of a strong magnetic field, cavitons in a moving plasma were observed by Antipov *et al.* (1976, 1977). Ion acoustic solitons had been detected earlier by Ikezi, Taylor, and Baker (1970) and Ikezi *et al.* (1971; see also the survey by Tran, 1979). For applications of the perturbation theory, of special concern are experimental investigations of ion acoustic solitons in inhomogeneous plasmas (e.g., Chang *et al.*, 1986). Another interesting perturbation-induced effect is the acceleration of a Langmuir soliton (caviton) under the action of nonlinear Landau damping (see Sec. III.B). In an experiment, this effect was observed by Watanabe (1977).

Solids. In elastic media, envelope solitons of the NS type may propagate. They were observed, for example, by Wu *et al.* (1987) in the form of flexural modes of thin shells. Korteweg–de Vries type solitons in an elastic rod were recently observed by Samsonov and co-workers (Dreiden *et al.*, 1988).

As we mentioned in the preceding section, solid-state physics gives rise to a number of nonlinear models in which solitons of different types may occur. An interesting example is the experiment of Harten *et al.* (1985), in which direct observation of a soliton in a layer of atoms adsorbed on a metallic surface was reported.

Long Josephson junctions. It was emphasized above that long Josephson junctions are unique physical objects for applications of the soliton theory. The first direct experimental observations of an isolated fluxon (equivalent to a SG kink) were achieved, by means of different techniques, by Matsuda and Uehara (1982), Matsuda and Kawakami (1983), and Scheuermann *et al.* (1983) [a detailed description of the techniques involved and experimental results has been given by Pedersen (1986)]. A very interesting object is a ringlike long Josephson junction. Such a junction with trapped fluxons was realized in an experiment by Davidson *et al.* (1985), although the observation of fluxons in this experiment was indirect.

Later, Akoh *et al.* (1985) constructed a device in which details of the dynamics of an isolated fluxon could be observed directly. In particular, acceleration of a fluxon under the action of the bias current, capture of a fluxon by a local inhomogeneity, and escape of the captured fluxon by a bias current pulse were observed.

Experimental techniques developed by the groups cited above have made possible, as well, the direct observation of interactions between fluxons. In particular, Matsuda and Kawakami (1984) have observed a fluxon-antifluxon collision, while Fujimaki, Nakajima, and Sawada (1987) observed a fluxon-antifluxon annihilation into a breather and subsequent dissipation-induced damping of the breather.

Magnetic systems. Commonly known domain walls in ferromagnets and antiferromagnets may be regarded as kinklike (topological) solitons. Their purport for experiments with magnetic systems is evident. Dynamical (nontopological) magnetic solitons are much more difficult to observe, since they are strongly damped by dissipation. However, some experiments with dynamical solitons have been successful. For instance, Kalinikos, Kovshikov, and Slavin (1983) were able to observe magnetostatic solitons directly in a thin film of a rare-earth orthoferrite (an yttrium iron garnet film); later, nonlinear excitations in the same system were discussed and detected indirectly by De Gasperis, Marchelli, and Miccoli (1987). Gornakov, Dedukh, and Nikitenko (1984) observed a solitonlike packet of spin waves propagating along a plane domain wall.

Nonlinear optics. Single-mode nonlinear fibers (optical waveguides) are remarkable physical objects for experiments with envelope solitons of the NS type. The first experimental observation of the optical solitons in fibers is due to Mollenauer, Stolen, and Gordon (1980). They detected one-soliton, two-soliton, and three-soliton states. Later, Mollenauer and Stolen (1984) constructed a fiber-based laser operating on soliton states (the soliton laser). It is also relevant to mention that Salin *et al.* (1988) observed a breatherlike oscillating bound state of two optical solitons. Finally, Krökel *et al.* (1988) and Weiner *et al.* (1988) have observed the so-called dark solitons in a fiber, i.e., a "dark spot" inside a broad light pulse. The dark solitons are described by the NS equation "with repulsion" [see below Eq. (9.1)].

As is well known, the NS equation describes, besides the self-modulation of a nonlinear one-dimensional nonstationary electromagnetic wave in an optical fiber, the self-focusing of a nonlinear stationary two-dimensional wave in a planar waveguide. In the latter case, a wave envelope in the two-dimensional space (x,z) is described by Eq. (1.6), where t must be replaced by z. Recently, stable self-trapping of laser beams in a nonlinear planar waveguide, which may be interpreted as the formation of self-focusing solitons, has been observed in an experiment of Maneuf, Desailly, and Froehly (1988).

Solitons also play an important role in other fields of nonlinear optics. McCall and Hahn (1967, 1969) have predicted and observed in an experiment optical solitons generated by a resonant interaction of light with a two-level medium (the so-called self-induced transparency). Generally, this interaction is described by the exactly integrable Maxwell-Bloch equations. In the case when broadening of a spectral line may be neglected, the Maxwell-Bloch equations reduce to the sine-Gordon equation. Drühl *et al.* (1983) have observed solitons in Raman scattering. The Raman scattering is described by an exactly integrable N-wave system, which also reduces to the SG equation in a certain particular case.

Mechanical and electric models. To demonstrate solitonic properties of the SG equation, Scott has invented an elegant mechanical model in the form of a chain of pendula connected by a long spring (Scott, 1969). This model may also be used for studies of a perturbed SG equation (see, for example, Cirillo *et al.*, 1981). A more sophisticated mechanical model of an important variant of the perturbed SG equation, viz., the so-called double SG equation (see Sec. IV.B.6), has been constructed by Salerno (1985).

The study of wave propagation of nonlinear and dispersive electrical transmission lines made up of LC circuits has received much attention since the pioneering work of Hirota and Suzuki (1970) on simulation of the Toda lattice. Such lines simulate the KdV system for sufficiently long and weakly nonlinear waves if a continuum approximation is introduced. Experimental studies of the KdV solitons have been performed by Kolosick *et al.* (1974), Noguchi (1974), etc. On the other hand, the propagation of envelope solitons, which are solutions of the NS equation, has been studied on electrical transmission lines by Kiyashko *et al.* (1975), Sakai and Kawata (1976), Yagi and Noguchi (1976), etc.

Electrical transmission lines can be readily modified to include various perturbing factors, e.g., dissipative losses or local inhomogeneities [see, for example, Stepanyants (1977) and Kako (1979)].

D. Aims and structure of the present work

Our basic aim is to present a detailed account of results obtained by means of the inverse scattering transform and allied techniques in perturbation theory for nearly integrable systems. In most cases, our purpose has been not merely to give a formulation of a problem and final results, but also to explain in sufficient detail the method of solution if that method is nontrivial.

We have tried to make the present survey comprehensive, for two reasons: First, we note that some significant results have been obtained repeatedly by different researchers due to apparent lack of information on previous work; second, it seems to us that the main problems in this field have already been solved, so that it is reasonable to attempt a comprehensive survey. If the survey seems lengthy, it is only because the number of physically interesting problems considered by different authors is large.

The rest of the paper is structured as follows: In Sec. II we review very briefly the basic components of the IST technique and give, without detailed derivation, general perturbation-induced evolution equations for the scattering data. These equations constitute the basis of IST perturbation theory. Sections III–VII are devoted to particular problems formulated in terms of the perturbed KdV, NS, and SG equations. As can be seen from the Contents, the problems are classified according to the number of solitons involved and the fundamental character of the problem, i.e., radiative problems are separated from adiabatic ones. The next four sections deal with special important problems (see the Contents). Among these problems, especially fundamental and difficult is the generation of new solitons under the action of perturbations. In Sec. XII we consider the perturbation theory for a more complicated integrable equation, viz., the Landau-Lifshitz equation, which plays a very important role in the dynamics of ferromagnets. Finally, in Sec. XIII we briefly discuss results obtained for other nearly integrable systems. We mention the following perturbed equations: Maxwell-Bloch, *N*-wave, derivative NS, Boussinesq, Benjamin-Ono, Joseph, Toda-lattice, Ablowitz-Ladik, and Kadomtsev-Petviashvili. In addition, we mention results obtained by means of perturbation theory for KdV and Toda-lattice equations with periodic boundary conditions, and for fully quantum NS and SG models. The appendixes contain several lengthy formulas excerpted from the basic text.

II. THE INVERSE SCATTERING TRANSFORM AND PERTURBATION THEORY

A. General remarks

The inverse scattering transform (IST) was introduced in the famous paper of Gardner *et al.* (1967). Further advances in the use of this method were made by Lax (1968), Zakharov and Shabat (1971), and Ablowitz *et al.* (1973b, 1974). Today the method is well elaborated; detailed accounts of it can be found in a number of books (Lamb, 1980; Zakharov *et al.*, 1980; Ablowitz and Segur, 1981; Calogero and Degasperis, 1982; Newell, 1985; Faddeev and Takhtadjan, 1986). The basic idea of the IST is to represent a nonlinear evolution equation for a function $u(x,t)$ in the form of the so-called (L,A) pair (Lax, 1968),

$$\hat{L}_t+[\hat{L},\hat{A}]=0\ , \tag{2.1}$$

where $\hat{L}$ and $\hat{A}$ are some linear operators with coefficients dependent on the function $u(x,t)$ and its derivatives (see Secs. II.B–II.D). In other words, the equation for $u(x,t)$ is a condition of compatibility of the auxiliary linear equations

$$\hat{L}\Psi=\lambda\Psi\ , \tag{2.2}$$

$$\Psi_t=\hat{A}\Psi\ , \tag{2.3}$$

where λ is a spectral parameter (generally speaking, complex), and Ψ is often called a *Jost function.*

The first step in using the IST is to solve the direct scattering problem, i.e., to find eigenfunctions of the spectral equation (2.2) (in other words, to find scattering data). Since coefficients of the L operator depend on the function $u(x,t)$, we can map this function into the *scattering data.* In general, the scattering data consist of two components $S(\lambda)$ and S_n pertaining to continuous and discrete spectra, n standing for a number of discrete eigenvalue. Temporal evolution of $u(x,t)$ generates evolution of the scattering data via Eq. (2.3). The efficiency of the IST rests upon the fact that the evolution of the scattering data proves to be trivial; evolution equations for $S(\lambda)$ and S_n have the following general form (see particular examples below in Secs. II.B–II.D):

$$\frac{\partial S(\lambda,t)}{\partial t}=i\Omega(\lambda)S(\lambda,t)\ , \tag{2.4}$$

$$\frac{dS_n(t)}{dt}=\Omega_n S_n(t)\ , \tag{2.5}$$

and, moreover, some components of $\Omega(\lambda)$ and Ω_n may be equal to zero. So, to solve the Cauchy problem for the underlying nonlinear evolution equation $u_t=F[u]$, we should first find the initial scattering data $S(\lambda,t=0)$ and $S_n(t=0)$ corresponding to an initial condition $u(x,t=0)$, i.e., solve *the direct scattering problem,* then find $S(\lambda,t)$ and $S_n(t)$ (for arbitrary $t>0$) from the trivial evolution equations (2.4) and (2.5) for the scattering data, and finally reconstruct $u(x,t)$ on the basis of $S(\lambda,t)$ and $S_n(t)$, i.e., solve *the inverse scattering problem.* In all cases, the discrete-spectrum scattering data correspond to solitons, and the continuous-spectrum scattering data correspond to radiation.

An important component of the IST is related to variational derivatives $\delta S/\delta u(x)$ that can be explicitly written in terms of the Jost functions and scattering data. With the aid of these quantities, one can easily obtain a generalization of the evolution equations (2.4) and (2.5) for the case when the original equations for $u(x,t)$ contains a perturbation, i.e., when it can be represented in the form

$$u_t=F[u]+\epsilon P[u]\ , \tag{2.6}$$

where $F[u]$ stands for the unperturbed part of the equation and $P[u]$ is a perturbation (u may be a multicomponent function). Indeed,

$$\begin{aligned}\frac{\partial S(\lambda,t)}{\partial t}&=\int_{-\infty}^{+\infty}dx\frac{\delta S(\lambda,t)}{\delta u(x,t)}u_t(x,t)\\&=\int_{-\infty}^{+\infty}dx\frac{\delta S(\lambda,t)}{\delta u(x,t)}F[u]+\epsilon\int_{-\infty}^{+\infty}\frac{\delta S(\lambda,t)}{\delta u(x,t)}P[u]\\&=i\Omega(\lambda)S(\lambda,t)+\epsilon\int_{-\infty}^{+\infty}dx\frac{\delta S(\lambda,t)}{\delta u(x,t)}P[u]\ .\end{aligned} \tag{2.7}$$

Analogously one can calculate $\partial S_n(t)/\partial t$:

$$\frac{\partial S_n(t)}{\partial t}=\Omega_n S_n(t)+\epsilon\int_{-\infty}^{+\infty}dx\frac{\delta S_n(t)}{\delta u(x,t)}P[u]\ . \tag{2.8}$$

If ϵ is a small parameter, one may substitute the unperturbed $u(x,t)$ and Jost functions into the right-hand side of Eq. (2.7), which yields the perturbed evolution equations for the scattering data in the lowest approximation of perturbation theory. Clearly, this procedure can be iterated to yield higher orders of perturbation theory. These ideas were independently put forward in two pioneer papers by Kaup (1976) and Karpman and Maslov (1977) [see also Karpman (1977)]. In the next three subsections we shall examine perturbation theory technique in more detail for the KdV, NS, and SG equations. We shall concentrate on first-order perturbation theory. The second order was developed by Maslov (1980) for the KdV, modified KdV, and NS equations, and by Kivshar (1984) for the SG equation.

B. Korteweg–de Vries equation

For the unperturbed KdV equation [i.e., Eq. (1.1) with $\epsilon=0$] the operators $\hat{L}$ and $\hat{A}$ in Lax's representation [Eq. (2.1)] are

$$\hat{L}=-\frac{\partial^2}{\partial x^2}+u(x,t)\ , \tag{2.9a}$$

$$\hat{A}=-4\frac{\partial^3}{\partial x^3}+3\left[u\frac{\partial}{\partial x}-\frac{\partial}{\partial x}u\right]\ . \tag{2.9b}$$

It is traditionally accepted that the spectral parameter for the auxiliary linear equation (2.2) with the operator (2.9a) be redesignated as $\lambda=k^2$. Since we are interested in the wave potential $u(x)$ decaying sufficiently quickly as $|x|\to\infty$, the eigenfunctions of the linear problem (2.2) can be chosen in the following form (the Jost functions):

$$\Psi^{(1)}(x,k)=e^{-ikx}+o(1)\quad (x\to+\infty)\ , \tag{2.10a}$$

$$\Psi^{(2)}(x,k)=e^{ikx}+o(1)\ , \tag{2.10b}$$

and

$$\Phi^{(1)}(x,k)=e^{-ikx}+o(1)\quad (x\to-\infty)\ , \tag{2.11a}$$

$$\Phi^{(2)}(x,k)=e^{ikx}+o(1)\ . \tag{2.11b}$$

As the operator (2.9a) is a linear operator of the second order, the two sets of eigenfunctions (2.10) and (2.11) are linearly dependent:

$$\Phi^{(i)}(x,k)=\sum_{l=1}^{2}T_{il}(k)\Psi^{(l)}(x,k),\quad i=1,2\ ,$$

where the monodromy matrix $[T_{il}(k)]$ has the well-known form

$$\hat{T}(k)=\begin{bmatrix} a(k) & b(k)\\ b^*(k) & a^*(k)\end{bmatrix}\ , \tag{2.12}$$

and where the Jost coefficients $a(k)$ and $b(k)$ satisfy the unitarity condition

$$|a(k)|^2-|b(k)|^2=1\ . \tag{2.12'}$$

The fundamental property of the functions $\Psi^{(2)}(x,k)$ and $\Phi^{(1)}(x,k)$ and of the Jost coefficient $a(k)$ is that they can be analytically continued into the upper half-plane of the complex parameter k. The function $a(k)$ may have zeros on the upper half-plane at the purely imaginary points $k_n=i\kappa_n$ $(n=1,2,\ldots)$. The set of real numbers κ_n represents the discrete spectrum of the operator $\hat{L}$ defined by Eq. (2.9a). At the points k_n the two eigenfunctions $\Phi^{(1)}(x,k)$ and $\Psi^{(2)}(x,k)$ are linearly dependent,

$$\Phi^{(1)}(x,k_n)=b_n\Psi^{(2)}(x,k_n),\quad b_n\equiv b(k=i\kappa_n)\ ,$$

and they decay exponentially as $|x|\to\infty$. The set of quantities $a(k)$ and $b(k)$ with real k constitutes the continuous-spectrum scattering data of the scattering problem, and the set of real numbers κ_n and complex numbers b_n constitutes the discrete-spectrum scattering data. The time dependence of the scattering data ensues from the second auxiliary linear equation (2.3) with operator $\hat{A}$ from Eq. (2.9b):

$$\begin{aligned} &a(k,t)=a(k,0),\quad b(k,t)=b(k,0)\exp(8ik^3t)\ ,\\ &\kappa_n(t)=\kappa_n(0),\quad b_n(t)=b_n(0)\exp(8\kappa_n^2t)\end{aligned} \tag{2.13}$$

[cf. Eqs. (2.4) and (2.5)].

The inverse scattering problem itself is the problem of reconstructing the wave potential $u(x)$ on the basis of the scattering data. For the KdV equation the problem is principally solved by the relation

$$u(x,t)=-2\frac{d}{dx}K(x,x;t)\ ,$$

where the implicitly time-dependent function $K(x,y)$ satisfies the Gelfand-Levitan-Marchenko equation

$$K(x,y)+F(x+y)=\int_x^\infty K(x,z)F(z+y)dz=0\ , \tag{2.14}$$

in which

$$F(x)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}r(k)dk+\sum_{n=1}^{N}\frac{b_ne^{-\kappa_nx}}{ia'(i\kappa_n)}\ , \tag{2.15}$$

$$a'(i\kappa_n)\equiv\left.\frac{\partial a(k)}{\partial k}\right|_{k=i\kappa_n}\ , \tag{2.16}$$

$$r(k)\equiv b(k)/a(k)\ . \tag{2.17}$$

An important particular case is that of the reflectionless potentials $u(x)$ for which $r(k)\equiv0$, and for which the Gelfand-Levitan-Marchenko equation degenerates into a system of linear algebraic equations. This class of potentials describes solitonic solutions of the KdV equation. The one-soliton solution, with a single zero of $a(k)$ at the point $k_1\equiv i\kappa$, has the form

$$u_s(z)=-2\kappa^2\mathrm{sech}^2z\ , \tag{2.18}$$

where

$$z=\kappa(x-\zeta)\ , \tag{2.19}$$

$\zeta=4\kappa^2t+\zeta_0$. The scattering data and Jost functions corresponding to the potential (2.18) are given in Appendix A.

As is well known, the KdV equation has an infinite set of local polynomial integrals of motion (see, for example, Zakharov *et al.*, 1980), which can be explicitly expressed in terms of the scattering data. We shall need these expressions for the three elementary integrals of motion: the "mass"

$$M \equiv \int_{-\infty}^{+\infty} u\,dx = -4\sum_{n=1}^{N}\kappa_n + \int_0^\infty \mathcal{M}(k)dk\ , \tag{2.20a}$$

$$\mathcal{M}(k) \equiv (2/\pi)\ln|a(k)|^2\ , \tag{2.20b}$$

the momentum

$$P \equiv \tfrac{1}{2}\int_{-\infty}^{+\infty} u^2 dx = \tfrac{8}{3}\sum_{n=1}^{N}\kappa_n^3 + \int_0^\infty dk\,\mathcal{P}(k)\ , \tag{2.21a}$$

$$\mathcal{P}(k) = (4/\pi)k^2\ln|a(k)|^2\ , \tag{2.21b}$$

and the energy

$$E \equiv \int_{-\infty}^{+\infty}(\tfrac{1}{2}u_x^2 + u^3)dx = -\tfrac{32}{5}\sum_{n=1}^{N}\kappa_n^5 + \int_0^\infty \mathcal{E}(k)dk\ , \tag{2.22a}$$

$$\mathcal{E}(k) \equiv (16/\pi)k^2\ln|a(k)|^2\ . \tag{2.22b}$$

In Eqs. (2.20)–(2.22) we have separated the contributions of the solitons ($\sum_n$) and that of the radiation, i.e., the dispersive waves described by the continuous-spectrum scattering data. From that viewpoint, the continuous real spectral parameter k is, simultaneously, the radiation wave number, and Eqs. (2.20b) and (2.22b) are the spectral densities of the "mass," momentum, and energy in the radiation wave field.

The general evolution equations (2.7) and (2.8) take the following form for the perturbed KdV equation (1.1) in the lowest approximation of perturbation theory:

$$\frac{\partial a(k,t)}{\partial t} = \frac{i\epsilon}{2k}\alpha(-k,k)\ , \tag{2.23}$$

$$\frac{\partial b(v,t)}{\partial t} = 8ik^3 b(k,t) - \frac{i\epsilon}{2k}\beta^*(-k,k)\ , \tag{2.24}$$

$$\frac{d\kappa_n}{dt} = -\frac{\epsilon}{2\kappa_n}\alpha(i\kappa_n,i\kappa_n)\Big/\int_{-\infty}^{\infty}[\Psi^{(2)}(x,i\kappa_n)]^2dx\ , \tag{2.25}$$

$$\frac{db_n}{dt} = 8\kappa_n^3 b_n + \frac{\epsilon}{2\kappa_n a'(i\kappa_n)} \times \frac{\partial}{\partial\kappa}[\alpha(-k,i\kappa_n)b_n - \beta(-k,i\kappa_n)]\Big|_{k=-i\kappa_n}\ , \tag{2.26}$$

where

$$\alpha(k',k) \equiv \int_{-\infty}^{+\infty}\Psi^{(2)*}(x,k')\Psi^{(2)}(x,k)P[u]dx\ ,$$

$$\beta(k',k) \equiv \int_{-\infty}^{+\infty}\Phi^{(1)*}(x,k')\Psi^{(2)}(x,k)P[u]dx\ ,$$

and where $a'(i\kappa_n)$ is defined in Eq. (2.16). It can be seen that the time dependence (2.13) of the unperturbed scattering data immediately follows from Eqs. (2.23)–(2.26) when one sets $\epsilon=0$.

C. Nonlinear Schrödinger equation

For the unperturbed NS equation [i.e., Eq. (1.6) with $\epsilon=0$] the operators $\hat{L}$ and $\hat{A}$ have the matrix form (Zakharov and Shabat, 1971)

$$\hat{L} = \begin{bmatrix} -i\dfrac{\partial}{\partial x} & u^*(x) \\ -u(x) & i\dfrac{\partial}{\partial x}\end{bmatrix}\ , \tag{2.27}$$

$$\hat{A} = \begin{bmatrix} -4i\lambda^3 + 2\lambda|u|^2 + u^*u_x - uu_x^* & -4i\lambda^2u^* - 2i\lambda u_x^* + iu_{xx}^* + 2iu^{*2}u \\ -4i\lambda^2 u + 2i\lambda u_x + iu_{xx} + 2iu^2u^* & 4i\lambda^3 - 2\lambda|u|^2 - u_xu^* + uu_x^*\end{bmatrix}\ . \tag{2.28}$$

The Jost functions for the real values of the spectral parameter λ are defined by the boundary conditions [cf. Eqs. (2.10) and (2.11)]

$$\Psi_\pm(x,\lambda) = e^{i\lambda\sigma_3 x} + o(1),\quad x\to\pm\infty\ , \tag{2.29}$$

where σ_3 is the Pauli matrix. The matrix Jost functions (2.29) can be represented in the form

$$\Psi_+ = (\psi,\tilde{\psi}),\quad \Psi_- = (-\tilde{\varphi},\varphi)\ , \tag{2.30}$$

where ψ and φ are independent vector columns, and the linear involution operation (designated by the tilde) acting on the column

$$\psi = \begin{bmatrix}\psi^{(1)}\\ \psi^{(2)}\end{bmatrix}$$

transforms it into

$$\tilde{\psi} = \begin{bmatrix}-\psi^{(2)*}\\ \psi^{(1)}\end{bmatrix}\ . \tag{2.31}$$

The monodromy matrix T relates the two fundamental solutions Ψ_+ and Ψ_-:

$$\Psi_-(x,\lambda) = \Psi_+(x,\lambda)T(\lambda)\ . \tag{2.32}$$

This matrix has the form

$$\hat{T}(\lambda)=\begin{bmatrix} a^*(\lambda) & b(\lambda) \\ -b^*(\lambda) & a(\lambda) \end{bmatrix}, \tag{2.33}$$

where the two Jost coefficients $a(\lambda)$ and $b(\lambda)$ are related by the unitarity condition

$$|a(\lambda)|^2+|b(\lambda)|^2=1 \tag{2.34}$$

[note the difference in the sign between Eqs. (2.12) and (2.31)].

The vector Jost functions $\psi(x,\lambda)$, $\varphi(x,\lambda)$ and the Jost coefficient $a(\lambda)$ admit an analytical continuation into the upper half-plane of the spectral parameter λ. The zeros $\lambda_n=\xi_n+i\eta_n$ $(n=1,2,\ldots)$ of the functions $a(\lambda)$ in the upper half-plane give the discrete spectrum of the corresponding linear problem (2.2), the Jost functions $\psi(x,\lambda_n)$ and $\varphi(x,\lambda_n)$ being linearly dependent:

$$\varphi(x,\lambda_n)=b_n\psi(x,\lambda_n)\ .$$

These functions decay exponentially as $|x|\to\infty$.

The Jost coefficients $a(\lambda)$ and $b(\lambda)$ with real λ constitute the continuous-spectrum scattering data, and the set of complex numbers λ_n and b_n constitutes the discrete-spectrum scattering data of the corresponding scattering problem. The time evolution of these scattering data ensues from the linear equation (2.3) with the operator $\hat{A}$ defined by Eq. (2.28):

$$\begin{aligned} &a(\lambda,t)=a(\lambda,0),\quad b(\lambda,t)=b(\lambda,0)\exp(4i\lambda^2 t)\ ,\\ &\lambda_n(t)=\lambda_n(0),\quad b_n(t)=b_n(0)\exp(4i\lambda_n^2 t)\ . \end{aligned} \tag{2.35}$$

The inverse scattering problem for the operator $\hat{L}$ defined by Eq. (2.27) reduces to a system of singular integral equations,

$$\tilde{\psi}(x,\lambda)e^{i\lambda x}=\begin{bmatrix}0\\1\end{bmatrix}+\sum_{n=1}^{N}\frac{b_n\psi(x,\lambda_n)e^{i\lambda_n x}}{(\lambda-\lambda_n)a'(\lambda_n)}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{r(\lambda')\psi(x,\lambda')e^{i\lambda' x}}{\lambda'-\lambda+i0}d\lambda'\ , \tag{2.36}$$

$$\tilde{\psi}(x,\lambda_n)e^{i\lambda_n^* x}=\begin{bmatrix}0\\1\end{bmatrix}+\sum_{m=1}^{N}\frac{b_m\psi(x,\lambda_m)e^{i\lambda_m x}}{(\lambda_n^*-\lambda_m)a'(\lambda_m)}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{r(\lambda)\psi(x,\lambda)e^{i\lambda x}}{\lambda-\lambda_n^*}d\lambda\ , \tag{2.37}$$

where

$$a'(\lambda_n)\equiv[\partial a(\lambda)/\partial\lambda]|_{\lambda=\lambda_n}\ , \tag{2.38}$$

$$r(\lambda)\equiv b(\lambda)/a(\lambda)\ . \tag{2.39}$$

Finally, the wave potential $u(x,t)$ is expressed in terms of the scattering data and solutions of Eqs. (2.36) and (2.37) as follows:

$$u^*(x,t)=-2\sum_{n=1}^{N}\frac{b_n}{a'(\lambda_n)}\psi^{(1)}(x,\lambda_n)e^{i\lambda_n x}+\frac{1}{\pi i}\int_{-\infty}^{\infty}r(\lambda)\psi^{(1)}(x,\lambda)e^{i\lambda x}d\lambda\ , \tag{2.40}$$

where the upper index designates a component of the vector column.

The reflectionless potentials $u(x,t)$, for which $b(\lambda)\equiv 0$, are solitonic solutions of the NS equation. The reflectionless scattering data with the single zero $\lambda_1=\xi+i\eta$ of the function $a(\lambda)$ correspond to one soliton,

$$u_s(x,t)=2i\eta\frac{\exp(-2i\xi x-i\phi)}{\cosh[2\eta(x-\zeta)]}\ , \tag{2.41}$$

where

$$\phi=4(\xi^2-\eta^2)t+\phi_0,\quad \zeta=-4\xi t+\zeta_0\ . \tag{2.42}$$

The one-soliton Jost functions and the corresponding scattering data are given in Appendix A.

Like the KdV equation, the NS equation possesses an infinite set of local polynomial integrals of motion. We shall need explicit expressions for the three elementary integrals of motion: the "charge" (or "number of quanta")

$$N\equiv\int_{-\infty}^{\infty}|u|^2dx=4\sum_n\eta_n+\int_{-\infty}^{\infty}\mathcal{N}(\lambda)d\lambda\ , \tag{2.43a}$$

$$\mathcal{N}(\lambda)\equiv(1/\pi)\ln|a(\lambda)|^{-2}\ , \tag{2.43b}$$

the momentum

$$P\equiv i\int_{-\infty}^{\infty}uu_x^*dx=-8\sum_n\xi_n\eta_n+\int_{-\infty}^{\infty}\mathcal{P}(\lambda)d\lambda\ , \tag{2.44a}$$

$$\mathcal{P}(\lambda)\equiv-2\lambda\mathcal{N}(\lambda)=-(2\lambda/\pi)\ln|a(\lambda)|^{-2}\ , \tag{2.44b}$$

and the energy

$$\begin{aligned} E&\equiv\int_{-\infty}^{\infty}(|u_x|^2-|u|^4)dx\\ &=16\sum_n(-\tfrac{1}{3}\eta_n^3+\xi_n^2\eta_n)+\int_{-\infty}^{\infty}\mathcal{E}(\lambda)d\lambda\ , \end{aligned} \tag{2.45a}$$

$$\mathcal{E}(\lambda)\equiv4\lambda^2\mathcal{N}(\lambda)=(4\lambda^2/\pi)\ln|a(\lambda)|^{-2}\ . \tag{2.45b}$$

In Eqs. (2.43)–(2.45) we have separated the soliton contribution ($\sum_n$) from that of the radiative component ($\int d\lambda$) of the NS wave field described by the continuous-spectrum scattering data. If we consider the radiation component as a superposition of free waves governed by the linear Schrödinger equation, the spectral parameter λ is, up to the multiplier $\frac{1}{2}$, the wave number of the radiation, and the quantities (2.43b), (2.40b), and (2.41b) have the sense of spectral densities of the "charge," momentum, and energy carried by the radiation wave field.

When a perturbation is present in Eq. (1.6), the evolution equations (2.7) and (2.8) take, in the lowest approximation, the form [cf. Eqs. (2.23)–(2.26)]

$$\frac{\partial a(\lambda,t)}{\partial t}=\epsilon\int_{-\infty}^{+\infty}dx\,P[u]\psi^{(1)}(x,\lambda)\varphi^{(2)}(x,\lambda)+\epsilon^*\int_{-\infty}^{\infty}dx\,P^*[u]\psi^{(2)}(x,\lambda)\varphi^{(1)}(x,\lambda)\ , \tag{2.46}$$

$$\frac{\partial b(\lambda,t)}{\partial t}=4i\lambda^2 b+\epsilon\int_{-\infty}^{\infty}dx\,P[u]\varphi^{(1)}(x,\lambda)\psi^{(2)*}(x,\lambda)-\epsilon^*\int_{-\infty}^{\infty}dx\,P^*[u]\varphi^{(2)}(x,\lambda)\psi^{(1)*}(x,\lambda)\ , \tag{2.47}$$

$$\frac{d\lambda_n}{dt}=-\frac{1}{a'(\lambda_n)}\int_{-\infty}^{+\infty}dx\{\epsilon P[u]\psi^{(1)}(x,\lambda_n)\varphi^{(2)}(x,\lambda_n)+\epsilon^*P^*[u]\psi^{(2)}(x,\lambda_n)\times\varphi^{(1)}(x,\lambda_n)\}\ , \tag{2.48}$$

$$\frac{db_n}{dt}=4i\lambda_n^2 b_n+\frac{b_n}{a'(\lambda_n)}\int_{-\infty}^{+\infty}dx\{\epsilon P[u]Q^{(1)}(x,\lambda_n)+\epsilon^*P^*[u]Q^{(2)}(x,\lambda_n)\}\ . \tag{2.49}$$

Here

$$Q^{(m)}(x,\lambda_n)\equiv\frac{\partial}{\partial\lambda}[\varphi^{(m)}(x,\lambda_n)\psi^{(m)}(x,\lambda)-\varphi^{(m)}(x,\lambda)\psi^{(m)}(x,\lambda_n)]|_{\lambda=\lambda_n}\ ,\quad m=1,2\ , \tag{2.50}$$

where $a'(\lambda_n)$ is defined in Eq. (2.38).

D. Sine-Gordon equation

For the unperturbed SG equation (1.15) with $\epsilon=0$, Lax's representation can be conveniently written in the form (see, for example, Zakharov *et al.*, 1980)

$$[\hat{L},\hat{A}]=0\ ,$$

where

$$\hat{L}=\hat{I}\frac{\partial}{\partial x}-\frac{i}{2}\left[\left[\lambda-\frac{1}{4\lambda}\cos u\right]\sigma_3-\frac{\sigma_2}{4\lambda}\sin u+\frac{\sigma_1}{2}(u_x-u_t)\right]\ , \tag{2.51}$$

$$\hat{A}=\hat{I}\frac{\partial}{\partial x}+\frac{i}{2}\left[\left[\lambda+\frac{1}{4\lambda}\cos u\right]\sigma_3+\frac{\sigma_2}{4\lambda}\sin u+\frac{\sigma_1}{2}(u_x-u_t)\right]\ . \tag{2.52}$$

Here $\hat{I}$ is the unit matrix, σ_α ($\alpha=1,2,3$) are the Pauli matrices, and λ is the real spectral parameter ($0\le\lambda<\infty$).

The boundary conditions for the wave potential are $u(x,t)\to 0$ (mod 2π), $u_t(x,t)\to 0$ at $|x|\to\infty$. The Jost functions are defined as solutions of the linear equation $\hat{L}\Psi=0$ with the boundary conditions

$$\Psi_\pm(x,\lambda)=\exp\left[\frac{i}{2}k(\lambda)\sigma_3 x\right]+o(1)\ , \tag{2.53}$$

where σ_3 is the Pauli matrix, and

$$k(\lambda)\equiv\lambda-1/4\lambda\ . \tag{2.54}$$

Let us introduce the two vector columns ψ and φ according to Eq. (2.30), and the involution operation according to Eq. (2.31). The monodromy matrix is defined by Eq. (2.32) and has the form (2.33); the unitarity condition retains the form (2.34).

The analytical properties of the Jost functions and the definition of the discrete spectrum also coincide with those for the NS equation, except for the additional reduction imposed by the reality of $u(x,t)$:

$$\begin{aligned}&\psi^{(1)}(x,\lambda)=\psi^{(1)*}(x,-\lambda)\ ,\\&\psi^{(2)}(x,\lambda)=\psi^{(2)*}(x,-\lambda)\ ,\\&\varphi^{(1)}(x,\lambda)=-\varphi^{(1)*}(x,-\lambda)\ ,\\&\varphi^{(2)}(x,\lambda)=\varphi^{(2)*}(x,-\lambda)\ ,\\&a(\lambda)=a^*(-\lambda),\quad b(\lambda)=-b^*(-\lambda)\ .\end{aligned} \tag{2.55}$$

The time dependence of the scattering data following from the linear equation $\hat{A}\Psi=0$ has the form

$$\begin{aligned}&a(\lambda,t)=a(\lambda,0)\ ,\\&b(\lambda,t)=b(\lambda,0)\exp[-i\omega(\lambda)t]\ ,\\&\lambda_n(t)=\lambda_n(0)\ ,\\&b_n(t)=b_n(0)\exp[-i\omega(\lambda_n)t]\ ,\end{aligned}$$

where

$$\omega(\lambda)=\lambda+1/4\lambda\ . \tag{2.56}$$

The inverse scattering problem reduces to a system of singular integral equations [cf. Eqs. (2.36) and (2.37)]:

$$\tilde{\psi}(x,\lambda)e^{ik(\lambda)x/2}=\begin{bmatrix}0\\1\end{bmatrix}+\sum_{n=1}^{N}\frac{b_n\psi(x,\lambda_n)}{a'(\lambda_n)(\lambda-\lambda_n)}e^{ik(\lambda_n)x/2}+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}d\lambda'\frac{r(\lambda')\psi(x,\lambda')}{\lambda'-\lambda+i0}e^{ik(\lambda')x/2}\ , \tag{2.57}$$

$$\tilde{\psi}(x,\lambda_n)e^{ik(\lambda_n^*)x/2}=\begin{bmatrix}0\\1\end{bmatrix}+\sum_{m=1}^{N}\frac{b_m\psi(x,\lambda_m)}{a'(\lambda_m)(\lambda_n^*-\lambda_m)}e^{ik(\lambda_m)x/2}+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}d\lambda\frac{r(\lambda)\psi(x,\lambda)}{\lambda-\lambda_n^*}e^{ik(\lambda)x/2}\ , \tag{2.58}$$

where definitions (2.38) and (2.39) are used. The wave potential $u(x,t)$ is expressed in terms of solutions of the system (2.57) and (2.58) as follows:

$$\cos(u/2)=(-1)^{u(+\infty)/2\pi}\left[1-\sum_{n=1}^{N}\frac{b_n}{\lambda_n a'(\lambda_n)}\psi^{(2)}(x,\lambda_n)e^{ik(\lambda_n)x/2}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\lambda\frac{r(\lambda)}{\lambda}\psi^{(2)}(x,\lambda)e^{ik(\lambda)x/2}\right]. \tag{2.59}$$

As above, reflectionless scattering data $[r(\lambda)\equiv 0]$ give rise to purely solitonic solutions, and zeros of the analytical function $a(\lambda)$ in the upper half-plane correspond to solitons. The important difference from the NS equation is that, due to the reduction (2.55), a zero lying at the point λ_n must have a corresponding twin zero at the point $-\lambda_n^*$. Consequently there may be zeros of two distinct types: individual zeros lying on the imaginary axis, and twin zeros lying at the symmetric points $(\pm\mathrm{Re}\lambda_n+i\,\mathrm{Im}\lambda_n)$.

An individual zero at the point

$$\lambda_1=i\nu\equiv\frac{i}{2}\sqrt{(1+V)/(1-V)} \tag{2.60}$$

gives rise to a topological soliton usually called a *kink*:

$$u_k(x,t)\equiv u_k(z)=4\tan^{-1}(e^{\sigma z}),\quad z=(x-\zeta)/\sqrt{1-V^2}, \tag{2.61a}$$

$$\zeta=Vt+\zeta_0, \tag{2.61b}$$

where V, as in Eq. (2.60), is the kink's velocity ($V^2<1$), and $\sigma=\pm1$ is the kink's polarity (topological charge). Solutions with $\sigma=+1$ and -1 are commonly referred to as kinks and antikinks. Note that the set of the numbers σ_n for all kinks constitutes additional information, which should be added to the aforementioned scattering data to completely specify a solution of the SG equation in terms of the IST. The scattering data and Jost functions corresponding to the kink solution (2.61) are given in Appendix A.

A pair of twin zeros can be conveniently represented in the form

$$\lambda_{1,2}=\pm\tfrac{1}{2}\sqrt{(1+V)/(1-V)}\exp(i\mu),\quad 0<\mu\leq\frac{\pi}{2},\quad V^2<1. \tag{2.62}$$

This pair gives rise to the so-called *breather* solution, which may be regarded as a bound kink-antikink state moving with the velocity V and performing internal oscillations with the frequency $\cos\mu$:

$$u_{\mathrm{br}}(x,t)=4\tan^{-1}\left[\tan\mu\frac{\sin(\Psi\cos\mu)}{\cosh(z\sin\mu)}\right], \tag{2.63}$$

$$z=(x-\xi)/\sqrt{1-V^2},\quad \xi=Vt+\xi_0, \tag{2.64a}$$

$$\Psi=(t-Vx)/\sqrt{1-V^2}+\Psi_0. \tag{2.64b}$$

The parameter μ is the breather's amplitude: in the case $\mu\ll1$, Eq. (2.63) simplifies to

$$\mu_{\mathrm{br}}(x,t)\approx4\mu\sin\left[\left[1-\frac{\mu^2}{2}\right]\Psi\right]\mathrm{sech}(\mu z), \tag{2.65}$$

and in the opposite case $\zeta\equiv\pi/2-\mu\ll1$, Eq. (2.63) may be represented in the form

$$u_{\mathrm{br}}(x,t)\approx4\tan^{-1}[\zeta^{-1}\sin(\zeta\Psi)\mathrm{sech}z]. \tag{2.66}$$

The approximate solutions (2.65) and (2.66) will hereafter be called, respectively, a small-amplitude breather and a low-frequency breather. In the limit case $\mu=\pi/2$ the exact solution represented by Eqs. (2.63) and (2.64) becomes (here we set $V=0$)

$$u=4\tan^{-1}(t\,\mathrm{sech}x). \tag{2.67}$$

Equation (2.67) may be interpreted as describing a kink-antikink pair separated by the distance L, which depends on the time t according to the following asymptotic law valid as $|t|\to\infty$:

$$L\approx\ln|t|. \tag{2.68}$$

Note that a solution obtained from Eqs. (2.63) and (2.64) (with $V=0$) by means of the analytical continuation $\mu=\pi/2+iW$ ($W\ll1$) describes as $|t|\to\infty$ a kink-antikink pair with a relative velocity W; the particular solution (2.67) corresponds to $W=0$.

Like other integrable equations, the SG equation possesses an infinite set of local polynomial conserved quantities. We shall need explicit expressions for the two elementary integrals of motion, viz., energy E and momentum P, in terms of the scattering data [cf. Eqs. (2.20)–(2.22) and (2.43)–(2.45)]:

$$E\equiv\int_{-\infty}^{+\infty}dx[\tfrac{1}{2}u_t^2+\tfrac{1}{2}u_x^2+(1-\cos u)]=8\sum_n(1-V_n^2)^{-1/2}+16\sum_m(1-V_m^2)^{-1/2}\sin\mu_m+\int_{-\infty}^{+\infty}\mathcal{E}(k)dk, \tag{2.69a}$$

$$\mathcal{E}(k)\equiv(4/\pi)\ln(1-|b[\lambda(k)]|^2), \tag{2.69b}$$

$$P\equiv-\int_{-\infty}^{\infty}dx\,u_xu_t=8\sum_nV_n(1-V_n^2)^{-1/2}+16\sum_mV_m(1-V_m^2)^{-1/2}\sin\mu_m+\int_{-\infty}^{+\infty}\mathcal{P}(k)dk, \tag{2.70a}$$

$$\mathcal{P}(k)\equiv\frac{4k}{\pi\sqrt{k^2+1}}\ln(1-|b[\lambda(k)]|^2), \tag{2.70b}$$

where k and $\lambda(k)$ are related by Eq. (2.54). In Eqs. (2.69a) and (2.70a) we have separated the contributions from the kinks ($\sum_n$), breathers ($\sum_m$), and radiation

($\int dk$). The quantities (2.69b) and (2.70b) have the sense of spectral densities of the energy and momentum carried by the radiative component of the SG wave field, while the continuous parameter k [see Eq. (2.54)] has the sense of the radiation wave number. Analogously, the parameter ω defined in Eq. (2.56) has the sense of the radiation frequency. Clearly, they are related by the dispersion relation of the linearized SG equation, $\omega^2=k^2+1$.

When a perturbation is taken into account in Eq. (1.15), the evolution equations for the scattering data take the form [cf. Eqs. (2.46)–(2.49)]

$$\frac{\partial a(\lambda,t)}{\partial t}=-\frac{i\epsilon}{4}\int_{-\infty}^{+\infty}dx\,P[u]W(\psi,\sigma_1\varphi)\,, \tag{2.71}$$

$$\frac{\partial b(\lambda,t)}{\partial t}=-i\omega(\lambda)b(\lambda,t)-\frac{i\epsilon}{4}\int_{-\infty}^{\infty}dx\,P[u]W(\sigma_1\varphi,\tilde{\psi})\,, \tag{2.72}$$

$$\frac{d\lambda_n}{dt}=\frac{i\epsilon}{4a'(\lambda_n)}\int_{-\infty}^{+\infty}dx\,P[u]W(\psi_n,\sigma_1\varphi_n)\,, \tag{2.73}$$

$$\frac{db_n}{dt}=-i\omega(\lambda_n)b_n-\frac{i\epsilon}{4a'(\lambda_n)}\int_{-\infty}^{+\infty}dx\,P[u]W(\sigma_1\varphi_n,\varphi_n'-b_n\psi_n')\,, \tag{2.74}$$

where the Wronskian of the two columns ψ and φ is $W(\psi,\varphi)\equiv\psi^{(1)}\varphi^{(2)}-\psi^{(2)}\varphi^{(1)}$, and $\psi_n'\equiv[\partial\psi(x,\lambda)/\partial\lambda]|_{\lambda=\lambda_n}$.

E. Adiabatic and radiative effects

The natural separation of the discrete and continuous spectra, i.e., solitonic and radiative degrees of freedom, in the framework of the IST suggests an analogous separation of modes in the presence of perturbations. Of course, in the latter case the separation may be only approximate; however, as the influence of emitted radiation on solitons appears in second-order perturbation theory, in the lowest (first) approximation we may study perturbation-induced dynamics of solitons ignoring excitation of the radiative degrees of freedom. In this approximation, it is sufficient to take into account only the evolution equations (2.8) for the discrete-spectrum scattering data; corresponding evolution equations for solitonic parameters are frequently called *adiabatic* (Karpman and Maslov, 1977). Typical examples of adiabatic effects are dissipative damping of a soliton, fusion of colliding solitons into a bound state on account of dissipative losses, and energy and momentum exchange in three- (or many-) soliton collisions under the action of conservative perturbations.

Typical *radiative* effects are energy emission by a soliton moving in a potential relief or by a soliton oscillating under the action of an external time-dependent field. Another important effect is emission of radiation by colliding solitons. The emitted radiation exerts reciprocal influence on the emitting solitons. The most natural and convenient way to take that influence into account is to use conservation laws. For instance, calculating net energy emitted by colliding solitons, one can find a threshold (maximum) value of their kinetic energy which admits fusion of the solitons into a bound state on account of the radiative losses.

III. ADIABATIC DYNAMICS OF ONE SOLITON

A. Korteweg–de Vries equation

1. Spatially homogeneous perturbations

Evolution of a KdV soliton under the action of dissipative perturbations was the first particular problem investigated in the framework of perturbation theory. Evolution equations for the parameters κ and ξ of the KdV soliton (2.18) can be easily derived from the general evolution equations (2.25) and (2.26) for the discrete-spectrum scattering data with the use of the one-soliton Jost functions (A1) and (A2) (Karpman and Maslov, 1977):

$$\frac{d\kappa}{dt}=-\frac{\epsilon}{4\kappa}\int_{-\infty}^{+\infty}dz\,P(u)\mathrm{sech}^2 z\,, \tag{3.1}$$

$$\frac{d\xi}{dt}=4\kappa^2-\frac{\epsilon}{4\kappa^3}\int_{-\infty}^{+\infty}dz\,P(u)[z+\tfrac{1}{2}\sinh(2z)]\mathrm{sech}^2 z\,. \tag{3.2}$$

As is well known (Karpman and Maslov, 1978; Kaup and Newell, 1978a; Newell, 1980), a dissipative perturbation acting upon a KdV soliton gives rise to a long shelf of amplitude $\sim\epsilon$ (see details in Sec. VI.N). Karpman and Maslov (1978) have demonstrated that consistent description of the shelf requires the use of an additional term $\tanh^2 z$ in the square brackets on the right-hand side of Eq. (3.2). However, mechanical insertion of this additional term into the soliton's equations of motion can sometimes result in contradictions; see the paper by Malomed (1988b).

The damping law of the soliton's amplitude ensuing from Eq. (3.1) is evident. For instance, for the perturbation (1.2a) it has the form

$$\frac{d\kappa}{dt}=-\tfrac{2}{3}\alpha\kappa\,. \tag{3.3}$$

Another important example is the so-called Korteweg–de Vries–Burgers equation with the perturbation (1.2b),

$$u_t-6uu_x+u_{xxx}=\gamma u_{xx}\,. \tag{3.4}$$

Insertion of the perturbation from the right-hand side of Eq. (3.4) into Eq. (3.1) yields the following damping law:

$$\frac{d\kappa}{dt}=-\tfrac{16}{15}\gamma\kappa^3\,. \tag{3.5}$$

It should be noted that the Korteweg–de Vries–Burgers equation (3.4) has an exact solution in the form of the so-called oscillatory shock wave, which may move with an arbitrary velocity V. This wave structure is distinguished by the boundary conditions $u(-\infty)=-V/3$, $u(+\infty)=0$ in the case $V>0$, and $u(-\infty)=0$, $u(+\infty)=-V/3$ in the case $V<0$. From the viewpoint of the perturbation theory for solitons, the oscillatory shock wave may be regarded as a bound state of a large number of KdV solitons (see details in Karpman, 1979b).

Let us proceed to consideration of evolution of a KdV soliton under the action of the linear Landau damping described by the perturbative term

$$-\mu \int_{-\infty}^{+\infty} u_{x'}(x-x')^{-1} dx' \tag{3.6a}$$

in the right-hand side of the KdV equation (where μ is a small parameter). According to Ott and Sudan (1969), this perturbation gives rise to an evolution equation for the soliton's amplitude κ which can be represented in the following form [cf. Eqs. (3.3) and (3.5)]:

$$\frac{d\kappa}{dt}=-c_1\mu\kappa^2, \quad c_1\approx 1.46 \tag{3.7a}$$

(by implication $\mu \ll \kappa^2$). Note that the dissipation (1.2a) results in the exponential decay of the amplitude, $\kappa(t)=\kappa_0\exp(-\frac{2}{3}\alpha t)$ [see Eq. (3.3)], while in cases (1.2b) and (3.6a) the decay law is algebraic: $\kappa(t)\sim t^{-1/2}$ in (1.2b), and $\kappa(t)\sim t^{-1}$ in (3.6a) [see Eqs. (3.5) and (3.7a)].

In studying the propagation of internal-wave solitons in a stratified liquid with an unstable shear flow, Ostrovskii, Stepanyants, and Tsimring (1984a) encountered the perturbative term (3.6a) with negative μ. They demonstrated that, in such a situation, the higher stabilizing term

$$\nu \int_{-\infty}^{+\infty} u_{x'x'x'}(x-x')dx' \tag{3.6b}$$

[and the term (1.2b)] should be taken into account. In the presence of the term (3.6b), Eq. (3.7a) takes the form

$$\frac{d\kappa}{dt}=-c_1\mu\kappa^2-c_2\nu\kappa^4, \quad c_2\approx 2.56 \; . \tag{3.7b}$$

While Eq. (3.7a) with $\mu<0$ describes explosive growth of the soliton's amplitude, $\kappa(t)=-(c_1\mu)^{-1}(t_0-t)^{-1}$, $t_0=|c_1\mu\kappa(0)|^{-1}$, Eq. (3.7b) with $\mu<0$, $\nu>0$ has the stable stationary solution $\kappa_0=\sqrt{-c_1\mu/c_2\nu}$.

Ostrovskii, Stepanyants, and Tsimring (1984a) have also derived evolution equations for parameters of the so-called cnoidal wave (a periodic chain of KdV solitons) subject to the action of the dissipative terms (1.2a), (1.2b), (3.6a), and (3.6b).

Another variant of a perturbed KdV equation is one with slowly varying coefficients:

$$v_\tau+\alpha(\epsilon\tau)vv_x+\beta(\epsilon\tau)v_{xxx}=0 \; , \tag{3.8}$$

where ϵ is a small parameter. As has been demonstrated by Karpman and Maslov (1982), the substitution

$$t=\int_0^\tau \beta(\epsilon\tau')d\tau' \; ,$$

$$u(x,t)=\tfrac{1}{6}\sigma v(x,\tau), \quad \sigma\equiv\alpha/\beta \; ,$$

transforms Eq. (3.8) into the form of Eq. (1.1) with the effective perturbation

$$P(u)=(\sigma_t/\sigma)u \; . \tag{3.8$'$}$$

Karpman and Maslov (1982) have also developed a similar analysis for a NS equation with variable coefficients.

Another approach to Eq. (3.8), based upon the method of matched asymptotic expansions, was put forward by Grimshaw (1979a; see also Grimshaw, 1983) and Ko and Kuehl [1978; later, Ko and Kuehl (1980) extended that approach to a modified KdV equation with variable coefficients]. Karpman and Maslov (1982) have demonstrated that both methods yield the same results.

Another tractable case is that when coefficients of the KdV equation (3.8) change by a jump at some $t=t_0$: $\alpha(t)=\alpha_\mp$, and $\beta(t)=\beta_\mp$ for $t<t_0$ and $t>t_0$, respectively. As noted first by Tappert and Zabusky (1971) [see also Johnson (1973), Zhong and Shen (1983)], this problem suggests direct application of the unperturbed inverse scattering transform: the wave field at $t-t_0=-0$ must be considered, after proper rescaling, as an initial condition for the unperturbed KdV equation at $t-t_0>0$. In particular, an initial soliton existing at $t<t_0$,

$$u_{\rm sol}=12(\beta_-/\alpha_-)\kappa^2\operatorname{sech}^2[\kappa(x-4\kappa^2\beta_- t)] \; ,$$

[cf. Eq. (2.18)] may split at $t>t_0$ into several secondary ones [plus a nonsoliton (radiation) component]. Tappert and Zabusky (1971) obtained an expression for the number N of secondary solitons:

$$N=[\tfrac{1}{2}\{\sqrt{1+4(\alpha_+\beta_-)/(\alpha_-\beta_+)}-1\}] \; ,$$

where square brackets indicate the integer part.

Tappert and Zabusky (1971) mentioned that the same approach could be applied to other integrable equations with steplike coefficients, e.g., the NS equation. Later, Djordjevic and Redekopp (1978) considered in this manner the Joseph equation, which is an integrable generalization of the KdV equation.

2. Interaction of a Korteweg–de Vries soliton with a moving dipole or a pump wave

The KdV equation with a moving dipole-type perturbation on its right-hand side,

$$u_t-6uu_x+u_{xxx}=\epsilon\delta'(x-Vt) \; , \tag{3.9}$$

has been the subject of intensive studies (Akylas, 1984; Cole, 1985; Mei, 1986; Wu, 1987). Equation (3.9) is the simplest model to describe propagation of unidirectional disturbances produced by a small moving body in a shallow liquid layer. The variable $u(x,t)$ has the sense of an elevation of a free surface. In this subsection we shall describe the interaction of a KdV soliton with a dipole moving with a positive velocity $V>0$. This discussion is taken from the recent paper of Malomed (1988b).

Due to the positiveness of the dipole's velocity, it moves in the same direction as the soliton (2.18), which will be written in the form

$$u(x,t)=-2\kappa^2\mathrm{sech}^2\{\kappa[x-\xi(t)]\}\ , \tag{3.10}$$

$$\frac{d\xi}{dt}=4\kappa^2\ . \tag{3.11}$$

The soliton-dipole interaction may be significant, provided the dipole's velocity V and the soliton's velocity $4\kappa^2$ are close. To consider that interaction by means of perturbation theory, one should assume $\epsilon^2 \ll V^3$. Then a straightforward analysis yields the following evolution equations for the parameters of the soliton:

$$\frac{dW}{dt}=\epsilon V \sinh\zeta\,\mathrm{sech}^3\zeta\ , \tag{3.12}$$

$$\frac{d}{dt}\zeta=\frac{\sqrt{V}}{2}W+\frac{\epsilon}{2}\mathrm{sech}^2\zeta\ , \tag{3.13}$$

where $W\equiv 4\kappa^2-V$, $\zeta\equiv\kappa(\xi-Vt)$. The dynamical system (3.12) and (3.13) has the stationary point

$$W=W_0\equiv-\frac{\epsilon}{\sqrt{V}},\quad \zeta=0\ . \tag{3.14}$$

This stationary point corresponds to a soliton pinned by the moving dipole. In the framework of the dynamical system (3.12) and (3.13), the point (3.14) is a center if $\epsilon<0$, and a saddle in the opposite case. [It will be shown in Sec. VI.M that, within the framework of the full equation (3.9), the pinned state of the soliton corresponding to Eqs. (3.14) (with $\epsilon<0$) is subject to a radiative instability.] The eigenfrequency of small oscillations in the vicinity of the center is $\Omega=V^{3/4}\sqrt{-\epsilon/2}$. A phase plane of the dynamical system (3.12) and (3.13) is depicted in Fig. 1. The boundary between free and trapped trajectories is the separatrix (shown as a heavier line in Fig. 1), which is approximately described by the equation

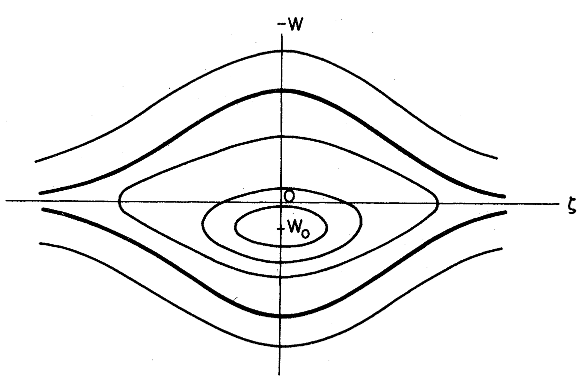

FIG. 1. The phase plane of the dynamical systems (3.12) and (3.13) describing the interaction of a KdV soliton with a moving dipole.

$$\sinh\zeta=(\epsilon V^{3/2}/\sqrt{2})^{1/2}t\ . \tag{3.14'}$$

Malomed (1988b) gives a full nonperturbative classification of solitary wave profiles pinned by the dipole moving with the positive velocity. In particular, he demonstrates that stationary pinned profiles exist in the parametric range $\epsilon\leq(2/3\sqrt{3})V^{3/2}$. In the range $\epsilon>(2/3\sqrt{3})V^{3/2}$ (as well as for $V<0$) there is no pinned profile, and one may expect to observe emission of solitons by the moving dipole. Such phenomena have indeed been seen in numerical experiments; see the papers quoted at the beginning of this subsection.

Another interesting example of KdV soliton dynamics in the presence of an inhomogeneous perturbation was considered by Gasch *et al.* (1986). They showed that the formation of stationary wave forms in a driven nonlinear microwave resonator with dissipation can be attributed to generation and parametric amplification of KdV solitons by a pump wave traveling in the same direction as the solitons. The interaction process and the soliton propagation may be described in terms of the perturbed KdV equation

$$u_t-6uu_x+u_{xxx}=-au+bu_{xx}+(fu)_x\ , \tag{3.15}$$

where, in the special case of a harmonic pump wave, f is given by

$$f(x,t)=V_p\cos[\omega_p(x-Wt)]\ ,$$

V_p being the pumping amplitude. Here t and x are the spatial coordinate and transformed time, respectively. As was demonstrated by Gasch *et al.* (1986), a KdV soliton can only be generated and stabilized if the amplitude of the pump wave V_p exceeds a threshold value $(V_p)_{\rm thr}$. Using the perturbation theory equations (3.1) and (3.2), one can find

$$(V_p)_{\rm thr}=(a-Wb+4b^3)/(\omega_p^2+b^2)^{1/2}\ . \tag{3.16}$$

The perturbed KdV equation (3.15) was solved numerically by Gasch *et al.* (1986). They demonstrated that Eq. (3.16) is in good agreement with the numerical calculations and, moreover, it describes experimentally observed solitonic modes of a high-frequency ring resonator.

B. Nonlinear Schrödinger equation

The general evolution equations for the parameters of one NS soliton (2.41), analogous to Eqs. (3.1) and (3.2), are

$$\frac{d\eta}{dt}=-\tfrac{1}{2}\mathrm{Re}\left[\epsilon\int_{-\infty}^{+\infty}dz\,P[u_s(z)]e^{2i\xi x+i\phi}\mathrm{sech}z\right]\ , \tag{3.17}$$

$$\frac{d\xi}{dt}=\tfrac{1}{2}\mathrm{Im}\left[\epsilon\int_{-\infty}^{\infty}dz\,P[u_s(z)]\frac{\tanh z}{\cosh z}e^{2i\xi x+i\phi}\right]\ , \tag{3.18}$$

$$\frac{d\xi}{dt}=-4\xi-\frac{1}{4\eta^2}\mathrm{Re}\left[\epsilon\int_{-\infty}^{\infty}dz\frac{zP[u_s(z)]}{\cosh z}e^{2i\xi x+i\phi}\right], \quad (3.19)$$

$$\frac{d\phi}{dt}=4(\xi^2-\eta^2)+\frac{1}{2\eta}\mathrm{Im}\left[\epsilon\int_{-\infty}^{\infty}dz\frac{1-2\eta x\tanh z}{\cosh z}P[u_s(z)]\times e^{2i\xi x+i\phi}\right]. \quad (3.20)$$

The simplest problem is the damping and braking of a NS soliton (2.41) under the action of the dissipative perturbation (1.8). The resulting evolution equations for the soliton's amplitude η and velocity $V=-4\xi$ are (Ott and Sudan, 1969; Karpman and Maslov, 1977; Nozaki and Bekki, 1983)

$$\frac{d\eta}{dt}=-2\eta(\alpha_1+\tfrac{4}{3}\alpha_2\eta^2+\tfrac{8}{3}\alpha_3\eta^2)-\tfrac{1}{2}\alpha_2\eta V^2, \quad (3.21a)$$

$$\frac{dV}{dt}=-\tfrac{16}{3}\alpha_2\eta^2 V. \quad (3.21b)$$

Note that the dissipative terms $\sim\alpha_1$ and $\sim\alpha_3$ in the first approximation do not affect the soliton's velocity.

The case when the coefficient α_1 in Eq. (1.8) is negative is also physically meaningful: it describes development of a long-wave instability in a nonlinear medium. Investigation of the dynamics of the NS soliton (2.41) under the action of this perturbation has demonstrated that, in contrast with the unperturbed case, in which the soliton's amplitude η and velocity -4ξ may take arbitrary values, in the presence of this perturbation their stationary values are determined uniquely: $\eta^2=\frac{3}{4}\alpha_1/(\alpha_2+2\alpha_3)$ and $\xi=0$ (Pereira and Stenflo, 1977). However, it is clear that the corresponding soliton is unstable, since as $|x|\to\infty$, $u(x,t)$ coincides asymptotically with the unstable trivial solution $u=0$. At the same time, it is interesting to consider the case in which a metastable trivial solution is unstable against finite perturbations (a system with hard excitation). The simplest model is (Petviashvili and Sergeev, 1984)

$$P(u)=-i\alpha_1 u+i\alpha_2 u_{xx}+i\alpha_3|u|^2-i\alpha_4|u|^4u,$$

where all the coefficients α_1 to α_4 are positive, and the last term is necessary to provide global stability. It is easy to determine that a hard excitation does indeed take place under the condition

$$\alpha_3\geq 2\sqrt{\alpha_1\alpha_4}. \quad (3.22)$$

As before, the solution (2.41) may exist with zero velocity only, and the evolution equation for its amplitude in the first order of perturbation theory is [cf. Eq. (3.21a)]

$$\frac{d}{dt}\eta=2\eta(-\alpha_1-\tfrac{4}{3}\alpha_2\eta^2+\tfrac{8}{3}\alpha_3\eta^2-\tfrac{128}{15}\alpha_4\eta^4). \quad (3.23)$$

According to Eq. (3.23), the stationary values of the amplitudes are

$$\eta^2=\frac{1}{64\alpha_2}(5(2\alpha_3-\alpha_2)\pm\{5[5(2\alpha_3-\alpha_2)^2-96\alpha_1\alpha_4]\}^{1/2}), \quad (3.23')$$

where the upper and lower signs correspond, respectively, to stable and unstable solitons (Malomed, 1987j). As can be seen from Eq. (3.23), the condition $\alpha_3\geq\alpha_2/2+\sqrt{(6/5)\alpha_1\alpha_4}$, necessary for the existence of a soliton, is more restrictive than the general hard-excitation condition (3.22).

Recently, Joets and Ribotta (1988) observed a stable solitonlike pulse of oscillatory convection in a layer of a nematic liquid crystal heated from below. Malomed (1988f) has interpreted this solitary pulse as a soliton in a system with hard excitation [see also Thual and Fauve (1988)].

The coefficient α_2 in Eq. (1.8) may be negative too. In this case, it is natural to assume $\alpha_1=0$, and it is necessary to supplement the perturbation (1.8) by the additional stabilizing term $-i\alpha_4 u_{xxxx}$ ($\alpha_4>0$). The corresponding perturbation-induced evolution equations for the soliton's parameters have been derived by Pismen (1987), who demonstrated that in this case the perturbation may support an equilibrium (stationary) soliton with a nonzero velocity. With a change of parameters α_j, bifurcations between the quiescent and steadily moving equilibrium solitons take place.

Another interesting example of a dissipatively perturbed NS equation is

$$iu_t+u_{xx}+2|u|^2u=\epsilon u\int_{-\infty}^{+\infty}|u(x')|^2(x-x')^{-1}dx', \quad (3.24)$$

where ϵ is a small real parameter that describes the nonlinear Landau damping of the Langmuir wave envelope (Ichikawa and Taniuti, 1973; see also Pécseli and Dysthe, 1977). Dynamics of one soliton described by Eq. (3.24) was studied in the adiabatic approximation by Ichikawa (1979). He derived the following evolution equation for the soliton velocity and amplitude:

$$\frac{dV}{dt}=\beta\epsilon\eta^2,\quad \frac{d\eta}{dt}=0, \quad (3.25)$$

where $\beta\equiv(192/\pi^3)\zeta(3)\approx 7.44$. Thus nonlinear Landau damping acts upon a soliton as a constant force. Analogous results have been obtained by Kodama and Hasegawa (1986) for the equation

$$iu_t+u_{xx}+2|u|^2u=-\tfrac{1}{2}\epsilon|u_x|^2u, \quad (3.26)$$

ϵ being again a small real parameter. Equation (3.26) describes the evolution of an electromagnetic wave envelope in a nonlinear one-mode waveguide with regard to nonlinear dissipation generated by the induced Raman scattering. According to Kodama and Hasegawa (1986), the evolution equations for the soliton's parameters are [cf. Eq. (3.25)]

$$\frac{dV}{dt}=\tfrac{128}{15}\epsilon\eta^4,\quad \frac{d\eta}{dt}=0\ . \tag{3.25'}$$

A problem of practical importance is to compensate for dissipative damping of optical solitons in a monomode fiber. According to Kodama and Hasegawa (1982), this compensation can be realized by means of a periodically acting stimulated Raman scattering. In the simplest version, a conformable model is

$$iu_t+u_{xx}+2|u|^2u=-i\gamma u+i\gamma a\sum_{n=-\infty}^{+\infty}\delta(x-an)u\ , \tag{3.27}$$

the second term on the right-hand side accounting for the Raman pump. In Eq. (3.27), as well as in Eq. (3.26), the variables t and x stand for the spatial coordinate and time, respectively (not vice versa). Inserting the right-hand side of Eq. (3.27) into the general perturbation-induced evolution equation (3.17), one can see that the chosen form of the pump term in Eq. (3.27) provides, on average, exact compensation of dissipative losses for all values of the soliton amplitude and velocity. Another compensation model, based on the small-amplitude Raman pumping wave, has been investigated numerically by Dianov *et al.* (1985) and analytically by Kivshar (1988b).

When dissipation acts in combination with the external time-periodic force [see Eq. (1.9)], a soliton may survive, being locked to the frequency Ω of the external force. One looks for a perturbed solution for a quiescent soliton in the form

$$u(x,t)=2i\eta(t)\frac{\exp[i\Omega t-i\chi(t)]}{\cosh[2\eta(t)x]}\ , \tag{3.28}$$

where $\chi(t)$ is a real phase [cf. Eqs. (2.41) and (2.42)]. Then Eqs. (3.17) and (3.19) reduce to the autonomous dynamical system (Kaup and Newell, 1978a, 1978b)

$$\frac{d\eta}{dt}=-2\alpha\eta+\tfrac{1}{2}\pi\epsilon\sin\chi\ , \tag{3.29}$$

$$\frac{d\chi}{dt}=\Omega-4\eta^2 \tag{3.30}$$

(see a phase portrait of this system in the paper cited above). The dynamical system (3.29) and (3.30) possesses two stable stationary points actually equivalent to each other:

$$\eta_n=\tfrac{1}{2}(-1)^n\sqrt{\Omega},\quad \chi_n=n\pi+\chi_0 \tag{3.31}$$

($n=0,1$), where $\sin\chi_0=2\alpha\sqrt{\Omega}/\pi\epsilon$. It follows from the expression for χ_0 that a stable soliton, frequency-locked to the external ac force, exists provided the force amplitude exceeds the threshold value (Kaup and Newell, 1978a, 1978b)

$$\epsilon_{\rm thr}=2\alpha_1\sqrt{\Omega}/\pi\ . \tag{3.32}$$

A related problem is the perturbation

$$\epsilon P(u)=\epsilon u^{*}e^{i\omega_0 t}-i\alpha u\ , \tag{3.33}$$

which described parametric pumping by a time-periodic external field in magnetic systems (Kivshar, 1988a). In this case the soliton is taken in the same form as Eq. (3.28), and the equations analogous to Eqs. (3.29) and (3.30) are

$$\frac{d\eta}{dt}=-2\alpha\eta-2\epsilon\eta\sin(2\chi)\ , \tag{3.34}$$

$$\frac{d\chi}{dt}=\omega_0-4\eta^2-\epsilon\cos(2\chi)\ . \tag{3.35}$$

The dynamical system (3.34) and (3.35) has stable stationary points [cf. Eq. (3.31)]

$$\eta_n=\tfrac{1}{2}(-1)^n(\omega_0+\sqrt{\epsilon^2-\alpha^2})^{1/2}\ , \tag{3.36}$$

$$\chi_n=\pi_n+\chi_0\ , \tag{3.37}$$

where $n=0,1$, and where $\sin(2\chi_0)=-\alpha/\epsilon$. Clearly a stable soliton exists for

$$\alpha<\epsilon<(\omega_0^2+\alpha^2)^{1/2} \tag{3.38}$$

[cf. Eq. (3.32)]. This problem is qualitatively different from that considered above: the dynamical system (3.34) and (3.35) possesses, in addition to the stationary points (3.36) and (3.37), the trivial stationary point $\eta=0$; $\chi=$const, which is also stable under the condition

$$\epsilon<(\omega_0^2+\alpha^2)^{1/2} \tag{3.39}$$

[note that conditions (3.38) and (3.39) are compatible]. This means that there is a separatrix on the phase plane (η,χ), which is a boundary between attraction basins of the two types of stationary points [a dynamical system with a qualitatively similar phase plane was investigated earlier by Bogdan, Kosevich, and Manzhos (1985)].

Karpman and Maslov (1982) have demonstrated how the perturbative approach can be applied to the NS equation with slowly varying coefficients [cf. Eq. (3.8)],

$$iv_\tau+\alpha(\epsilon\tau)|v|^2v+\beta(\epsilon\tau)v_{xx}=0\ , \tag{3.40}$$

where ϵ is a small parameter. The substitution

$$t=\int_0^\tau\beta(\epsilon\tau')d\tau'\ ,\qquad u(x,t)=\left[\frac{\sigma}{2}\right]^{1/2}v(x,\tau),\quad \sigma\equiv\alpha\beta\ , \tag{3.41}$$

transforms Eq. (3.40) into the standard form (1.6) with the perturbation

$$P(u)=\frac{i}{2}\frac{\sigma_t}{\sigma}u$$

[cf. Eq. (3.8′)]. Grimshaw (1979b) investigated the one-soliton dynamics governed by the same equation (3.40) with slowly varying coefficients, with the aid of a multiscale expansion technique.

Another example of a NS equation with a variable coefficient,

$$iu_t+u_{xx}+2|u|^2u=n(x)u\ , \tag{3.42}$$

is known in plasma physics, where it describes Langmuir waves in a plasma with inhomogeneous density (see, for example, Chen and Liu, 1978; Cow, Tsintsadze, and Tskhakaya, 1982). In particular, assuming that the profile $n(x)$ is linear, $n(x)=2\alpha x$, Chen and Liu (1978) have demonstrated that the corresponding perturbed NS equation can be reduced to the unperturbed one by means of the transformation $u=\tilde{u}\exp[-2i(\alpha t\tilde{x}-4\alpha^2t^3/3)]$, $\tilde{x}=x+2\alpha t^2$. In a more general case ($d^2n/dx^2\neq 0$), this perturbation is irreducible; for instance, a soliton may perform small oscillations near a density minimum described by $n(x)=\beta x^2$, $\beta>0$. The oscillations may be regarded in the framework of an adiabatic approximation similar to that described above (Cow, Tsintsadze, and Tskhakaya, 1982).

C. Sine-Gordon equation

1. A kink under the action of spatially homogeneous perturbations

As described in Sec. II.D, the sine-Gordon equation admits two fundamental types of solitonic solutions: a kink and a breather. For a kink the general evolution equations for the SG discrete-spectrum scattering data yield (Kaup and Newell, 1978a; McLaughlin and Scott, 1978)

$$\frac{dV}{dt}=-\frac{\epsilon\sigma}{4}(1-V^2)^{3/2}\int_{-\infty}^{+\infty}dz\frac{P[u_s(z)]}{\cosh z}\ , \qquad (3.43a)$$

$$\frac{d\xi}{dt}=V-\frac{\epsilon\sigma}{4}V(1-V^2)\int_{-\infty}^{+\infty}dz\frac{zP[u_s(z)]}{\cosh z}\ . \qquad (3.43b)$$

However, there is a simpler and more popular approach to evolution equations for a kink in the adiabatic approximation. This approach is based on the use of modified conservation laws (Christiansen and Olsen, 1982; Bergman *et al.*, 1983; Olsen and Samuelsen, 1983; Levring, Samuelsen, and Olsen, 1984; Salerno *et al.*, 1985; Sakai, Samuelsen, and Olsen, 1987). Taking, for instance, the expression for the kink's momentum

$$P=-\int_{-\infty}^{+\infty}u_tu_x dx=8V/\sqrt{1-V^2} \qquad (3.44a)$$

and using the perturbed SG equation (1.15), one can find

$$\frac{dP}{dt}=-\epsilon\int_{-\infty}^{+\infty}dx\, u_xP[u]\ . \qquad (3.44b)$$

Then, substituting into the right-hand side of Eq. (3.44b) $u(x,t)$ in the form of the unperturbed kink (2.61a), one can express the right-hand side in terms of the kink's velocity. Together with Eq. (3.44a), this yields a particle-like equation of motion for the kink.

In some applications, a perturbed SG equation arises naturally in the so-called light-cone coordinates $\tau=\frac{1}{2}(x+t)$ and $\chi=\frac{1}{2}(t-x)$:

$$u_{\tau\chi}+\sin u=\epsilon\Pi(u) \qquad (3.45)$$

(see Drühl and Alsing, 1986). In this notation, the unperturbed kink takes the form

$$u_k(\tau,\chi)=4\tan^{-1}[e^{2\nu(\chi-\xi)}],\quad \xi\equiv\tau/4\nu^2\ . \qquad (3.46)$$

In Eq. (3.46), the parameter ν is related to the usual kink velocity V from Eqs. (2.61): $\nu=\frac{1}{2}\sqrt{(1+V)/(1-V)}$.

General perturbation-induced evolution equations for the quantities ν and ξ have been derived (by means of a conformable variant of IST perturbation theory) by Spatschek (1979):

$$\frac{d\nu}{d\tau}=\frac{\epsilon}{4}\int_{-\infty}^{\infty}\Pi[u_k(z)]\text{sech}z\, dz\ ,$$

$$\frac{d\xi}{d\tau}=\frac{1}{4\nu^2}+\frac{\epsilon}{8\nu^2}\int_{-\infty}^{\infty}z\Pi[u_k(z)]\text{sech}z\, dz\ ,$$

where $z=2\nu(\chi-\xi)$. Spatschek (1979) has also derived general perturbation-induced evolution equations for continuous-spectrum scattering data in the light-cone coordinates.

A simple but physically important problem pertains to the motion of the kink in the external field (1.17) or in the field (1.21) in the presence of dissipation (1.16). In particular, if the external field (1.17) is constant, the kink may move steadily with the velocity (McLaughlin and Scott, 1978)

$$V_0=\sigma\left[1+\left[\frac{4\gamma}{\pi\epsilon}\right]^2\right]^{-1/2}\ . \qquad (3.47)$$

Analogously, for the constant field (1.21),

$$V_0=\sigma\,\text{sgn}\epsilon\left[1+\left[\frac{2\gamma}{\epsilon}\right]^2\right]^{-1/2}\ . \qquad (3.48)$$

When a kink moves steadily with velocity V_0, the energy dissipation rate is equal exactly to the rate of energy input from the external drive. If the external field (1.21) contains a variable component, a kink's law of motion can be found in the same manner as for the perturbation (1.21) (Bar'yakhtar *et al.*, 1985). As to the perturbation (1.17) with a variable $f(t)$, to determine a law of motion for the kink one should take into account that the full wave field must be sought in the form

$$u(x,t)=u_0(t)+U(x,t)\ , \qquad (3.49)$$

where $u_0(t)$ is a spatially uniform solution of the SG equation far from the kink, and $U(x,t)$ is the kink proper (Olsen and Samuelsen, 1983). The perturbed SG equation for the redefined wave field $U(x,t)$ in the first order in ϵ takes the form

$$U_{tt}-U_{xx}+\sin U=u_0(t)(1-\cos U)\ . \qquad (3.50)$$

If, for instance, $f(t)=\cos(\Omega t)$, the transformation of the original perturbed SG equation into the form of Eq. (3.50) is actually equivalent to a change of the perturbation parameter ϵ in the resultant equation of motion for a kink by the renormalized parameter $\epsilon/|\Omega^2-1|$. For the near-resonant case, when $|\Omega^2-1|\ll 1$, the renormalizing factor takes a more complicated form, which has been calculated by Malomed (1987c) and will be given below in subsection VI.C.1 [see Eq. (6.16)].

2. A kink under the action of spatially inhomogeneous perturbations

In this subsection we shall deal with perturbing terms of the form

$$P(u)=-h(x)G'(u) . \qquad (3.51)$$

Inserting Eq. (3.51) into the general adiabatic evolution equations (3.43), in the first approximation in ϵ one can transform the kink's equation of motion into that for a (generally speaking, relativistic) particle of mass $m=8$ moving in the potential

$$U(\xi)=\int_{-\infty}^{\infty} dx\, h(x)G\{4\tan^{-1}[\exp(x-\xi)]\} . \qquad (3.52)$$

The analogy between perturbed solitons and particles has been stressed by many authors, including Kaup and Newell (1978a), Gorshkov and Ostrovskii (1981), and Bergman *et al.* (1983).

a. A local inhomogeneity in a long Josephson junction

The perturbed SG equation

$$u_{tt}-u_{xx}+\sin u=-f-\gamma u_t-\beta\delta(x)u_t+\epsilon\delta(x)\sin u \qquad (3.53)$$

(with $\beta=0$) is a standard model of a damped dc-driven long Josephson junction with a local inhomogeneity of the maximum Josephson current density (McLaughlin and Scott, 1978; see also Barone and Paterno, 1982, and Likharev, 1985). In Eq. (3.53) f stands for the dc bias current density, γ is a phenomenological dissipative constant, and ϵ is a "strength" of the local inhomogeneity [$\epsilon>0$ and $\epsilon<0$ correspond, respectively, to a microresistor and microshort (microshunt)]. Local inhomogeneities of a more general form may affect not only the local value of a maximum Josephson current, which is accounted for by the term $\sim\epsilon$ in Eq. (3.53), but also a local value of the dissipative coefficient, which is taken into account by the term $\sim\beta$ ($\beta>0$). A local inhomogeneity of this kind can be created, for example, by the action of a concentrated laser beam on a long Josephson junction (Scheuermann *et al.*, 1983) [Eq. (3.53) as a model of such an inhomogeneity was proposed by Chang (1985)]. The same model may be employed to describe a local inhomogeneity created by superimposing a short resistor on the junction (Akoh *et al.*, 1985). Finally the model describes a microshunt (i.e., a narrow bridge connecting two bulk superconductors) made of a normal (nonsuperconducting) metal (Kivshar and Malomed, 1988e; Kivshar, Malomed, and Nepomnyashchy, 1988). A fluxon in a long Josephson junction is described by the following approximate solution to Eq. (3.53):

$$u_f(x,t)=-f+u_k(x,t;V_0) , \qquad (3.54)$$

where u_k is the SG kink (2.61a), and V_0 is the equilibrium velocity (3.47).

Interaction between a fluxon and the microshort was considered by McLaughlin and Scott (1978), who demonstrated that the fluxon may be regarded as a particle of mass $m=8$ with position $\xi(t)$ moving in the repulsive potential

$$U(\xi)=-2\epsilon\,\mathrm{sech}^2\xi \qquad (3.55)$$

[cf. Eq. (3.52)] in the presence of a friction force $F_{\mathrm{fr}}=-8\gamma(d\xi/dt)$ and a constant driving force $F=2\pi\sigma f$. The driving force has its own potential $-2\pi\sigma f\xi$, so that the full effective one-fluxon potential is

$$U_{\mathrm{tot}}(\xi)=-2\epsilon\,\mathrm{sech}^2\xi-2\pi\sigma f\xi . \qquad (3.55')$$

Note that the repulsive character of the effective potential (3.55) does not depend on the fluxon's polarity.

An important dynamical process is the capture of a moving fluxon by a microshort. Under the condition

$$|\epsilon|\gg\gamma^2 , \qquad (3.56)$$

the maximum (threshold) value f_{thr} of the bias current density allowing the capture can be found in a very simple way by equating the kinetic energy of the free fluxon $E_{\mathrm{kin}}\approx 4V_0^2$ to the height $U_0=2|\epsilon|$ of the potential barrier (3.55):

$$f_{\mathrm{thr}}=(2\gamma/\pi)\sqrt{2|\epsilon|} \qquad (3.57)$$

(McLaughlin and Scott, 1978). Kivshar and Malomed (1988e; see also Kivshar, Malomed, and Nepomnyashchy, 1988) have found a dynamical correction to the kinematic result (3.57):

$$f_{\mathrm{thr}}^{(1)}=-4\left[\frac{\gamma^2}{\pi}\right]\ln 2 . \qquad (3.58)$$

This correction is much smaller than the basic expression (3.57) due to the adoption of condition (3.56). The significance of the correction (3.58) can be seen by comparing it with results of a direct numerical solution of the equation of motion for a full equation of motion for a fluxon (see Fig. 2).

In contrast to the case of a microshort, calculation of the threshold bias current density admitting the capture of a fluxon by a microresistor must be based upon dynamical considerations in the lowest approximation (Kivshar and Malomed, 1988e; Kivshar, Malomed, and Nepomnyashchy 1988). A microresistor gives rise to the same one-fluxon effective potential (3.55) with positive ϵ. Clearly, this time the potential is attractive irrespective of the fluxon's polarity. The calculation relies upon the same fundamental assumption (3.56). The threshold value of f is determined by the energy balance in the following form: the total dissipative energy loss

$$|\Delta E|=8\gamma\int_{-\infty}^{+\infty}V^2dt \qquad (3.59)$$

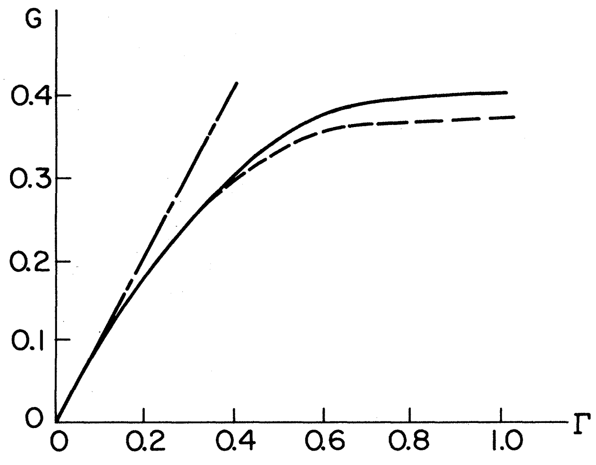

FIG. 2. The dependence of the threshold value $f_{\rm thr}$ of the bias current density for capture of a moving fluxon by a microshort on the parameters γ and ϵ: $G\equiv(\pi/2|\epsilon|)f_{\rm thr}$ and $\Gamma\equiv\gamma\sqrt{2/|\epsilon|}$. Dot-dashed curve, Eq. (3.57); dashed curve, Eq. (3.57) with the correction (3.58); solid curve, numerical results of Kivshar, Malomed, and Nepomnyashchy (1988).

must be equal to the kinetic energy $4V_0^2$ corresponding to the velocity (3.47) with $f=f_{\rm thr}$. Due to condition (3.56), one may insert into (3.59) the law of motion

$$\frac{d\xi}{dt}=\sqrt{\epsilon/2}\,{\rm sech}\xi\ , \tag{3.60}$$

which corresponds to the motion of a fluxon with zero velocity at infinity in the absence of bias current and dissipation. Thus one can obtain

$$|\Delta E|=8\gamma\pi\sqrt{\epsilon/2} \tag{3.61}$$

and, inserting this expression into the energy balance equation, one finds eventually

$$f_{\rm thr}^2=\frac{16\sqrt{2}}{\pi}\sqrt{\epsilon}\gamma^3\ . \tag{3.62}$$

Note that the threshold value (3.62) is much smaller than that of Eq. (3.57). A correction to (3.62) has also been found by Kivshar, Malomed, and Nepomnyashchy (1988). These authors have demonstrated that the analytical results are in very good agreement with numerical ones.

In a real experiment, a microshort or a microresistor may have a size comparable to the Josephson penetration length, i.e., a length characteristic of a fluxon. In this relation, it is meaningful to find the quantity $f_{\rm thr}$ for a finite-size inhomogeneity. This has been done by Kivshar, Kosevich, and Chubykalo (1988).

In the theory of long Josephson junctions there also occurs another localized perturbation,

$$P=\delta'(x)\ . \tag{3.63}$$

The term (3.63) multiplied by a small ϵ describes a local inhomogeneity created by an Abrikosov vortex perpendicular to the junction (Aslamasov and Gurovich, 1984). It is easy to find that the perturbation (3.63) acts upon a fluxon as an effective potential of the form

$$U(\xi)=2\sigma\epsilon\,{\rm sech}\xi \tag{3.64}$$

[cf. Eq. (3.55)]. As follows from Eq. (3.64), in contrast with a microshort or microresistor, the Abrikosov vortex attracts or repulses a fluxon depending upon its polarity. It is evident that in the case of attraction a fluxon may be pinned by the vortex. Less trivial is the fact that there also exists an equilibrium state in which two unipolar fluxons attracted by the vortex rest at distances $\approx\ln\epsilon^{-1}$ to the right and to the left of the vortex (Gurovich and Mikhalev, 1987; in the case of an attractive microresistor, such an equilibrium state is not possible). However, this state has no physical meaning because it is unstable against antisymmetric perturbations, i.e., shifts of both fluxons to the left or to the right with the distance between them kept constant.

One can also find the threshold values of the bias current admitting the capture of a fluxon by the Abrikosov vortex. In the case of repulsion, $f_{\rm thr}$ coincides, in the lowest approximation, with Eq. (3.57), while in the attraction case dynamical consideration yields $f_{\rm thr}^2=[16\Gamma^2(\frac{1}{4})/\pi^{5/2}]\gamma^3\sqrt{\epsilon}$.

According to Kosevich, Kivshar, and Chubykalo (1987b) and Sakai, Samuelsen, and Olsen (1987), a long Josephson junction with an inductance step gives rise to a localized perturbation described by the following equation:

$$u_{tt}-u_{xx}-\sin u=-\gamma u_t-f+\epsilon\delta(x)u_x\ . \tag{3.65}$$

The last term on the right-hand side of Eq. (3.65) acts upon a nonrelativistic fluxon as the effective potential

$$U(\xi)=4\epsilon\tanh\xi\ . \tag{3.66}$$

With regard to the presence of the terms $-\gamma u_t-f$, the potential (3.66) creates an effective potential well that can capture a moving fluxon. The threshold value of the bias current density f, which admits the capture, has been found by Kivshar and Malomed (1988l) under the same condition (3.56):

$$f_{\rm thr}=\frac{4\gamma}{\pi}\sqrt{2\epsilon}-\frac{\gamma^2}{\pi}\ln(\epsilon/\gamma^2)\ ,$$

where the first term takes its origin in a kinematic approximation analogous to that of McLaughlin and Scott (1978), while the second is a dynamical correction similar to Eq. (3.58).

Another physically interesting problem is to find threshold conditions for the capture of a fluxon by a combined inhomogeneity described by the two terms $\sim\epsilon$ and $\sim\beta$ on the right-hand side of Eq. (3.53). The most interesting analytical results can be obtained under the condition

$$\beta \gg \sqrt{|\epsilon|} \gg \gamma \tag{3.67}$$

(Kivshar and Malomed, 1988e; Kivshar, Malomed, and Nepomnyashchy, 1988). First let us consider the case $\epsilon<0$ (a combination of a microshort with a dissipative inhomogeneity). In this case the fluxon's law of motion in the vicinity of the inhomogeneity takes the approximate form

$$\frac{d^2\xi}{dt^2} = -\frac{\beta}{2}\left[\frac{d\xi}{dt}\right]\operatorname{sech}^2\xi . \tag{3.68}$$

Integrating Eq. (3.68) yields

$$V(\xi)=V_0-(\beta/2)(\tanh\xi+1) . \tag{3.69}$$

According to Eq. (3.69), as $t\to+\infty$ the fluxon approaches the stop point

$$\xi_0=\operatorname{arctanh}\left[\frac{2V_0}{\beta}-1\right] , \tag{3.70}$$

provided $V_0<\beta$. We may regard the fluxon as captured if the stop point lies to the left of the maximum $\xi=0$ of the repulsive potential (3.55) (for definiteness, we assume $\sigma=+1$, i.e., the fluxon moves from left to right). So, $f_{\rm thr}$ is determined by the condition $\xi_0=0$, or, with regard to Eq. (3.47),

$$f_{\rm thr}=(2/\pi)\gamma\beta . \tag{3.71}$$

Note that Eq. (3.71) does not explicitly contain the parameter ϵ; recall, however, that this result is true under the condition (3.67).

Let us proceed to the case $\epsilon>0$ (a "hybrid" of a microresistor with the dissipative inhomogeneity). We shall assume that condition (3.67) holds. In this case we again base our consideration on Eq. (3.70). The fluxon will be captured provided $\xi_0<\xi_{\rm max}$, where $\xi_{\rm max}$ is the coordinate of the maximum of the total effective potential

$$U(\xi)=-2\epsilon\,\operatorname{sech}^2\xi-2\pi f\xi \tag{3.72}$$

(we assume $\sigma=+1$). The value $\xi_{\rm max}$ is defined by the equation

$$\pi f=2\epsilon\tanh\xi_{\rm max}(1-\tanh^2\xi_{\rm max}) . \tag{3.73}$$

Setting $\xi_{\rm max}=\xi_0$, we obtain an equation for $f_{\rm thr}$. Straightforward analysis based on Eqs. (3.72) and (3.73) yields the following results: if $\gamma\beta/\epsilon<(\sqrt{3}-1)/3$,

$$f_{\rm thr}=(\gamma\beta/\pi)(3+\sqrt{1-4\gamma\beta/\epsilon}) . \tag{3.74}$$

At $\gamma\beta/\epsilon=(\sqrt{3}-1)/3$ Eq. (3.74) attains the value

$$(f_{\rm thr})_{\rm max}=4\epsilon/3\sqrt{3}\pi , \tag{3.75}$$

at which the extrema of the potential (3.72) merge and disappear, so that the value (3.75) is the threshold bias current for

$$\gamma\beta/\epsilon\geq(\sqrt{3}-1)/3 .$$

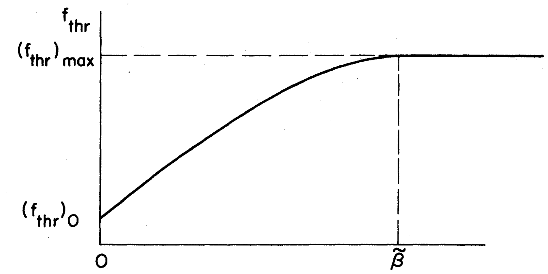

FIG. 3. The dependence of the threshold bias current density $f_{\rm thr}$ on the "strength" β of the dissipative part of the micro-inhomogeneity in the case $\epsilon>0$, $\beta\gg\sqrt{\epsilon}\gg\gamma$.

The full dependence $f_{\rm thr}(\beta)$ for fixed ϵ and γ (in the considered regime $\epsilon>0$, $\beta\gg\sqrt{\epsilon}\gg\gamma$) is shown in Fig. 3. In the picture, the value of $f_{\rm thr}$ corresponding to $\beta=0$ is $(f_{\rm thr})_0=4(\sqrt{2}\gamma^3\sqrt{\epsilon}/\pi)^{1/2}$. In the range $\sqrt{\epsilon}\ll\beta\leq\tilde\beta\equiv(\sqrt{3}-1)\epsilon/3\gamma$ the dependence is given by Eq. (3.74), while at $\beta\geq\tilde\beta$, $f_{\rm thr}\equiv(f_{\rm thr})_{\rm max}$; see Eq. (3.75). Note that, as follows from Eq. (3.74), $df_{\rm thr}/d\beta=0$ at $\beta=\tilde\beta$, so that the graph in Fig. 3 is smooth at the point $\beta=\tilde\beta$.

b. A lattice of local inhomogeneities in a long Josephson junction

A physically relevant generalization of Eq. (3.53) is

$$u_{tt}-u_{xx}+\sin u=\epsilon\sum_{n=-\infty}^{\infty}\delta(x-na)\sin u-f-\gamma u_t , \tag{3.76}$$

which describes a long Josephson juntion with periodically installed micro-inhomogeneities (McLaughlin and Scott, 1978). As far as we know, the first realization of a long Josephson junction with a periodic lattice of microresistors ($\epsilon>0$) was achieved in an experiment performed by Serpuchenko and Ustinov (1987) (see also Golubov, Serpuchenko, and Ustinov, 1988), while a lattice of microshorts has not as yet been constructed. Motion of a fluxon in the model (3.76) has been analyzed by Malomed (1988a). The lattice gives rise to the effective one-fluxon potential

$$U(\xi)=-2\epsilon\sum_{n=-\infty}^{\infty}\operatorname{sech}^2(\xi-an) = -\frac{8\epsilon}{a^2}\left\{K^2(q)\left[1-q^2\operatorname{sn}^2\left[\frac{2K(q)}{a}\xi,q\right]\right] -K(q)E(q)+\frac{a}{2}\right\} , \tag{3.77}$$

where sn is the Jacobi elliptic sine, and the auxiliary modulus q is uniquely determined by the equation $\pi a^{-1}=K(\sqrt{1-q^2})/K(q)$, $K(q)$ and $E(q)$ being the complete elliptic integrals. Equilibrium positions of the fluxon are determined by the equation

$$2\pi f = \frac{\partial U}{\partial \xi} \ . \tag{3.78}$$

Proceeding from Eq. (3.78), it is easy to find the static quantity $f_{\rm cr}$, which is a minimum bias current density that causes a pinned fluxon to move:

$$f_{\rm cr}^2 = \frac{2}{27}\left[\frac{16}{\pi}\epsilon K^3(q)/qa^3\right]^2 \times[(1+q^2)(2-q^2)(q^2-\tfrac{1}{2})+(1+q^4-q^2)^{3/2}] \ . \tag{3.79}$$

The threshold value $f_{\rm thr}$ defined as above can be found under the assumption that the potential force corresponding to Eq. (3.77) is much larger than the friction force $-8\gamma\, d\xi/dt$. For $a \gtrsim 1$, this requirement amounts to the inequality $\gamma \ll \sqrt{|\epsilon|}$ [cf. Eq. (3.56)], while for $a \ll 1$ it is more restrictive. To find $f_{\rm thr}$, one needs a threshold trajectory that connects two adjacent maxima of the potential (3.77), the driving and friction forces being neglected. In the cases $\epsilon<0$ (the lattice of microshorts) and $\epsilon>0$ (the lattice of microresistors), the equations determining the threshold trajectory take the forms, respectively,

$$\frac{d\xi^{(-)}}{dt} = \frac{\sqrt{2|\epsilon|}}{a}K(q)q\ \mathrm{sn}\left[\frac{2K(q)}{a}\xi, q\right] \tag{3.80}$$

and

$$\frac{d\xi^{(+)}}{dt} = \frac{\sqrt{2\epsilon}}{a}K(q)q\,\mathrm{cn}\left[\frac{2K(q)}{a}\xi, q\right] \ . \tag{3.81}$$

Expedient boundary conditions to Eqs. (3.80) and (3.81) are, respectively,

$$\xi_i^{(-)} \equiv \xi(t=-\infty)=0, \quad \xi_f^{(-)} \equiv \xi(t=+\infty)=a \tag{3.82}$$

and

$$\xi_i^{(+)} = -\frac{a}{2}, \quad \xi_f^{(+)} = \frac{a}{2} \ . \tag{3.83}$$

The value $f_{\rm thr}$ is determined by equating the dissipative energy loss along the threshold trajectory,

$$E_{\rm diss} = 8\gamma\int_{-\infty}^{+\infty}\left[\frac{d\xi}{dt}\right]^2 dt = 8\gamma\int_{\xi_i}^{\xi_f}\frac{d\xi}{dt}d\xi \ , \tag{3.84}$$

to the compensating work $2\pi fa$ of the bias current. Inserting Eqs. (3.80)–(3.83) into (3.84) yields, for $\epsilon<0$ and $\epsilon>0$,

$$f_{\rm thr}^{(-)} = \frac{E_{\rm diss}^{(-)}}{2\pi a} = \frac{2\gamma\sqrt{2|\epsilon|}}{\pi a}\ln\frac{1+q}{1-q} \ , \tag{3.85}$$

$$f_{\rm thr}^{(+)} = \frac{E_{\rm diss}^{(+)}}{2\pi a} = \frac{4\gamma\sqrt{2\epsilon}}{\pi a}\sin^{-1}q \ . \tag{3.86}$$

The dependence of the threshold values (3.85) and (3.86) on the lattice spacing a is shown in Fig. 4. In the limit $a\to\infty$, when we return to an isolated local inhomogeneity, Eq. (3.85) goes over into (3.57). As to (3.86), it gives zero in the same limit. The reason is that the corresponding nonzero expression (3.62) can only be obtained in the next order with respect to the small parameter $\gamma/\sqrt{\epsilon}$. In the opposite limit $a\to 0$, Eqs. (3.85) and (3.86) become exponentially small in a^{-1}.

In an experiment, the values $f_{\rm thr}$ and $f_{\rm min}$ can be observed as terminal points of two branches of a current-voltage characteristic of the Josephson junction, i.e., the dependence of the mean fluxon velocity $\overline{V}$ (proportional to dc voltage across the junction) upon the bias current density f. In principle, one can find the whole current-voltage characteristic, proceeding from the fluxon's equation of motion corresponding to the effective potential (3.77). In an analytical form, this can be done for a model with the term $\epsilon\sin(\kappa x)\sin u$ instead of $\epsilon\sum_{n=-\infty}^{+\infty}\delta(x-an)\sin u$ on the right-hand side of Eq. (3.76) (we again assume $\gamma\ll\sqrt{\epsilon}$). This model has been proposed by Mkrtchyan and Shmidt (1979). In the "nonrelativistic" range $V^2\ll 1$, the resultant current-voltage characteristic can be written in a parametric representation (Malomed, 1988a)

$$\overline{V}^2 = \epsilon(\pi^4/2)\left[a\sinh\frac{\pi^2}{a}\right]^{-1}[K(q)q]^{-2} \ , \tag{3.87}$$

$$f^2 = \epsilon\frac{128}{\pi^2}\left[a\sinh\frac{\pi^2}{a}\right]^{-1}[E(q)]^2 q^{-2}\gamma^2 \ , \tag{3.88}$$

where q is an auxiliary elliptic modulus. The current-voltage characteristic (3.87) and (3.88) is monotonic; see Fig. 5. The value

$$f_{\rm thr} = \frac{8\gamma}{\pi}\left[\frac{2\epsilon}{a\sinh(\pi^2/a)}\right]^{1/2}$$

corresponds to $q=1$. As can be seen from Eq. (3.87), at

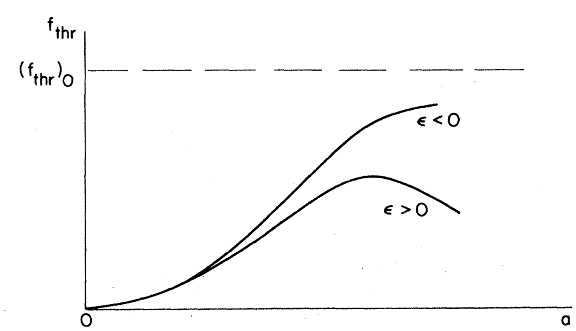

FIG. 4. The dependences (3.85) and (3.86) of the threshold bias current density upon the lattice spacing a. At $a\to\infty$, Eq. (3.86) vanishes, while (3.85) approaches the value $(f_{\rm thr})_0=(2\gamma/\pi)\sqrt{2|\epsilon|}$ corresponding to an isolated microshort; see Eq. (3.57).

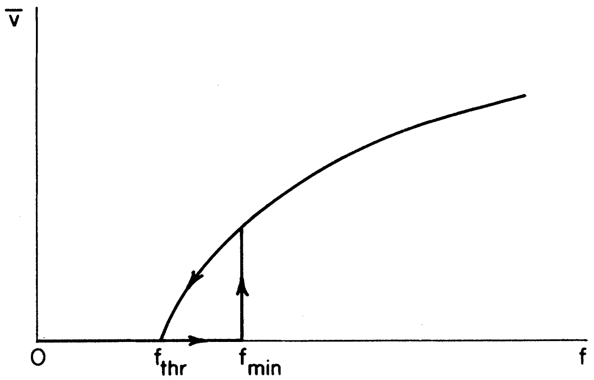

FIG. 5. The current-voltage characteristic (3.87) and (3.88) of a long Josephson junction with a critical supercurrent density subjected to sinusoidal modulation.

this value of q the mean velocity vanishes. The value f_{cr} in this model can be found with no difficulty: $f_{cr}=4\pi^2\epsilon/a^2\sinh^2(\pi^2/a)$. As is shown in Fig. 5, the range $f_{thr}<f<f_{cr}$ is hysteretic, the arrows on the two branches pointing out their sense. In the relativistic range $(1-V^2)^{-1}\gtrsim 1$ the current-voltage characteristic of the present model practically coincides with that of the homogeneous long Josephson junction.

It is also of interest to consider a model describing a lattice of dissipative inhomogeneities (Kivshar and Malomed, 1988e):

$$u_{tt}-u_{xx}+\sin u=-\gamma u_t-\sum_{n=-\infty}^{+\infty}\beta_n\delta(x-x_n)u_t-\epsilon\ , \tag{3.76'}$$

where the lattice may be either regular or random (all β_n are positive). Kivshar and Malomed (1988e) have demonstrated that in either case the average velocity $\langle V\rangle$ of a fluxon (here, averaging is realized with respect to *time*) is related to the bias current density ϵ according to Eq. (3.47), where γ must be replaced by $\gamma_{\rm eff}=\gamma+\bar{\beta}/\bar{a}$. Here $\bar{\beta}$ and $\bar{a}$ are the *spatial* averages of the inhomogeneity "strength" and the lattice spacing ($\bar{a}\equiv x_{n+1}-x_n$), respectively.

c. Local impurities in a commensurate charge-density-wave system

A long commensurate charge-density-wave (CDW) system with a localized impurity is described by the equation [cf. Eq. (3.53)]

$$u_{tt}-u_{xx}+\sin u=-\epsilon\delta(x)\sin\left[\frac{u}{M}+\theta\right]-f-\gamma u_t \tag{3.89}$$

(Malomed, 1988e), where θ is an effective coordinate of a charged impurity relative to the underlying lattice, f is the external dc electric field (voltage), α is again a phenomenological dissipative coefficient, and M is an integer ≥ 1. A commensurate charge-density-wave system proper corresponds to $M\geq 3$ (see, for example, Horovitz, 1986), while the cases $M=1$ and 2 correspond to a noncommensurate system containing an ionic superlattice that induces effective commensurability [the case $M=2$ was discussed by Fukuyama (1978); the SG model with $M=1$ was introduced by Hansen and Carneiro (1984) and Apostol and Baldea (1985)]. In what follows we shall concentrate on the case $M=1$, following Malomed (1988e); results for higher M are qualitatively analogous (Malomed and Nepomnyashchy, 1989b).

Recall that Eq. (3.53) is a particular case of the model (3.89) with $M=1$ and $\theta=0$ or π. We shall demonstrate that generalization to arbitrary θ yields nontrivial results.

In charge-density-wave theory, a kink represents a charged soliton (Grüner and Zettl, 1985; Horovitz, 1986). A charged impurity, described by the term $\sim\epsilon$ from the right-hand side of Eq. (3.89) with $M=1$, acts upon the kink as an effective potential,

$$U(\xi)=2\epsilon\cos\theta\,\mathrm{sech}^2\xi+2\epsilon\sin\theta\sinh\xi\,\mathrm{sech}^2\xi\ ; \tag{3.90}$$

see Fig. 6, where, for convenience, the constant $\epsilon\cos\theta$ has been added to the potential. In what follows we shall assume, for definiteness, $\epsilon>0$ (the case $\theta=0$, $\epsilon<0$ mentioned above is then equivalent to $\theta=\pi$, $\epsilon>0$). The potential (3.90) has a minimum at the point $\xi_{\min}$,

$$\sinh\xi_{\min}=-\cot(\theta/2)\ , \tag{3.91}$$

and a maximum at the point $\xi_{\max}$,

$$\sinh\xi_{\max}=\tan(\theta/2)\ . \tag{3.92}$$

The maximum value of $U(\xi)$ attained at $\xi=\xi_{\max}$ is

$$U_{\max}=\epsilon(1+\cos\theta)\ . \tag{3.93}$$

Using assumption (3.56), it is straightforward to find

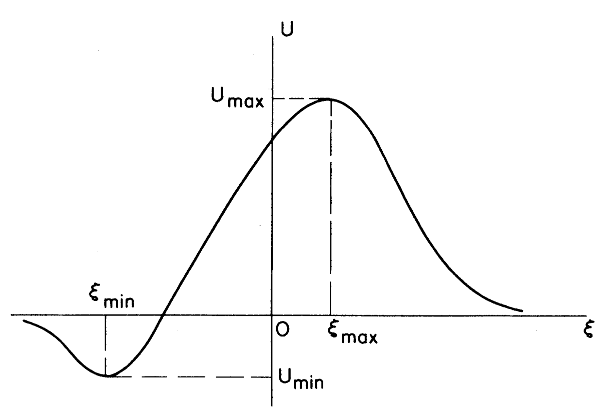

FIG. 6. The shape of the effective potential (3.90). For definiteness, the case $0<\theta<\pi/2$ is shown.

the threshold (maximum) value of the external drive f which allows the capture of a kink moving with the equilibrium velocity (3.47) by the potential trap (3.90):

$$f_{\rm thr} \approx (2\gamma/\pi)\sqrt{2\epsilon}|\cos(\theta/2)| \tag{3.94}$$

[cf. Eq. (3.57)]. The first dynamical correction to Eq. (3.94) [analogous to correction (3.58) to Eq. (3.57)] has also been found by Malomed (1988e). The corresponding value of the energy ΔE dissipated by a kink moving according to the threshold law of motion is [cf. Eq. (3.61)]

$$\Delta E = 8\gamma\sqrt{\epsilon/2}\left\{\cos(\theta/2)\ln\frac{1}{\cos^2(\pi/4\pm\theta/4)} \mp \tfrac{1}{2}(\pi\pm\theta)\sin(\theta/2)\right\} . \tag{3.95}$$

The sign $\pm$ should be taken opposite to the sign of the kink's polarity σ. The reason for the polarity-dependent character of ΔE is quite clear: kinks with opposite polarities arrive at the point $\xi_{\max}$ from opposite directions and, due to the asymmetry of the potential in the general case ($\sin\theta \neq 0$; see Fig. 6), their laws of motion are different. In general, when θ is not too close to π, Eq. (3.95) yields

$$f_{\rm thr} \approx f_{\rm thr}^{(0)} - 2\gamma^2 \Delta E/\pi^2 f_{\rm thr}^{(0)} , \tag{3.96}$$

where $f_{\rm thr}^{(0)}$ is given by Eq. (3.94). The dependence of the corrected expression for $f_{\rm thr}$ on the kink's polarity is a new feature, absent in the particular case $\theta=0$ [when Eq. (2.10) goes over into (3.58)].

In the special case $\theta=\pi-\Delta$, $\Delta^2 \lesssim \gamma\sqrt{\epsilon}$,

$$f_{\rm thr} = (\gamma/\pi)[2(\epsilon\Delta^2 + 8\pi\gamma\sqrt{2\epsilon})]^{1/2} . \tag{3.97}$$

In particular, at $\Delta=0$ [when the maximum of the potential (3.90) disappears], one recovers Eq. (3.62).

The coordinate ξ_0 of a kink pinned by the inhomogeneity is determined by equating the total force acting upon it to zero:

$$2\pi\sigma f - \frac{\partial}{\partial\xi}U(\xi)\bigg|_{\xi=\xi_0} = 0 . \tag{3.98}$$

At $f=0$ Eq. (3.98) has the two roots $\xi=\xi_{\max}$ and $\xi=\xi_{\min}$ [see Eqs. (3.91) and (3.92)]. With an increase in f, depending on the sign of σf and the value of θ, these two roots may remain, or two more roots may appear which formally lie at infinity when $f=0$. In either case, half of the roots correspond to stable equilibria of the kink, and the other half correspond to unstable ones. With the further growth of f, the stable and unstable equilibria will inevitably merge and disappear in pairs. If, at $f=0$, a CDW system contains a sufficiently large number of isolated impurities, it is natural to suppose that all the kinks are pinned, i.e., the solitonic contribution to the conductivity of the system is zero. With the growth of f, the pinned state of a given kink will disappear at some value $f_{\max}$ which depends on θ, i.e., at $f > f_{\max}(\theta)$ the kink will be a free charge carrier. Since a pinned state disappears through a merger with an unstable equilibrium, at $f=f_{\max}(\theta)$ Eq. (3.98) must have a double root, i.e., a pinned state disappears at a point where

$$\frac{\partial^2 U}{\partial\xi^2} = 0 . \tag{3.99}$$

Inserting Eq. (3.90) into Eq. (3.99), one eventually finds the relation between θ and $f_{\max}$ to be

$$16\cot^2\theta = \{[(9\pi^2 f_{\max}^2 + 16\epsilon^2)^{1/2} - 5\pi f_{\max}]/(2\pi f_{\max})\} \times \{[(9\pi^2 f_{\max}^2 + 16\epsilon^2)^{1/2} - 15\pi f_{\max}]/[(9\pi^2 f_{\max}^2 + 16\epsilon^2)^{1/2} - 6\pi f_{\max}]\}^2 . \tag{3.100}$$

For a given f, kinks pinned originally by inhomogeneities with θ such that $f_{\max}(\theta) < f$ become free. What, then, is the fraction of free kinks for a given f, assuming that θ is a random quantity distributed uniformly over the interval [0;2π]? Analysis based on Eq. (3.100) with regard to necessary sign relations, and averaging over θ yield the following expressions for the density of free kinks n as a function of the external force (dc voltage) f: (a) in the range $0 < f < \sqrt{2/27}\,\epsilon/\pi$

$$n(f) = n_0 F(\pi f/\epsilon) ; \tag{3.101a}$$

(b) in the range $\sqrt{2/27}\,\epsilon/\pi < f < (4/\sqrt{27})\epsilon/\pi$

$$n(f) = n_0[\tfrac{1}{2} - F(\pi f/\epsilon)] ; \tag{3.101b}$$

(c) in the range $(4/\sqrt{27})\epsilon/\pi < f < \epsilon/\pi$

$$n(f) = n_0[\tfrac{1}{2} + 2F(\pi f/\epsilon)] , \tag{3.101c}$$

where n_0 is the total density of kinks of both polarities in the system, and the function $F(x)$ is

$$F(x) = \frac{1}{2\pi}\tan^{-1}|\tfrac{1}{4}\{[\sqrt{1+(3x/4)^2} - 5x]/(2x)\}^{1/2}[\sqrt{1+(3x/4)^2} - 15x]/[\sqrt{1+(3x/4)^2} - 6x]| . \tag{3.102}$$

The last pinned state disappears at $f=\epsilon/\pi$, i.e.,

$$n(f)\equiv n_0 \quad (3.101d)$$

at $f\geq\epsilon/\pi$. The full dependence $n(f)$ is depicted in Fig. 7.

The released kinks may be regarded as free provided f exceeds the maximum value of the threshold force (3.94),

$$(f_{\mathrm{thr}})_{\max}\approx(2\alpha/\pi)\sqrt{2\epsilon}\ , \quad (3.103)$$

attained at $\theta=0$. Due to assumption (3.56), this value is small compared to the values $f\sim\epsilon$, a fact which is crucial from the viewpoint of the dependence (3.101) and (3.102).

In an experiment, the dependence $n(f)$, along with Eq. (3.42), can manifest itself through the current-voltage characteristic of the system, i.e., through an experimentally measurable dependence of the current

$$j=n(f)V_0(f)$$

on the dc voltage f.

In fact, the model described above is a model of nonlinear dc conductivity of a quasi-one-dimensional CDW system, which may be regarded as supplementary to the better known model of Maki (1977) based upon calculation of a quantum production rate of kink-antikink pairs in the external field. The present model can be extended to include ac conductivity as well (Malomed, 1988e). It is also worth noting that many real CDW systems should be described by the overdamped SG equation (i.e., with $\gamma\gg1$, when the term u_{tt} may be neglected), rather than by the usual SG equation (see, for example, Weger and Horovitz, 1982; Horovitz and Trullinger, 1984). Calculation of the dc and ac conductivities along the lines outlined above can be accomplished within the framework of the overdamped model (Malomed and Nepomnyashchy, 1989b).

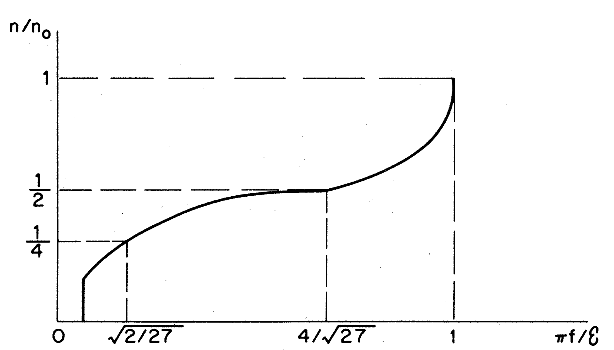

FIG. 7. The share of free kinks $n(f)/n_0$ vs the external force (dc voltage) f according to Eqs. (3.101) and (3.102). The graph terminates at $f=(2\gamma/\pi)\sqrt{2\epsilon}$, i.e, the maximum value of f_{thr}; see Eq. (3.94).

3. Solitonic conductivity in a continuum model of randomly inhomogeneous charge-density-wave systems and long Josephson junctions

a. General analysis

In real doped CDW systems the mean distance l between the charged impurities is small compared to the kink's size, $l\ll1$ (Fukuyama, 1978). In this case Eq. (3.89) with a large number of impurities may be approximated by the equation (Fukuyama, 1978; Malomed, 1988e, 1989a)

$$u_{tt}+\gamma u_t-u_{xx}+\sin u+f=\zeta_1(x)\sin\frac{u}{M}+\zeta_2(x)\cos\frac{u}{M}\ , \quad (3.104)$$

where $\zeta_{1,2}(x)$ are Gaussian random functions determined by the correlators

$$\langle\zeta_{1,2}(x)\rangle=\langle\zeta_1(x)\zeta_2(x')\rangle=0\ , \quad (3.105a)$$

$$\langle\zeta_1(x)\zeta_1(x')\rangle=\langle\zeta_2(x)\zeta_2(x')\rangle=\tilde{\epsilon}^2\delta(x-x')\ , \quad (3.105b)$$

and where $\tilde{\epsilon}^2\equiv\epsilon^2/(2l)$. In what follows, $\tilde{\epsilon}^2$ will also be regarded as a small parameter. The model (3.104) and (3.105) with $M=1$, $\zeta_2\equiv0$ was proposed by Mineev, Feigel'man, and Shmidt (1981) to describe a long Josephson junction with maximum supercurrent density subject to random modulation.

In many one-dimensional metals in which CDW conductivity takes place dissipation may be very strong; see Weger and Horovitz (1982), Horovitz and Trullinger (1984). In this case, Eq. (3.104) must be replaced by its overdamped version,

$$\gamma u_t-u_{xx}+\sin u+f=\zeta_1(x)\sin\frac{u}{M}+\zeta_2(x)\cos\frac{u}{M}\ . \quad (3.106)$$

Equation (3.106) with $M=1$, $\zeta_2=0$ describes a randomly inhomogeneous Josephson junction of the *SNS* type, i.e., two bulk superconductors separated by a thin layer of a normal metal.

The solitonic ac/dc conductivity of the driven damped and overdamped continuum models (3.104) and (3.106) has been investigated by Malomed (1988e, 1989b). Here we shall give the basic results.

An effective one-kink potential corresponding to these models is [cf. Eq. (3.90)]

$$U(\xi)=\int_{-\infty}^{+\infty}dx\sum_{i=1,2}\zeta_i(x)U_i(\xi-x)\ ,$$
$$U_1(z)\equiv M\{1+\cos[M^{-1}u_k(z)]\}\ , \quad (3.107)$$
$$U_2(z)\equiv M\{1-\sin[M^{-1}u_k(z)]\}\ ,$$

where $u_k(z)$ is the kink wave form (2.61). With an increase in f, a given trapped kink escapes at some $f=f_{\mathrm{cr}}$ and $\xi=\xi_{\mathrm{cr}}$, at which $d^2U/d\xi^2=0$ [cf. Eq. (3.99)]. The basic idea of the model is to find the fraction $\mu(f)$ of points ξ_{cr} for which $f_{cr}<f$. Proceeding from the probability-density functional for Gaussian random fields,

$$P[\zeta_{1,2}(x)]\sim\exp\left[-\int_{-\infty}^{+\infty}(2\tilde{\epsilon}^2)^{-1}[\zeta_1^2(x)+\zeta_2^2(x)]dx\right],$$

one can readily find a probability distribution for the quantity $|f_{cr}|$ corresponding to Eq. (3.107) [a similar problem was solved by Mineev, Feigel'man, and Shmidt (1981)]:

$$p(f_{cr})=2(2\pi/I_1)^{1/2}\tilde{\epsilon}^{-1}\exp(-2\pi^2f_{cr}^2/I_1\tilde{\epsilon}^2)\,, \quad (3.108)$$

where

$$I_j\equiv\int_{-\infty}^{+\infty}dx\sum_{k=1,2}[d^jU_k(x)/dx^j]^2\quad(j=1,2,3)\,. \quad (3.109)$$

Using Eq. (3.108), it is straightforward to find the abovementioned quantity $\mu(f)$ of kinks released at a given f:

$$\mu(f)=\int_0^f p(f_{cr})df_{cr}=\mathrm{erf}(\sqrt{2}\pi f/\sqrt{I_1}\tilde{\epsilon})\,. \quad (3.110)$$

Further analysis differs for the two models (3.104) and (3.106).

b. The overdamped continuum model

In the case of the overdamped model (3.106), we need to take account of the fact that a released kink may be trapped again in the vicinity of another point ξ_{cr} with $f_{cr}>f$. So, to find the density of free kinks that contribute to the solitonic conductivity, it is necessary to know the maximum density $n_t(f)$ of kinks that may be trapped at a given f. We know that $n_t(f)$ is proportional to the density ν of the points ξ_{cr}. It is easy to find $\nu=\pi^{-1}(I_3/I_2)$, where $I_{2,3}$ is defined in Eq. (3.109). The conductivity problem may be considered in the one-kink approximation provided the density n_0 of kinks trapped at $f=0$ is small, $n_0\ll1$. In this case the range of concern is $f\gg\tilde{\epsilon}$, where one can easily obtain from Eqs. (3.108) and (3.110)

$$n_t(f)=\nu\mu(f)\approx(2\pi^5)^{-1/2}\sqrt{I_1I_3/I_2}\,\tilde{\epsilon}f^{-1}\exp(-2\pi^2f^2/I_1\tilde{\epsilon}^2)\,. \quad (3.111)$$

The system becomes conductive at $f=f_0$ such that $n_t(f_0)=n_0$. According to Eq. (3.111),

$$f_0^2\approx(I_1/2\pi^2)\tilde{\epsilon}^2\ln n_0^{-1}\,. \quad (3.112)$$

At $f>f_0$ the density of free kinks is $n(f)=n_0-n_t(f)$. The mean velocity of a free kink in the overdamped model is given by the "nonrelativistic" limit of Eq. (3.47):

$$V(f)=\pi f/4\gamma\,. \quad (3.113)$$

Thus at $f>f_0$ the dependence of the solitonic current j on the dc voltage f, i.e., the current-voltage characteristic of the overdamped continuum model, takes the form

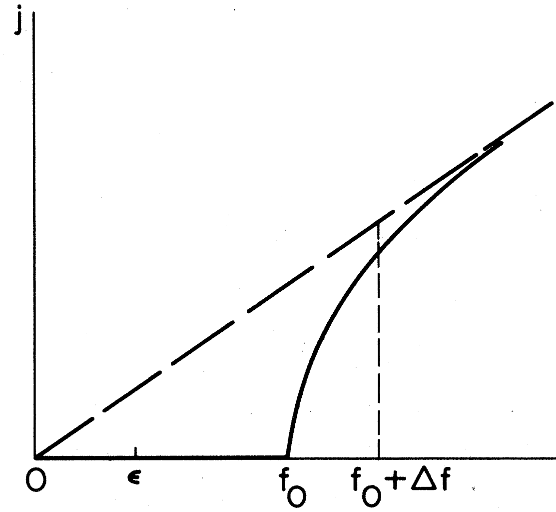

FIG. 8. The solitonic current-voltage characteristic (3.114) and (3.112) of the overdamped continuum model (3.106). The dashed line depicts the solitonic current-voltage characteristic $j=n_0(\pi f/4\gamma)$ of the overdamped homogeneous system.

$$j=V(f)[n_0-n_t(f)]\approx n_0(\pi f/4\gamma)\{1-\exp[-(4\pi^2f_0/I_1\tilde{\epsilon}^2)(f-f_0)]\}\,. \quad (3.114)$$

The whole current-voltage characteristic is shown schematically in Fig. 8. As follows from Eq. (3.114), the transient region shown in Fig. 8 is $0<f-f_0\lesssim\Delta f=I_1\tilde{\epsilon}^2/4\pi^2f_0$.

c. The damped continuum model

Let us proceed to the model represented by Eq. (3.104) with $\gamma\ll\tilde{\epsilon}$. If the trapped kinks are distributed uniformly at $f=0$, the fraction of free kinks at $f>0$ is equal to $\mu(f)$ defined by Eq. (3.110). Let us formulate conditions that guarantee that the released kinks will not be trapped again. A potential hill of height U_0 will trap a free kink if $f^2\lesssim\gamma^2U_0$; see Eq. (3.57). As can be seen from Eq. (3.110), the range of basic interest is $f\sim\tilde{\epsilon}$, so, to avoid repeated capture of the released kinks, we must require that the values taken by $|\zeta_{1,2}(x)|$ be limited by some ζ_m such that $\gamma^2\zeta_m\ll\tilde{\epsilon}^2$. Due to the assumption $\gamma\ll\tilde{\epsilon}$, this limitation is not significant.

A current-voltage characteristic determined by Eqs. (3.110) and (3.47) takes the form [cf. Eq. (3.114)]

$$j=V(f)n_0\mu(f)\approx n_0\mathrm{erf}(\sqrt{2}\pi f/\sqrt{I_1}\tilde{\epsilon}) \quad (3.115)$$

in the range $f\sim\tilde{\epsilon}$. A full current-voltage characteristic is hysteretic (Fig. 9): The branch describing Eq. (3.115) (lower curve in Fig. 9) is observed if f increases from zero, and the usual branch corresponding to $\mu(f)\equiv1$ (upper curve in Fig. 9) is observed if f decreases from

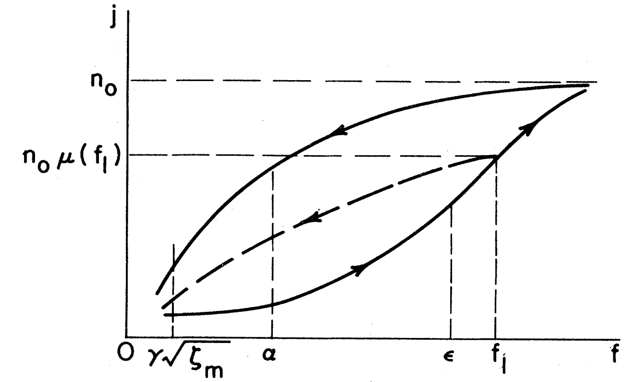

FIG. 9. The hysteretic current-voltage characteristic of the continuum model (3.104). The arrows indicate the senses of different branches of the current-voltage characteristic.

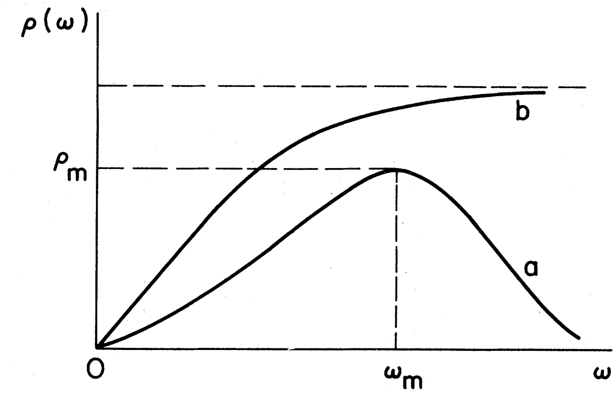

FIG. 10. The dependence of the ac conductivity ρ on the frequency ω: (a) the model (3.104); (b) the overdamped model (3.106).

large values $f \gg \tilde{\epsilon}$. If f increases along the lower branch up to some $f_1 \sim \tilde{\epsilon}$, and then turns back, one will observe an intermediate branch (dashed curve in Fig. 9) corresponding to $\mu \equiv \mu(f_1)$. The current-voltage characteristic shown in Fig. 9 terminates at $f \sim \gamma\sqrt{\zeta_m}$, where, as described above, the kinks will be trapped by maxima of the random potential.

It was implied tacitly that collisions between free kinks and trapped ones do not release the latter. We shall demonstrate in Sec. IV.B.6 that this is true under the above assumption $\tilde{\epsilon} \gg \gamma$ if the kinks are unipolar; if both polarities are present, one needs $\tilde{\epsilon} \gg \gamma^{2/5}$.

Comparing the current-voltage characteristics shown in Figs. 7 and 9, we infer that they are qualitatively similar, i.e., kinks trapped initially by an effective disordered potential escape gradually upon increase of the dc drive, and the qualitative consequences of this escape seem insensitive to details of the model.

d. ac conductivity

If the ac drive (ac electric field) $f(t) = F\cos(\omega t)$ is applied to the system described by Eq. (3.104) or (3.106), a trapped kink oscillates in the vicinity of a local minimum ξ_0 of the potential (3.107) according to the equation[2]

$$\ddot{\xi} + \gamma\dot{\xi} + (\kappa/8)(\xi - \xi_0) = (\pi/4)F\exp(i\omega t)\ , \qquad (3.116)$$

where $\kappa \equiv U''(\xi_0)$ (in the overdamped model, the term $\ddot{\xi}$ is absent). A solution to Eq. (3.116) is

$$\xi = \Xi\exp(i\omega t)\ ,$$
$$\Xi = (\pi/4)(\kappa/8 - \omega^2 + i\gamma\omega)^{-1}F\ . \qquad (3.117)$$

The ac current can be defined as follows [cf. Eqs. (3.114) and (3.115)]: $j(t) \equiv J(\omega)\exp(i\omega t) = n_0\langle\dot{\xi}\rangle$, where averaging is carried out according to Eq. (3.105). The probability density for the distribution of values κ analogous to Eq. (3.108) is

$$p(\kappa) = 2(2\pi I_2)^{-1/2}\tilde{\epsilon}^{-1}\exp(-\kappa^2/2I_2\tilde{\epsilon}^2)\ . \qquad (3.118)$$

Using Eqs. (3.117) and (3.118), it is possible to calculate the ac conductivity,

$$\rho(\omega) \equiv n_0\langle\dot{\xi}/F\rangle = in_0\omega\langle\Xi/F\rangle\ . \qquad (3.119)$$

In particular,

$$\rho(\omega) \approx (2\pi/I_2)^{1/2}(n_0\omega/\tilde{\epsilon})\ln[\tilde{\epsilon}^2/\omega^2(\gamma^2+\omega^2)]$$

at $\omega^2 \ll \tilde{\epsilon}$, and $\rho(\omega) \approx -i(\pi/4)n_0\omega^{-1}$ at $\omega^2 \gg \tilde{\epsilon}$ [Fig. 10(a)]. As a matter of fact, the latter expression pertains to a homogeneous system. A maximum $\rho_m \sim n_0\tilde{\epsilon}^{-1/2}$ attained at $\omega = \omega_m \sim \sqrt{\tilde{\epsilon}}$. For the overdamped model (3.106), $\rho(\omega) \approx (\pi/2\gamma)n_0$ at $\gamma\omega \gg \tilde{\epsilon}$, and $|\rho(\omega)| \approx 2(2\pi/I_2)^{1/2}(n_0\omega/\tilde{\epsilon})\ln(\tilde{\epsilon}/\gamma\omega)$ at $\gamma\omega \ll \tilde{\epsilon}$ [Fig. 10(b)].

The frequency dependence of the ac conductivity is qualitatively similar in a model of the doped CDW system [Eq. (3.89)] with large mean distance between impurities, $l \gg 1$, discussed by Malomed (1988e, 1989a).

4. Adiabatic interaction of a periodic chain of fluxons with a periodic lattice of impurities

A direct generalization of the one-kink solution to the unperturbed SG equation describes a periodic chain of unipolar kinks:

[2]Strictly speaking, Eq. (3.117) is valid for $\omega^2 \ll 1$. Otherwise, as explained above in Sec. III.C.1, F must be replaced by the renormalized value $F_R = F(1-\omega^2)^{-1}$, see Eq. (3.50). Accordingly, the expressions for the ac conductivity $\rho(\omega)$ given below must be multiplied by $(1-\omega^2)^{-1}$. The near-resonant case $|1-\omega^2| \ll 1$ requires special consideration (see Secs. VI.C.1 and VI.C.2 below). Analogously, in the overdamped model the renormalizing multiplier is $(1+i\gamma\omega)^{-1}$.

$$u=\pi-2\,\mathrm{am}\left[\frac{x-\xi}{k},k\right], \tag{3.120}$$

where am is the Jacobi elliptic amplitude, k is its modulus ($0<k<1$), and ξ is an arbitrary phase constant. The period of the chain is

$$L=2kK(k), \tag{3.121}$$

where $K(k)$ is a complete elliptic integral of the first kind. In the limit $k\rightarrow 1$ the solution (3.120) goes over into the usual kink (2.61a). In the opposite limit $k\rightarrow 0$, Eq. (3.120) describes a "closely packed" chain that is close to a linear function:

$$u=\pi-\frac{2}{k}(x-\xi)-\frac{k^2}{4}\sin\left[\frac{x-\xi}{2k}\right]+O(k^4). \tag{3.122}$$

A moving chain can be obtained from Eq. (3.120) by the transformation

$$\xi\rightarrow Vt+\xi_0 \tag{3.123}$$

(we consider only the nonrelativistic case $V^2\ll 1$).

In the theory of long Josephson junctions, Eqs. (3.120)–(3.123) represent a periodic chain of fluxons. A chain of arbitrary density L^{-1} can be created by a magnetic field $H\sim L^{-1}$ applied to the junction. From the practical standpoint, a periodic chain of fluxons is a more important object than a solitary fluxon. Although the chain is a multisoliton aggregate, we consider its dynamics in the present "one-soliton" section of this paper since we shall regard the chain as absolutely rigid, neglecting its deformation under the action of perturbations. If perturbative parameters are small, this assumption is applicable for a "sufficiently dense" chain, viz., $k^2(1-k^2)^{-1}\lesssim 1$.

A physically important problem is the interaction of the chain with a periodic lattice of impurities in a long Josephson junction described by the model (3.76). Recently, this problem was solved analytically and was realized in an experiment with a real Josephson junction by Malomed and Ustinov (1989). Here we give the main results.

First of all, let us consider the simplest static problem: If the chain and the lattice have commensurate periods, i.e.,

$$a=(p/q)L \tag{3.124}$$

[where L and a are defined by Eqs. (3.121) and (3.76), and where p and q are integers], it is necessary to find the minimum (critical) value $f_{\rm cr}$ of the bias current density f that frees the fluxon chain from the pinning lattice. To tackle the problem, one uses the Hamiltonian of the chain-lattice interaction,

$$H_\epsilon=-2\epsilon\sum_{n=-\infty}^{+\infty}\mathrm{cn}^2\left[\frac{\xi-an}{k},k\right], \tag{3.125}$$

where cn is the Jacobi elliptic cosine. The Hamiltonian accounting for the driving force is

$$H_f=f\int_{-\infty}^{+\infty}u(x)dx=\int_{-\infty}^{+\infty}dx\frac{\pi f\xi}{kK(k)}+\text{const}. \tag{3.126}$$

If a chain is pinned by a commensurate lattice, the corresponding value of the chain's phase ξ is determined by the equation

$$\frac{\partial}{\partial\xi}(H_\epsilon+H_f)=0. \tag{3.127}$$

The value $f_{\rm cr}$ is that at which two roots of Eq. (3.127) merge and disappear. It can be found in an explicit form for the particular case $q=1$ [see Eq. (3.124)]:

$$f_{\rm cr}=\frac{4\epsilon^2}{\pi^2p^2}\frac{(5k^2-2k^4-2)(1+k^2)+2(1+k^4-k^2)^{3/2}}{27k^6}. \tag{3.128}$$

In particular, in the limit $k\rightarrow 0$ one obtains from Eq. (3.128)

$$f_{\rm cr}\approx\epsilon/\pi pk. \tag{3.129}$$

If k is small (a "dense" chain), the value $f_{\rm cr}$ can be found for arbitrary p and q:

$$f_{\rm cr}\approx\frac{\epsilon q^2}{4\pi p}\left[\frac{k}{4}\right]^{2q-3}. \tag{3.130}$$

In the particular case $q=1$ Eq. (3.130) coincides with (3.129).

To compare the theoretical results with an experiment, it is necessary to express the modulus k in terms of p, q, and the lattice spacing a according to Eq. (3.124). In particular, for $k\ll 1$ and $q=1$, Eq. (3.129) can be cast in the form

$$f_{\rm cr}\approx\frac{\epsilon}{a}\left[1+\frac{5}{4}\left[\frac{a}{2\pi p}\right]^4\right], \tag{3.131}$$

where by implication $a\ll 2\pi p$. Comparison of Eq. (3.131) with experimental data has demonstrated a fairly good agreement.

Malomed and Ustinov (1989) have also found a minimum (threshold) value of f that admits free motion of the chain through a commensurate lattice. For $q=1$ and arbitrary k the problem is, in fact, equivalent to that for one kink considered above [see Eqs. (3.85) and (3.86)]. For $k^2\ll 1$ (p and q arbitrary),

$$f_{\rm thr}^2=\frac{4\gamma^2|\epsilon|q^2}{\pi^3pk}\left[\frac{k}{4}\right]^{2(q-1)},$$

where γ is, as above, the dissipative constant.

5. Long-scale envelope solitons in an array of fluxons in a Josephson junction with a periodic lattice of impurities

In the preceding subsection we considered an array of fluxons [Eq. (3.120)], assuming it to be absolutely rigid. Now, following the work of Malomed, Oboznov, and Ustinov (1989), we shall demonstrate that a weak deformation of such an array interacting with a commensurable lattice of defects according to Eq. (3.76) gives rise to kinklike solitons of a new type.

We shall confine ourselves to the case in which $q=1$ in Eq. (3.124). Let us assume that the array is subject to a long-scale modulation. To that end, we replace the phase constant ξ in Eq. (3.120) by a function $\xi(x,t)$ varying on a large scale $l \gg L$, L being the spatial period (3.121) of the array. Proceeding from the fluxon-defect and fluxon-current interaction Hamiltonians [Eqs. (3.125) and (3.126)], and taking account of the SG Hamiltonian proper $H_0=\int_{-\infty}^{+\infty}dx\,(1-\cos u)$, it is straightforward to find an effective Hamiltonian written in terms of the modulation function $\xi(x,t)$:

$$H_{\rm eff}=\int_{-\infty}^{+\infty}dx\, h_{\rm eff}\ , \qquad h_{\rm eff}=\tfrac{1}{2}\rho(\xi_t^2+\xi_x^2)-2\epsilon a^{-1}{\rm cn}^2(\xi/k,k)-G\xi\ , \tag{3.132}$$

where $\rho\equiv 4k^{-2}E(k)/K(k)$ is the mass density of the array (3.120), and $G\equiv-\pi f/[kK(k)]$ is the density of the force exerted by the bias current upon the array. The Hamiltonian (3.132) gives rise to the equation

$$\xi_{tt}-\xi_{xx}+4\epsilon a^{-1}\rho^{-1}k^{-1}{\rm sn}(\xi/k){\rm cn}(\xi/k){\rm dn}(\xi/k)=\rho^{-1}G-\gamma\xi_t\ , \tag{3.133}$$

where sn, cn, and dn are the standard Jacobi elliptic functions. The dissipative term in Eq. (3.133) [γ is the same as in Eq. (3.76)] can be derived by means of an energy-balance analysis. Note that the nonlinearity in the left-hand side of Eq. (3.133) is solely due to the fluxon-defect interaction, and the limit (Swihart) velocity $V_{\rm Sw}=1$ is the same as in Eq. (3.76).

Equation (3.133) with $G=\gamma=0$ goes over into the SG equation in the limit $k\to 0$. At finite k, the equation is not exactly integrable. Nevertheless, it supports an exact kinklike solution (an envelope soliton, or "superkink"). The quiescent "superkink" has the form

$$\xi(x)=kF(\sin^{-1}[(1-k^2)\cosh^2(2k^{-1}\sqrt{\epsilon/a\rho}\,x)+k^2]^{-1/2},k),\quad -\infty<x<0\ , \tag{3.134a}$$

$$\xi(x)=2kK(k)-\xi(-x),\quad 0<x<+\infty\ , \tag{3.134b}$$

if $\epsilon>0$, and

$$\xi(x)=-kF(\cos^{-1}\{(1-k^2)/[\cosh^2(2\sqrt{1-k^2}\,k^{-1}\sqrt{|\epsilon|/a\rho}\,x)-k^2]\}^{1/2},k),\quad -\infty<x<0\ , \tag{3.135a}$$

$$\xi(x)=-\xi(-x),\quad 0<x<+\infty\ , \tag{3.135b}$$

if $\epsilon<0$. Here $F(z,k)$ is an incomplete elliptic integral of the first kind. The "superkinks" (3.134) and (3.135) are distinguished by the property

$$\xi(x=+\infty)-\xi(x=-\infty)=2kK(k)\equiv L\ , \tag{3.136}$$

L being the period (3.121) of the underlying array of fluxons. Thus the superkink may be interpreted as a hole in the infinite array, i.e., a gap where one fluxon is lacking. Quite analogously, a superkink of the opposite polarity, which is given by the same solutions (3.134) and (3.135) with the opposite sign, may be interpreted as a surplus fluxon in the array. A moving superkink can be obtained from Eqs. (3.134) and (3.135) by the Lorentz transformation.

In the case $L\gg 1$ [see, however, the inequality (3.140) below] one may regard the array (3.120) as consisting of individual weakly interacting particlelike fluxons. This is, in fact, a variant of the Frenkel-Kontorova model, and the superkink may be identified with the well-known dislocation soliton in that model; see, for example, Pokrovskii and Talapov (1978).

Let us proceed to the case $G,\gamma\neq 0$. We shall be interested in the range $f<f_{\rm cr}$ [$f_{\rm cr}$ is determined by Eqs. (3.128) and (3.131)], where the array as a whole remains pinned. An energy-balance analysis neglecting a distortion of the superkink's shape yields an equilibrium velocity V_0 of the moving superkink [cf. the fluxon's equilibrium velocity, Eq. (3.47)]

$$V_0^2/(1-V_0^2)=(\pi^2p/2)k^3[K^2(k)/E(k)](f^2/\gamma^2\epsilon)\times\{\ln[(1+k)/(1-k)]\}^{-2} \tag{3.137}$$

if $\epsilon>0$, and

$$V_0^2/(1-V_0^2)=(\pi^2p/8)k^3[K^2(k)/E(k)](f^2/\gamma^2|\epsilon|)\times(\sin^{-1}k)^{-2}\ , \tag{3.138}$$

if $\epsilon<0$, where p is the commensurability index defined by Eq. (3.124) (recall that we have set $q=1$).

In Josephson-junction theory, the averaged quantity $\overline{u_t}$ is the voltage across the junction. Making use of the underlying equation (3.120) with regard to Eq. (3.136) yields eventually

$$\overline{u_t}=2\pi V_0/R\ , \tag{3.139}$$

where R is the total length of the junction. This equation, together with Eq. (3.137) or (3.138), gives a superkink current-voltage characteristic of the long Josephson junction containing the pinned array of fluxons. It is noteworthy that a single fluxon moving with velocity V_0 gives rise to the same current-voltage characteristic (3.139).

In conclusion, let us estimate the limits of validity of the envelope-soliton concept. According to Eqs. (3.134) and (3.135), the size l of the superkink may be estimated as follows: $l^2 \sim pk|\epsilon|^{-1}/\ln[2(1-k)^{-1}]$. Comparing this expression to the array's period [Eq. (3.121)] and making use of Eq. (3.124) with $q=1$, we conclude that the underlying assumption $L^2 \ll l^2$ amounts to the requirement

$$L^3 \ll a|\epsilon|^{-1} . \tag{3.140}$$

This means that the array must be sufficiently dense.

The relevance of this kind of envelope-soliton analysis has been supported by numerical simulations of both Eqs. (3.76) and (3.133) carried out by Malomed, Oboznov, and Ustinov (1989) [see also Ustinov (1989)]. It is interesting to note that, although Eq. (3.133) with $G=\gamma=0$ is not exactly integrable, a numerically simulated collision between a superkink and a super-antikink proves to be almost elastic for a fairly wide range of values of the modulus k.

6. Perturbed dynamics of a breather

The dynamics of a sine-Gordon breather subjected to the action of perturbations is more complicated than that of a kink. General perturbation-induced evolution equations for the parameters μ, V, ξ, and Ψ of the breather (2.63) and (2.64) have been derived by Kosevich and Kivshar (1982) and Karpman, Maslov, and Solov'ev (1983):

$$\frac{d\mu}{dt}=\epsilon(1-V^2)^{1/2}(4\cos\mu)^{-1}I_1 , \tag{3.141}$$

$$\frac{dV}{dt}=-\epsilon(1-V^2)^{3/2}(4\cos\mu)^{-1}I_2 , \tag{3.142}$$

$$\frac{d\xi}{dt}=V+\epsilon(1-V^2)(2\sin\mu)^{-2}(I_3-V\tan\mu I_4) , \tag{3.143}$$

$$\begin{aligned}\frac{d\Psi}{dt}=&(1-V^2)^{1/2}\cos\mu\\ &-\epsilon(1-V^2)^{1/2}(4\sin\mu\cos^2\mu)^{-1}\\ &\times[V\cot\mu I_3+(1-V^2)\cos^2\mu I_4-I_5] ,\end{aligned} \tag{3.144}$$

where $\zeta\equiv\sin\mu z$,

$$I_1\equiv\int_{-\infty}^{+\infty}d\zeta\frac{\cosh\zeta\cos(\cos\mu\Psi)}{\cosh^2\zeta+\tan^2\mu\sin^2(\cos\mu\Psi)}\times P[u_{\rm br}(\zeta,\Psi)] , \tag{3.145}$$

$$I_2\equiv-\int_{-\infty}^{+\infty}d\zeta\frac{\sinh\zeta\sin(\cos\mu\Psi)}{\cosh^2\zeta+\tan^2\mu\sin^2(\cos\mu\Psi)}\times P[u_{\rm br}(\zeta,\Psi)] , \tag{3.146}$$

$$I_3\equiv\int_{-\infty}^{+\infty}d\zeta\frac{\zeta\cosh\zeta\cos(\cos\mu\Psi)}{\cosh^2\zeta+\tan^2\mu\sin^2(\cos\mu\Psi)}\times P[u_{\rm br}(\zeta,\Psi)] , \tag{3.147}$$

$$I_4\equiv-\int_{-\infty}^{\infty}d\zeta\frac{\zeta\sinh\zeta\sin(\cos\mu\Psi)}{\cosh^2\zeta+\tan^2\mu\sin^2(\cos\mu\Psi)}\times P[u_{\rm br}(\zeta,\Psi)] \tag{3.148}$$

$$I_5\equiv-\int_{-\infty}^{\infty}d\zeta\frac{\cosh\zeta\sin(\cos\mu\Psi)}{\cosh^2\zeta+\tan^2\mu\sin^2(\cos\mu\Psi)}\times P[u_{\rm br}(\zeta,\Psi)] . \tag{3.149}$$

If the unperturbed internal oscillations of the breather are much faster than the perturbation-induced changes of the breather's parameters, one can readily derive evolution equations averaged in the unperturbed oscillations. This has been done with the use of modified conservation laws by Pagano *et al.* (1987).

The simplest problem is to calculate the rate of dissipative damping of the breather under the action of the perturbing term $-\gamma u_t$ on the right-hand side of the SG equation. The damping law takes a simple form for a quiescent ($V=0$) small-amplitude ($\mu \ll 1$) breather (McLaughlin and Scott, 1978):

$$\frac{d\mu}{dt}=-2\gamma\mu .$$

A more sophisticated problem is synchronization of a breather by the external time-periodic force (1.17) [$f(t)=\sin(\omega t)$, $0<\omega<1$] in the presence of dissipation. The "small-amplitude" version of this problem reduces to the one considered above in the context of the NS equation with the perturbation (1.9), where ϵ is the same as now, and $\Omega\equiv 1-\omega$. Like a small-amplitude breather, a breather of a general form may be locked to the driving frequency if ϵ exceeds the threshold value (Lomdahl and Samuelsen, 1986)

$$\epsilon_{\rm thr}=\frac{2\gamma(1-\omega^2)\sin^{-1}(\sqrt{1-\omega^2})}{K(1-\omega^2)-E(1-\omega^2)} , \tag{3.150}$$

where γ is the dissipative constant, and $K(z)$ and $E(z)$ are complete elliptic integrals of the first and second kinds, respectively. The frequency-locking problem for the perturbation (1.21) can be considered analogously. In the "small-amplitude" limit the latter problem goes over into a similar problem for the perturbation (3.33) considered above in the context of the NS equation.

A synchronized breather may also be supported by the perturbation

$$P(u)=\delta(x)\sin(\omega t),\quad \omega<1 \tag{3.151}$$

[it is implied that the dissipation (1.16a) is present too]. Physically, this perturbation describes the action of an oscillating magnetic field applied to the edge of a semi-infinite Josephson junction (Olsen and Samuelsen, 1986b). As has been demonstrated by Olsen and Samuelsen in the paper just cited in the framework of the energy approach, the perturbation (3.129) supports a phase-locked breather with the internal frequency ω if the perturbation amplitude ϵ exceeds the threshold value [for simplicity, it is set $\beta=0$; see Eq. (1.16)]

$$\epsilon_{\text{thr}}=2\gamma\left[\frac{1+\omega}{1-\omega}\right]^{1/2}\sin^{-1}\sqrt{1-\omega^2}\ .$$

Due to the threshold character of the excitation of a breather by the driving force considered, this system is hysteretic: depending on initial conditions, one will observe as a final state either the synchronized breather or a quasilinear standing plasma wave, which is an obvious solution of the linearized SG equation with the perturbation (3.151),

$$u_{\text{pl}}(x,t)\approx\frac{\epsilon}{2\sqrt{1-\omega^2}}e^{-\sqrt{1-\omega^2}x}\sin(\omega t)\ .$$

It is clear that a similar hysteresis takes place in a system driven by the external force (1.17) with $f(t)=\sin(\omega t)$.

Constant perturbations, i.e., those described by Eq. (1.17) or (1.21) with $f(t)\equiv 1$, can result in a decay of the low-frequency breather (2.66) into a kink-antikink pair. In other words, these perturbations alter the interval of the breather's internal frequencies $\omega\equiv\cos\mu$ equal to $0<\omega<1$ in the absence of perturbations. According to Karpman, Maslov, and Solov'ev (1983), for the constant perturbation (1.17) the threshold (minimum) value of ω at which the breather does not decay is

$$\omega_{\text{thr}}^2=\frac{\pi\epsilon}{4}\left[\ln\left[\frac{16}{\pi\epsilon}\right]+1\right]\ . \qquad (3.152)$$

It is interesting to note that the same result, provided $\ln\epsilon^{-1}\gg 1$, has been obtained by Lomdahl, Olsen, and Samuelsen (1984) in the framework of a very simple "direct" approach. As to the perturbation (1.21), a similar result has been obtained by Kosevich and Kivshar (1982).

It should be emphasized that, in the presence of the perturbations mentioned, a breather is stable in the adiabatic approximation only: In fact, it slowly decays due to emission of radiation (see Sec. VI.F).

The interaction of a breather with a localized inhomogeneity requires special consideration. If the breather is not close to a threshold of decay into a kink-antikink pair (more accurately, we need $\cos\mu\gg\epsilon$), it is natural to average the perturbation Hamiltonian over the fast internal oscillations of the breather. As a result, for the perturbation $\epsilon\delta(x)\sin u$ [see Eq. (1.19)] one can obtain the averaged effective interaction potential (Malomed, 1987d, 1987e)

$$U(\xi)=-8\epsilon\cot\mu\cosh(\sin\mu\xi)\times[1+\cot^2\mu\cosh^2(\sin\mu\xi)]^{-3/2}\ . \qquad (3.153)$$

According to the paper mentioned, the breather may be considered as a nonrelativistic particle of mass $m=16\sin\mu$ with the coordinate $\xi(t)$ moving in the potential (3.153). The shape of the potential is shown in Fig. 11. Equation (3.153) can be readily generalized for the more general localized perturbation $-\epsilon\delta\theta(x)\sin(u+\theta)$ [see Eq. (3.89)]:

$$U(\xi)=-8\epsilon\cos\theta\cot\mu\cosh(\sin\mu\xi)\times[1+\cot^2\mu\cosh^2(\sin\mu\xi)]^{-3/2}\ .$$

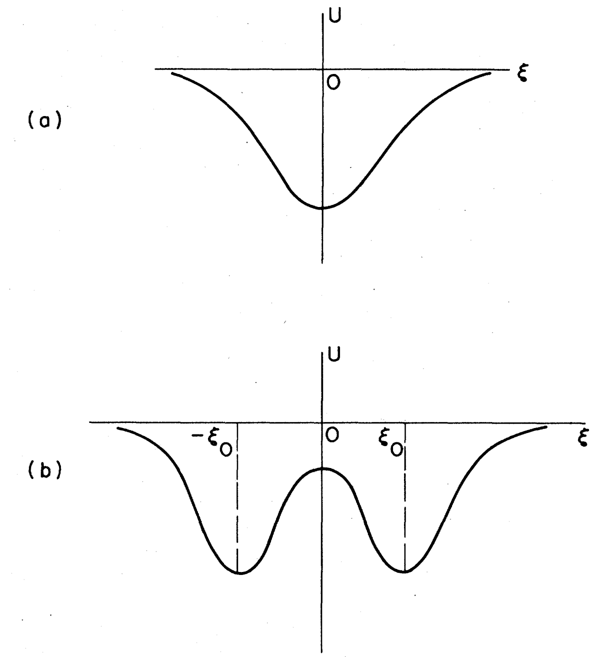

FIG. 11. The shape of the potential (3.153) for $\epsilon>0$: (a) $\cos^2\mu>\frac{1}{3}$; (b) $\cos^2\mu<\frac{1}{3}$. As can be seen from the two graphs, in case (a) a bound state (at $\xi=0$) exists for positive ϵ only, while in case (b) there are two bound states at $\xi=\pm\xi_0\equiv\pm(\sin\mu)^{-1}\ln[(\tan\mu+\sqrt{\tan\mu-2})/\sqrt{2}]$ for $\epsilon>0$, and one bound state at $\xi=0$ for $\epsilon<0$ (Malomed, 1987d, 1987e).

If the frequency $\cos\mu$ of the internal breather becomes sufficiently small $\sim\epsilon$, the averaging becomes irrelevant. In this case the interaction of internal oscillations with "external" oscillations of the breather as a whole near the inhomogeneity may result in stochastization of the breather's dynamics. This problem is analyzed in Sec. V.E.2.

Finally, interaction of a moving breather that has a sufficiently small value of $\cos\mu$ (i.e., a low-frequency breather) with a localized inhomogeneity may result in breakup of the breather into a kink-antikink pair. The possibility of this inelastic process was first noted by Malomed (1985). This effect becomes most noticeable for the perturbation $P=\delta'(x)$ (an Abrikosov vortex intersecting a long Josephson junction): According to Gurovich and Mikhalev (1987), this is possible provided $\cos^2\mu < \frac{1}{2}\epsilon V$, where V is the initial velocity[3] of the breather.

IV. MANY-SOLITON ADIABATIC EFFECTS

A. Inelastic interactions of solitons in dissipatively perturbed nonlinear Schrödinger and sine-Gordon equations

1. Preliminary remarks

Dissipative perturbations give rise to inelastic interactions of solitons in a direct way. In this section we shall consider two typical interactions of this kind: fusion of two free solitons into a breather, and breakup of a breather into two free solitons due to collision with another (fast) soliton. Both problems will be considered within the framework of the SG equation perturbed by the dissipative term (1.16a) and the NS equation with the dissipative perturbations (1.8). Since the NS breather is a less popular object than the SG, we shall also describe its structure.

2. Fusion of a kink-antikink pair into a breather

The simplest perturbation-induced inelastic interaction is the fusion of a free SG kink and antikink into a breather (Kaup and Newell, 1978a). The basic characteristic of this process is the maximum (threshold) value $W_{\rm thr}$ of the relative velocity W of the colliding kinks allowing their fusion. A natural way to find $W_{\rm thr}$ is to employ the energy balance. Indeed, if one evaluates the collision-induced dissipative energy loss $E_{\rm diss}$, the threshold velocity may be found by equating $E_{\rm diss}$ to the kinetic energy $E_{\rm kin}$ of the kink-antikink pair before the collision. The expression for $E_{\rm diss}$ can be obtained directly from the Eq. (2.69a) for the energy of the SG wave field, with regard to the form of the perturbations (1.16):

$$E_{\rm diss} = \left| \int_{-\infty}^{+\infty} dt \frac{dE}{dt} \right| = \gamma \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dx\, u_t^2 + \beta \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dx\, u_{tx}^2 \; . \tag{4.1}$$

Since the parameters γ and β are small, $W_{\rm thr}$ will be small too, i.e., one may employ the "nonrelativistic" approximation $E_{\rm kin} \approx 2W^2$. For the same reason, in the first approximation one may describe the colliding kink and antikink by the exact unperturbed solution (2.67). Indeed, this solution describes a pair consisting of a kink and an antikink with zero relative velocity at $t = -\infty$. Due to mutual attraction, the kinks accelerate to "relativistic" relative velocities ~ 1, and the presence of a small nonzero initial velocity may be neglected. Inserting Eq. (2.67) into Eq. (4.1) yields

$$E_{\rm diss} = 8\pi^2\gamma + 6\pi^2\beta \; . \tag{4.1'}$$

Then, equating $E_{\rm kin}$ to Eq. (4.1′) yields

$$W_{\rm thr} = 2\pi\sqrt{\gamma + (3/4)\beta} \tag{4.2}$$

(an estimate that $W_{\rm thr}^2 \simeq \gamma$ was first obtained by Kaup and Newell, 1978a).

According to Eq. (2.60), a pair consisting of a kink and antikink moving with small velocities $\pm W/2$ correspond to a pair of points $\lambda_{1,2} = (i/2)(1 \pm W/2)$ on the complex plane of the spectral parameter λ, while the low-frequency breather described by Eq. (2.66) with $V=0$ corresponds to the pair of points (2.62) with $V=0$, $\mu = \pi/2 - \zeta$. Trajectories of motion of the points $\lambda_{1,2}$ on the complex plane during perturbation-induced fusion of the pair into the breather is shown in Fig. 12.

The result (4.2) has an important physical application in the theory of long damped dc-driven Josephson junctions. Indeed, the velocities of the steady motion of a free kink and antikink (fluxon and antifluxon) are, according to Eq. (3.47), $V_0 \approx \pm \pi\epsilon/4$ (we assume $\epsilon \ll \gamma$), i.e., the corresponding relative velocity $W \approx \pi\epsilon/2\gamma$. Equating this to Eq. (4.2), we see that a "fluxon-antifluxon plasma" may exist provided the bias current ϵ exceeds the threshold value

$$\epsilon_{\rm thr} = 4\gamma^{3/2} \tag{4.3}$$

(we have set $\beta=0$). Equation (4.3) was obtained by Malomed (1985b). Independently and a little bit earlier, a similar expression was obtained by Pedersen *et al.* (1984). Unfortunately, the latter paper contains a com-

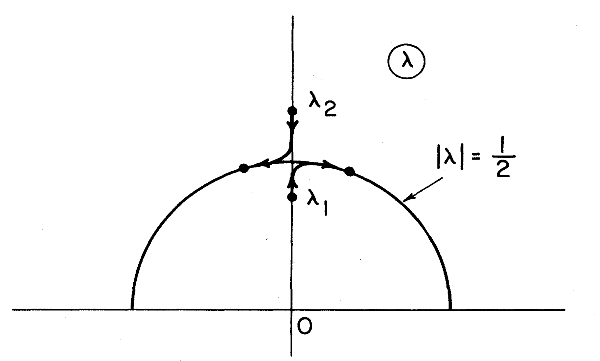

FIG. 12. Motion of the points $\lambda_{1,2}$ in the complex plane of the spectral parameter λ corresponding to the fusion of a kink-antikink pair into a low-frequency breather.

[3]The *averaged* potential of the interaction between a breather and this type of local inhomogeneity is zero, in contrast with the potential (3.153).

putational error, due to which the numerical coefficient in front of $\gamma^{3/2}$ in that paper differs from ours in Eq. (4.3).

Calculation of dissipative losses in the kink-antikink collision has another interesting application, to the computation of a one-fluxon current-voltage characteristic of a finite-length Josephson junction (Olsen *et al.*, 1986). Indeed, reflection of a fluxon from a junction's edge may be regarded as the collision of a fluxon with its "mirror image." In this case the genuine dissipative energy loss is half the full loss (4.1′). Since the current-voltage characteristic, in fact, reflects the dependence of the mean fluxon velocity on the bias current ϵ, it is clear that, for a finite-length Josephson junction, taking this loss into account results in a departure of the current-voltage characteristic from that obtained in an infinitely long Josephson junction [usually called a zero-field step (Fulton and Dynes, 1973)]. A full analytical expression can be found in the paper of Olsen *et al.* (1986), where the effect of the unperturbed collision-induced phase shift on the form of the current-voltage characteristic has also been taken into account.

It is interesting to consider a full phase portrait of the evolution of a kink-antikink pair in the presence of a combination of perturbing terms (1.16a) and (1.17) with $f(t)\equiv 1$. The solution of the unperturbed SG equation describing a free kink-antikink pair in the rest reference frame of its center of mass is (Zakharov *et al.*, 1980)

$$u = 4\tan^{-1}[T(t)\operatorname{sech}(x/\sqrt{1-V^2})]\ , \tag{4.4a}$$

$$T(t)\equiv V^{-1}\sinh(Vt/\sqrt{1-V^2})\ , \tag{4.4b}$$

where the velocities of the kink and antikink at $t\to\pm\infty$ are $\pm V$. Note that the analytical continuation $V\to i\cot\mu$ transforms Eq. (4.4) into the quiescent breather (2.63) and (2.64).

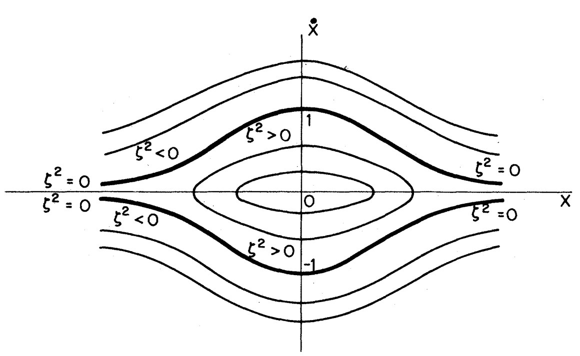

FIG. 13. The phase portrait of the kink-antikink pairs (4.4) and breathers (2.63) in the coordinates (4.5) for the unperturbed SG equation. The separatrix (bold line) corresponds to the solution (2.67). The closed trajectories inside the separatrix correspond to the breathers (2.63), while the open trajectories outside it correspond to the free pairs (4.4). The point O is a center, in terms of the dynamical systems theory.

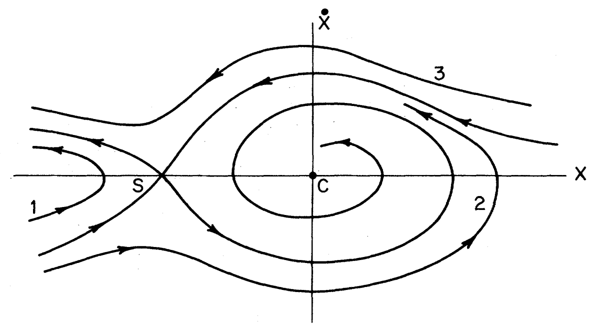

FIG. 14. The phase portrait of the evolution of the kink-antikink pairs (4.4) and breathers (2.63) in the coordinates (4.5) for the SG equation with the combined perturbation $-\gamma u_t+\epsilon$. The points C and S are, respectively, a center and a saddle. The bold separatrices of the saddle are boundaries between (1) reflected, (2) captured, and (3) passing trajectories.

The perturbed phase portrait was constructed by Karpman *et al.* (1983). They depicted only a part of the portrait that represented breathers in the coordinates $(\omega\equiv\cos\mu,\Psi)$; see Eqs. (2.63) and (2.64). We find it more visual to give the portrait in other coordinates,

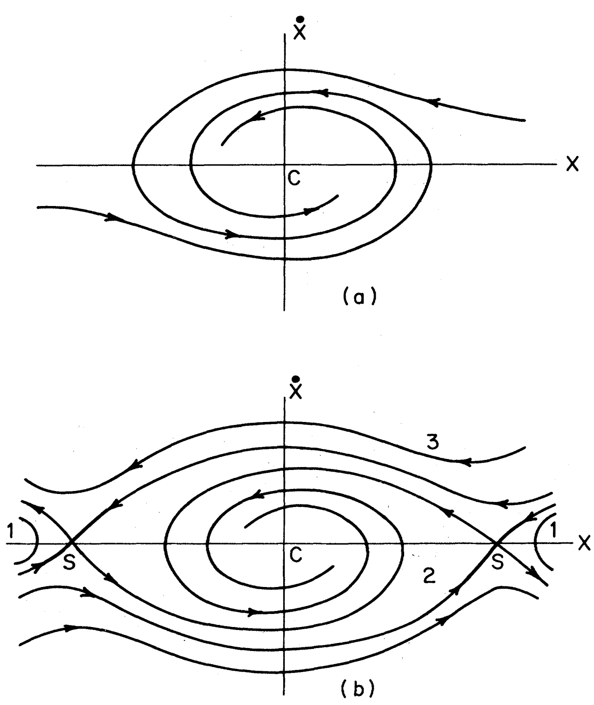

FIG. 15. The phase portrait of the evolution of the kink-antikink pairs (4.4) and breathers (2.63) in the coordinates (4.5) for the SG equation with the combined perturbation $-\gamma u_t+\epsilon\sin(u/2)$: (a) $\epsilon<0$; (b) $\epsilon>0$. The points S and C are saddles and centers, respectively. The bold separatrices of the saddles [in case (b)] are boundaries between (1) reflected, (2) captured, and (3) passing trajectories.

$$X \equiv \ln(T+\sqrt{1+T^2}), \quad \dot{X} \equiv \dot{T}/\sqrt{1+T^2}\ , \tag{4.5}$$

where for a kink-antikink pair T is defined in Eqs. (4.4), and for a breather [cf. Eq. (2.63)]

$$T \equiv \tan\mu \sin(\Psi \cos\mu)\ . \tag{4.6}$$

In particular, for the low-frequency breather (2.66) with $V=0$,

$$T \approx \zeta^{-1}\sin(\zeta t)\ . \tag{4.6$'$}$$

In these coordinates, one can depict both the breathers and the free pairs on the same phase plane. For the unperturbed case $\gamma=\epsilon=0$ the phase portrait is shown in Fig. 13 (Malomed, 1987d; 1987e). In the presence of the aforementioned perturbation, the portrait takes the form shown in Fig. 14.

A combination of the dissipative term $-\gamma u_t$ and the perturbation (1.21) with $f(t)\equiv 1$ also finds important physical applications. For instance, this is the simplest model of a weak ferromagnet in an external magnetic field (Zvezdin, 1979; Bar'yakhtar *et al.*, 1980). Separate phase portraits for pairs and breathers subject to the action of this combined perturbation were constructed by Kosevich and Kivshar (1982). In Fig. 15 we show the portrait in the coordinates (4.5), which again enable us to depict both the free pairs and the breathers on the same phase plane. However, we must distinguish the two variants [Figs. 15(a) and 15(b)] corresponding to different signs of ϵ.

3. Breakup of a breather due to collision with a fast kink

Once a breather has been created, it will be gradually damped by dissipation [Eq. (1.16a)]. The damping rate can be evaluated approximately with the aid of Eq. (4.1′). Indeed, in the first approximation the perturbation-induced evolution of the low-frequency breather is dominated by the overlap stage, i.e., a relatively short portion (of duration ~ 1) of the large period $\tau \approx 2\pi/\zeta$ of the breather's internal oscillations, when the kink and antikink that are bound inside the breather are strongly overlapping. At the overlapping stage, the low-frequency breather may be approximated by the solution (2.67), i.e., a contribution from that stage to the dissipative energy loss $E_{\rm diss}$ per period τ is twice Eq. (4.1′) (the kinks overlap twice during a period). The binding energy E_b of the low-frequency breather, defined as the double rest energy of a kink minus the breather's total energy, is [see Eq. (2.69a)]

$$E_b = 16 - 16\sin\mu \approx 8\zeta^2\ . \tag{4.7}$$

Thus the damping rate averaged over many periods may be defined as follows: $8\langle d(\zeta^2)/dt\rangle \approx 16\pi^2\gamma/\tau$, or

$$\left\langle \frac{d\zeta}{dt}\right\rangle = \pi\gamma/2\ . \tag{4.8}$$

The approach based on approximating the low-frequency breather by the solution (2.67) will be repeatedly employed below.

In the presence of the same perturbation (1.16a) the breather may decay into a kink-antikink pair due to a collision with another kink (Malomed, 1985b). To consider this process, we employ the general evolution equation (2.73), which takes the following form for a low-frequency breather:

$$\frac{d\lambda_1}{dt} = -\frac{\epsilon}{4\zeta}e^{-i\zeta\Psi} \times \int_{-\infty}^{\infty} dx\, P[u(x)]\{[\psi^{(1)}(x,\lambda=\lambda_1)]^2 - [\psi^{(2)}(x,\lambda=\lambda_1)]^2\}\ , \tag{4.9}$$

where λ_1 is defined in Eq. (2.60), $\zeta \equiv \pi/2-\mu \ll 1$, and the relevant Jost functions can be cast in the form

$$\psi^{(1)}(x,\lambda=\lambda_1) = \tfrac{1}{2}\Delta^{-1}\exp(x/2) \times [\cosh x - (i/\zeta)\sin(\zeta\Psi)e^{i\zeta\Psi}\exp(-x)]\ , \tag{4.10a}$$

$$\psi^{(2)}(x,\lambda=\lambda_1) = \frac{i}{2}\Delta^{-1}\exp(-x/2) \times [\exp(i\zeta\Psi)\cosh x + \zeta^{-1}\sin(\zeta\Psi)e^{x}]\ , \tag{4.10b}$$

$$\Delta \equiv \cosh^2 x + \zeta^{-2}\sin^2(\zeta\Psi)\ . \tag{4.10c}$$

Let us consider a collision of a quiescent low-frequency breather with a fast ("ultrarelativistic") kink moving with velocity V, $1-V^2 \ll 1$. This condition greatly simplifies further analysis. As was demonstrated by Malomed (1985), in the first approximation in the small parameter $(1-V^2)$ the full SG wave field splits into the sum of the usual kink and breather wave forms:

$$u(x,t) \simeq u_k(x,t) + u_{\rm br}(x,t)\ . \tag{4.11}$$

Combinations of the Jost functions in the evolution equation (2.73) for both the breather and the kink may be taken, in the same approximation, in the "one-particle" form, ignoring the presence of another soliton.

The splitting of the wave field and combinations of the Jost functions, introduced by Malomed (1985) in terms of the inverse scattering transform, have, in fact, a simple interpretation. If one considers a collision problem with an "ultrarelativistic" relative velocity V, in the lowest approximation the unperturbed SG equation turns into the linear d'Alembert equation, and particular shapes of the solitons involved, formed by the full SG equation, play the role of "initial conditions" for that effective d'Alembert equation. A wave field described by a linear equation may be taken in the form of the linear superposition (4.11). The approximate splitting of the wave potential and necessary combinations of the Jost functions are concepts that will be repeatedly invoked hereafter in the context of the SG and other equations considered.

The evolution equation for the parameter ζ^2, when a low-frequency breather collides with a fast kink in the presence of a dissipative perturbation, takes the form (Malomed, 1985)

$$\frac{d}{dt}\zeta^2 = -\pi\sigma\gamma\cos(\zeta\Psi)\cosh t/(\cosh^2 t + T^2) , \quad (4.12)$$

where σ is the kink's polarity, and the quantity $T \equiv \zeta^{-1}\sin(\zeta\Psi)$ [see Eq. (4.6′)] varies in time according to

$$\left[\frac{dT}{dt}\right]^2 = 1 - \zeta^2 T^2 . \quad (4.13)$$

If the parameter T is large, $T^2 \gg 1$, i.e., $\sin^2(\zeta\Psi_0) \gg \zeta$, where Ψ_0 is the value of the breather's internal oscillation phase Ψ at the moment $t=0$ (when the kink's center coincides with the breather's center), one may neglect the change of T^2 during the collision. In this case integrating Eq. (4.12) immediately yields the change of the quantity ζ^2 due to the collision:

$$\Delta(\zeta^2) \equiv \int_{-\infty}^{+\infty} \frac{d(\zeta^2)}{dt} dt \approx -\pi^2\gamma\sigma\cos(\zeta\Psi_0)/(1+T_0^2)^{1/2} , \quad (4.14)$$

where

$$T_0 \equiv \sin(\zeta\Psi_0)/\zeta_0 . \quad (4.15)$$

In the opposite case, when $T_0^2 \lesssim 1$, the evolution of T^2 should be taken into account. As can be seen from Eq. (4.13), in this case we may write

$$T^2 = (T_0 + t)^2 . \quad (4.16)$$

Inserting Eq. (4.16) into Eq. (4.12) yields

$$\Delta(\zeta^2) = -2\pi\gamma\sigma\cos(\zeta\Psi_0) F(T_0) , \quad (4.17)$$

where the even function is

$$F(T_0) \equiv \int_0^{\infty} \cosh t [\cosh^2 t + (T_0 + t)^2]^{-1} dt \quad (4.18)$$

[the multiplier $\cos(\zeta\Psi_0)$ on the right-hand side of Eq. (4.18) affects only the sign of that expression since, at $T_0^2 \lesssim 1$, $|\cos(\zeta\Psi_0)|$ is close to one]. In particular,

$$F(0) \approx 1.35 . \quad (4.19)$$

The asymptotic form of the function $F(T_0)$ at large T_0^2 is $F(T) \approx (\pi/2)(T_0^2+1)^{-1/2}$. Substituting this into Eq. (4.17) recovers Eq. (4.14).[4]

[4]If one set $T_0=0$ in Eq. (4.14), one would obtain the value of $\Delta(\zeta^2)$ corresponding to the value $F(0)=\pi/2\approx 1.57$ instead of Eq. (4.19). Note that $1.35/1.57 \approx 0.86$, i.e., Eq. (4.14) is, as a matter of fact, relevant not only for $T_0^2 \gg 1$ but also for $T_0^2 \ll 1$.

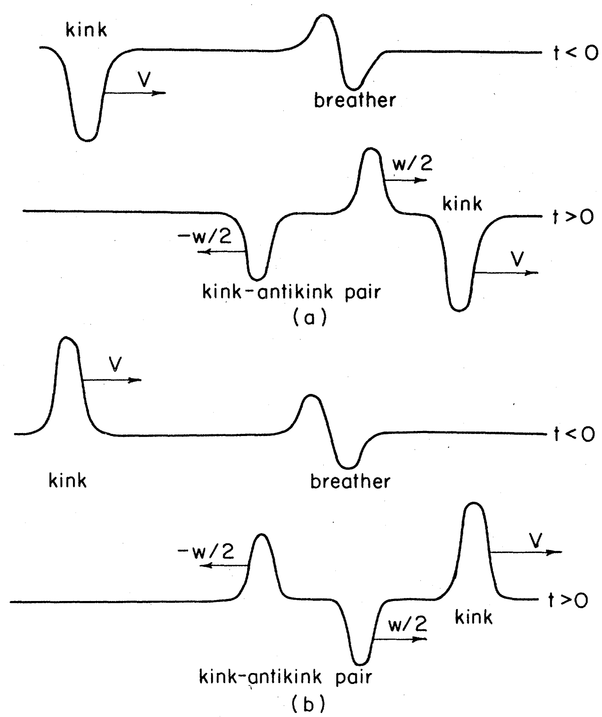

FIG. 16. The breakup of a low-frequency breather due to collision with a fast kink of polarity σ in the presence of the dissipative perturbation $-\gamma u_t$. The figure represents u_x vs x: (a) $\sigma<0$; (b) $\sigma>0$. In both (a) and (b), the upper and lower pictures correspond to initial and final states, respectively.

A collision with a fast kink may break a breather into a kink-antikink pair if $\Delta(\zeta^2)<0$, i.e., according to Eqs. (4.14) and (4.17), if $-\sigma\cos(\zeta\Psi_0)<0$.[5] The quantity $\zeta_{\rm thr}$, i.e., the maximum initial value ζ_0 of ζ which makes the breakup possible, corresponds to $\Delta(\zeta^2)=-\zeta_0^2$, and, according to Eq. (4.17),

$$\zeta_{\rm thr} = \sqrt{2\pi\gamma F(T_0)} .$$

This value corresponds to the threshold binding energy [see Eq. (4.7)] $(E_{\rm br})_{\rm thr} = -16\pi\gamma F(T_0)$.

If $\zeta<\zeta_{\rm thr}$, the collision breaks the breather into a kink-antikink pair which corresponds to a pair of points

$$\lambda_{1,2} = i/2 \pm iW/4 \quad (4.20)$$

in the complex spectral parameter plane, W being the relative kink-antikink velocity. Thus the quantity $(\lambda_1-\lambda_2)$, which is real and equal to ζ for the breather, is imaginary for the pair; see Fig. 12. The velocity W can be found from Eq. (4.18) with regard to Eq. (4.20):

[5]This condition implies that the escaping kink, which moves in the same direction as the fast one, has a polarity opposite to that of the fast kink (Fig. 16).

$$W^2=8\pi\gamma F(T_0)-4\xi_0^2 \, . \tag{4.21}$$

The solution which describes the emerging kink-antikink pair is [cf. Eq. (4.4)]

$$u=4\tan^{-1}\{W^{-1}[\delta\exp(-\tfrac{1}{2}Wt) -\delta^{-1}\exp(\tfrac{1}{2}Wt)]/\cosh x\} \, , \tag{4.22a}$$

where δ is a real constant determined by comparing Eq. (2.63) (with the parameters $\zeta=\zeta_0$ and $\Psi=\Psi_0$) to Eq. (4.22):

$$\delta-\delta^{-1}=(W/\zeta_0)\sin\Psi \, . \tag{4.22b}$$

4. The nonlinear Schrödinger breather and its stability in the presence of dissipative perturbations

The NS breather consists of two solitons (2.41) with zero velocities, i.e., $\xi_1=\xi_2=0$ (Zakharov and Shabat, 1971). We shall consider the most interesting case when the amplitudes η_1 and η_2 of the two solitons are close:

$$0<\zeta\equiv(\eta_1-\eta_2)/2\eta\ll 1, \quad \eta\equiv\tfrac{1}{2}(\eta_1+\eta_2) \, . \tag{4.23}$$

Under this condition, the breather takes the form (Malomed, 1985)

$$u(x,t)=-4\eta\zeta(\gamma_1^*+\gamma_2^*)[|\gamma_1+\gamma_2|^2+2\zeta^2\cosh(4\eta x)]^{-1} \times\{[1+\gamma_1^*\gamma_2^*(\gamma_1+\gamma_2)(\gamma_1^*+\gamma_2^*)^{-1}]\cosh(2\eta x)+[1-\gamma_1^*\gamma_2^*(\gamma_1+\gamma_2)(\gamma_1^*+\gamma_2^*)^{-1}]\sinh(2\eta x)\} \, , \tag{4.24}$$

where

$$\gamma_n(t)\equiv\gamma_n(0)\exp(-4i\eta_n^2 t) \, , \quad n=1,2, \ \gamma_2(0)=1/\gamma_1(0) \, , \tag{4.25}$$

$\gamma_1(0)$ being a real positive parameter.

The internal structure of the breather (4.24) and (4.25) can be described as follows: the maxima of $|u(x,t)|$ are located at the points

$$x=x_{1,2}=\pm\frac{1}{2\eta}\ln(|\gamma_1+\gamma_2|/\zeta) \, , \tag{4.26}$$

and the minimum of $|u(x,t)|$ is located at $x=0$. At the maximum points $|u(x_{1,2},t)|=2\eta$, while at the minimum point $|u(x=0,t)|\sim\zeta^2\eta$. As follows from Eq. (4.26), the size L of the breather is large,

$$L\equiv|x_1-x_2|=\frac{1}{\eta}\ln(|\gamma_1+\gamma_2|/\zeta)\gg\frac{1}{\eta} \, , \tag{4.27}$$

i.e., the NS breather consists of two remote slightly overlapping solitons. The period T of internal oscillations of the breather is

$$T=\pi/(8\eta^2\zeta) \, . \tag{4.28}$$

As can be seen from Eq. (4.27), the amplitude ΔL of the size oscillations is

$$\Delta L=\tfrac{1}{2}\eta^{-1}\ln[(|\gamma_1|+|\gamma_2|)^2/(|\gamma_1|-|\gamma_2|)^2] \, . \tag{4.29}$$

Thus the breather's structure is determined by the two dimensionless parameters ζ and $\gamma_1(0)$. In what follows we shall concentrate on the most interesting case, $|1-\gamma_1(0)|\lesssim\zeta$; otherwise, the two constituent solitons are weakly overlapped during the entire period (4.28) [see Eqs. (4.27) and (4.29)], and the breather's dynamics reduce to a trivial superposition of the evolution of two almost noninteracting solitons.

When

$$|\gamma_1(t)+\gamma_2(t)|\lesssim\zeta \tag{4.30}$$

[which, of course, is not possible unless $|1-\gamma_1(0)|\lesssim\zeta$], the breather's shape becomes more complicated than (4.24):

$$u(x,t)=-4\eta\exp(-2\eta x)\{\zeta^{-1}[(\gamma_1^*+\gamma_2^*)-(\gamma_1^*)^2(\gamma_1+\gamma_2)\exp(-4\eta x)]+2\gamma_1^*[(1-2\eta x)+|\gamma_1|^2(1+2\eta x)\exp(-4\eta x)]\} \times[1+\zeta^{-2}|\gamma_1+\gamma_2|^2\exp(-4\eta x)+2|\gamma_1|^2(1+8\eta^2x^2)\exp(-4\eta x)+|\gamma_1|^4\exp(-8\eta x)]^{-1} \, . \tag{4.31}$$

As can be seen from Eq. (4.31), the two constituent solitons strongly overlap as long as Eq. (4.30) holds, i.e. [see Eq. (4.25)], during the time $t\sim\eta^{-2}$ per one period (4.23).

In the unperturbed case ($\epsilon=0$) the NS breather, in contrast to the SG breather, is not a physically interesting object, since its binding energy, as follows from Eq. (2.45a), is zero. In the presence of dissipative perturbations, there is no conserved energy, and the breather's stability against decay into two solitons must be defined dynamically [the breather's stability problem plays a significant role, for instance, in the theory of optical soliton propagation in a one-mode nonlinear waveguide; see, for example, Golovchenko *et al.* (1985), Wai *et al.* (1986), Kodama and Nozaki (1987)].

Analogously to the case of the SG breather considered above, the perturbation-induced evolution of a NS breather is dominated by a strong overlapping stage (4.30). The evolution equation (2.48) takes the following form for the breather under consideration:

$$\frac{d\lambda_j}{dt}\equiv\frac{d}{dt}(\xi_j+i\eta_j) =2i\eta_j[\gamma_j(t)/\zeta]M_j, \quad j=1,2 \, , \tag{4.32}$$

where

$$M_j \equiv \int_{-\infty}^{+\infty} dx \{[\psi_j^{(1)}(x)]^2 \epsilon P[u(x)] - [\psi_j^{(2)}(x)]^2 \epsilon^* P^*[u(x)]\} , \quad (4.33)$$

and the Jost functions are

$$\psi_j^{(1)}(x) \equiv \psi^{(1)}(x, \lambda = \lambda_j) = \Delta^{-1} D_{1j} e^{-\eta x} , \quad (4.34a)$$

$$\psi_j^{(2)}(x) \equiv \psi^{(2)}(x, \lambda = \lambda_j) = i\Delta^{-1} D_{2j} e^{-\eta x} , \quad (4.34b)$$

where

$$D_{11} = D_{12} \equiv 1 - \zeta^{-1} e^{-4\eta x} \gamma_1 (\gamma_1^* + \gamma_2^*) + |\gamma_1|^2 (1 + 4\eta x) \exp(-4\eta x) , \quad (4.34c)$$

$$D_{21} = D_{22} \equiv \zeta^{-1} (\gamma_1^* + \gamma_2^*) \exp(-2\eta x) + \gamma_1^* (1 - 4\eta x) \exp(-2\eta x) + \gamma_1^* |\gamma_1|^2 \exp(-6\eta x) , \quad (4.34d)$$

$$\Delta \equiv 1 + \zeta^{-2} |\gamma_1 + \gamma_2|^2 \exp(-4\eta x) + 2|\gamma_1|^2 (1 + 8\eta^2 x^2) \exp(-4\eta x) + |\gamma_1|^4 \exp(-8\eta x) . \quad (4.34e)$$

Inserting Eqs. (4.33)–(4.35) and the perturbation (1.8) into Eq. (4.32), one obtains the following evolution equations for the parameters determining the breather's stability [cf. Eq. (4.12)]:

$$\frac{d}{dt}(\xi_1 - \xi_2) = 0 ,$$
$$\frac{d\zeta}{dt} = -\zeta^{-1} \left[\alpha_1 f_1 \left[\frac{|\gamma_1(t) + \gamma_2(t)|}{\zeta} \right] + \eta^2 \sum_{j=2}^{3} \alpha_j f_j \left[\frac{|\gamma_1(t) + \gamma_2(t)|}{\zeta} \right] \right] , \quad (4.35)$$

where f_j are some regular functions of a rather cumbersome form which take values ~ 1 when their argument is $\lesssim 1$, and vanish sufficiently rapidly when the argument becomes large. The change of the parameter ζ^2 during one overlapping can be obtained from Eqs. (4.35), just as we obtained Eq. (4.14) from (4.12):

$$\Delta(\zeta^2) = -\alpha_1 \int_{-\infty}^{+\infty} f_1(t) dt - \eta^2 \sum_{j=2}^{3} \alpha_j \int_{-\infty}^{+\infty} f_j(t) dt . \quad (4.36)$$

Numerical evaluation of the integrals $I_j \equiv \int_{-\infty}^{+\infty} f_j(t) dt$ demonstrates that these integrals are positive for $j=1$ and 3, and negative for $j=2$. Thus the perturbing terms $\sim\alpha_1$ and α_3 from Eq. (1.8) render the breather stable, while the term $\sim\alpha_2$ may break it into a pair of free solitons. Indeed, ζ^2 may be redefined in terms of the solitons' parameters $\lambda_n = \xi_n + i\eta_n$ as follows [see Eq. (4.23)]:

$$\zeta^2 = -(\lambda_1 - \lambda_2)^2 / 4\eta^2 . \quad (4.37)$$

Due to the negativity of I_2, the perturbation $\sim\alpha_2$, if it dominates, will render the quantity (4.37) negative, i.e., according to Eq. (4.37), the quantity $(\lambda_1 - \lambda_2)^2$ will become positive. This means that $(\eta_1 - \eta_2)$ vanishes and there appears $\xi_1 - \xi_2 \neq 0$. In other words, the breather will decay into two solitons with equal amplitudes η and with a relative velocity $\sim(\xi_1 - \xi_2)$. This explains the results of the numerical experiments performed by Pereira and Chu (1979), who observed the stability of a NS breather subject to the action of dissipative perturbation $\sim\alpha_1$, and the decay into a soliton pair under the action of a perturbation $\sim\alpha_2$. Numerical data obtained in those experiments clearly demonstrate that the perturbation-induced dynamics of the breather are dominated just by that stage of internal oscillations when the two constituent solitons are strongly overlapped.

5. Fusion of a soliton pair into a nonlinear Schrödinger breather

As we have seen above, the perturbing terms $\sim\alpha_1$ and $\sim\alpha_3$ from Eq. (1.8), in contrast to the one $\sim\alpha_2$, do not break the NS breather. Thus it is natural to assume, by analogy with the situation for the perturbed SG equation considered above in Sec. IV.A.2, that a collision of two NS solitons in the presence of perturbations which render the breather stable may result in binding the colliding solitons into a breather. In order to demonstrate this, we shall consider the collision of two solitons with equal amplitudes η and small velocities $\pm V$. A simple investigation based on Eqs. (4.33) and (A5) yields the following equation for the dissipative perturbing term $\epsilon P[u] = -i\alpha |u|^{2N} u$ [see Eq. (1.7b)]:

$$\frac{d(V^2)}{dt} = \alpha (4\eta)^{2N+1} \frac{N!(N+1)!}{(2N+2)!} \frac{\sin\Psi}{\cosh(4\eta z) + \cos\Psi} , \quad (4.38)$$

where $z(t) \equiv \int_0^t V(t') dt'$, and Ψ is a relative phase of the internal oscillations of the two solitons ($-\pi \leq \Psi \leq \pi$). It follows from Eq. (4.38) that, when the initial velocity V_0 is sufficiently large ($V_0^2 \gg \eta^{2N} \alpha$), the change of velocity due to the collision is (Malomed, 1985)

$$\Delta V = \frac{\alpha}{V} (4\eta)^{2N} \frac{N!(N+1)!}{(2N+2)!} \Psi . \quad (4.38')$$

This result is nontrivial in the sense that it is stipulated by the overlapping of the solitons; as is well known (Karpman and Maslov, 1977), the amplitude of a single NS soliton decreases under the action of a dissipative perturbation, but its velocity remains constant in the first approximation.

A threshold velocity for binding the colliding solitons into a breather is defined by the condition $V_{\text{thr}} = -\Delta V$. Equation (4.38) will not give an accurate calculation of V_{thr}, since it actually assumes $|\Delta V| \ll V$. However, one may use Eq. (4.38) to obtain the estimate (Malomed, 1985)

$$V_{\rm thr} \sim \eta^N \sqrt{\alpha} \tag{4.39}$$

[cf. Eq. (4.2)]. Recall that this estimate concerns the collision of solitons with equal amplitudes. When the amplitudes are dissimilar the threshold velocity is much smaller.

6. Breakup of a nonlinear Schrödinger breather due to collision with a fast soliton

Following the approach of Sec. IV.A.3, we now consider the possibility of a breakup of the NS breather (4.24)–(4.31) due to its collision with a fast NS soliton [see Eq. (2.41)] in the presence of the dissipation described by Eq. (1.8). We shall designate V and η_3 as the fast soliton's velocity and amplitude; it is assumed that V is a sufficiently large parameter (see below). On this assumption, one may expect the full wave form to split, in the lowest approximation, into breather and soliton wave forms [cf. Eq. (4.11)]:

$$u(x,t)=u_{\rm br}(x,t)+u_s(x,t) . \tag{4.40}$$

In the same approximation, the Jost functions for a breather and for a soliton may be taken in the "unperturbed" form [in contrast to the case of the SG equation (Sec. IV.A.3), now not only the combinations of Jost functions that enter the evolution equation (2.48), but also the Jost functions themselves, are split]. Corrections to Eq. (4.40) and to the "split" Jost functions are of order V^{-1}.

Our aim is to calculate the relative velocity ΔV of the solitons, which are the "splinters" of the breather broken up by collision with the fast soliton. This quantity, together with the difference $\Delta\eta$ of the amplitudes of the splinters, can be obtained from Eq. (4.32):

$$\begin{aligned}-\tfrac{1}{4}\Delta V+i\Delta\eta &= \int_{-\infty}^{+\infty} dt \left[\frac{d\lambda_1}{dt}-\frac{d\lambda_2}{dt}\right]\\ &= -2i\zeta^{-1}\int_{-\infty}^{+\infty} dt[\gamma_1(t)M_1(t)\\ &\quad -\gamma_2(t)M_2(t)] ,\end{aligned} \tag{4.41}$$

where M_1 and M_2 are determined by Eq. (4.33). Inserting Eq. (4.40) into the perturbing term $-i\alpha_1 u$ from Eq. (1.8a), and that term into Eq. (4.41), one obtains divergent integrals. In fact, to consider accurately the effect of the perturbing term, one needs to go beyond the approximation afforded by Eq. (4.40). Then the integrals become convergent, but the resultant expressions are very ponderous. We shall give only the estimates,

$$\Delta V, \Delta\eta \sim \alpha\eta_3 V^{-2}\zeta^{-3}\ln(V^2/\eta\eta_3) .$$

Approximation (4.40) is sufficient to analyze the effects produced by the perturbing term $-i\alpha_3|u|^2u$ from Eq. (1.8b). Details of the calculations are presented by Malomed (1985). For the case $|\gamma_1(t)+\gamma_2(t)| \gg \zeta$ [see Eq. (4.24)], when the two constituent solitons inside the breather are slightly overlapped at the moment of collision, the result is almost trivial, since ΔV and $\Delta\eta$ are, as a matter of fact, mere differences in the velocity and amplitude of each constituent soliton produced by their collision with the fast soliton.

The strong overlapping case $|\gamma_1(t)+\gamma_2(t)| \lesssim \zeta$ is, however, nontrivial [see Eq. (4.31)]. In this case the result is (Malomed, 1985)

$$\begin{aligned}\Delta\eta &= 64\eta_3\eta V^{-1}\zeta^{-1}\alpha_3 f(|\gamma_1+\gamma_2|/\zeta) ,\\ \Delta\xi &= 64\eta_3\eta V^{-1}\zeta^{-1}\alpha_3 g(|\gamma_1+\gamma_2|/\zeta) ,\end{aligned} \tag{4.42}$$

where the functions f and g, determined as certain integrals, take values of order one. In particular,

$$f(0)\approx 0.428, \quad f'(0)=0, \quad g(0)=0, \quad g'(0)\approx 0.016 .$$

Recoil effects, i.e., changes in the velocity and amplitude of the fast soliton, have also been investigated by Malomed (1985).

B. Many-soliton interactions in the presence of conservative perturbations

As is well known (Zakharov *et al.*, 1980), in the absence of perturbations the interaction of solitons results only in their phase shifts, with the shift due to collision with a pair of solitons being equal to the sum of partial shifts resulting from separate collisions with each soliton of the pair; this situation is commonly referred to as the absence of many- (three-) particle effects. It is, moreover, easy to verify that the collision of two solitons in the presence of a conservative perturbation does not cause energy and momentum exchange between the solitons in first-order perturbation theory. In the first order, exchange *is* possible in the three-particle situation: dealing with the collision of a fast soliton with two slower ones, we can calculate the changes in velocity of the three "particles," i.e., the changes of their energies and momenta, using the energy and momentum conservation laws (Kivshar and Malomed, 1986a, 1986b, 1987a). The three-particle nature of this effect manifests itself in the dependence of the exchanged energy and momentum on the distance between the slow solitons at the moment of their interaction with the fast one. The effect vanishes when the distance tends to infinity. Three-particle effects are of principal interest for field theory problems, as well as for elucidating the very notion of a nearly integrable system.

To analyze the nontrivial two- and three-particle effects, it is necessary to employ as a zeroth approximation an exact form of a two- or three-soliton solution, since it will have a crucial effect during the period of strong overlap between the colliding solitons. Interactions between remote (weakly overlapped) kinks can be considered more simply as an interaction between classical particles. In this connection, it is pertinent to mention the early paper by Rubinstein (1970), who demonstrated (prior to the discovery of the exact integrability of the SG equation) that for two weakly overlapped kinks an effective interaction potential is $U(r)=32e^{-r}$, where r ($r \gg 1$) is the distance between the kinks. A general quasiparticle description of systems of weakly overlapped solitons has been elaborated by Gorshkov *et al.* (1974) and by Gorshkov and Ostrovskii (1981). In that case, proximity to exact integrability is not needed.

The results to be set forth in this subsection were obtained by Kivshar and Malomed (1986a, 1986b, 1987a).

1. Two- and three-soliton collisions in the sine-Gordon equation

a. Three-kink collision

In this subsection we shall deal with the perturbation (1.25). We shall consider the interaction of the fast kink (2.61) with a pair of slower ones described by the unperturbed solution [cf. Eqs. (4.4) and (4.22a)]

$$u_{sl}(x,t)=4\tan^{-1}\left[\frac{\sigma_1\exp(-z_1+D/2\nu_1)+\sigma_2\exp(-z_2-D/2\nu_2)}{1-\sigma_1\sigma_2 b_{12}\exp(-z_1-z_2+D/2\nu_1-D/2\nu_2)}\right], \tag{4.43a}$$

$$z_n \equiv \nu_n(x-V_n t),\quad \nu_n\equiv(1-V_n^2)^{-1/2},\quad n=1,2,$$
$$b_{12}\equiv[\nu_1(1+V_1)-\nu_2(1+V_2)]^2[\nu_1(1+V_1)+\nu_2(1+V_2)]^{-2}, \tag{4.43b}$$

where $V_1=W$, $V_2=-W$ ($W>0$) are the velocities of the two kinks, σ_1 and σ_2 are their polarities, and D (which may be sign-changing) is the distance between them determined as the difference between coordinates of the first and second kinks at the moment $t=0$.

The evolution of a kink's velocity under the action of perturbation (1.25) in the SG equation is determined by Eq. (2.73). Explicit results can be obtained in the case

$$1-V^2 \ll 1-W^2 . \tag{4.44}$$

We assume that the slower kinks (4.43) are also relativistic, i.e.,

$$1-W^2 \ll 1 . \tag{4.45}$$

As is shown by Malomed (1985; see also Sec. IV.A.3), condition (4.44) provides for the "splitting" of the full wave field [cf. Eq. (4.11)],

$$u=u_f+u_{sl}+O(\nu^{-1}), \tag{4.46a}$$

where ν is determined in Eq. (4.43b). Analogously, condition (4.45) provides for the "secondary" splitting of the wave field (4.43a) into two kink wave forms:

$$u_{sl}=(u_{sl})_1+(u_{sl})_2 . \tag{4.46b}$$

As for the Jost functions, it was explained in the preceding section that one may insert them into the adiabatic equations (2.73) in the "one-particle" approximation for each $n=1,2,3$.

Using these simplifications, we obtain from Eq. (2.73) an evolution equation for the fast kink,

$$\frac{d\nu}{dt}=\frac{\epsilon}{4}\nu\sigma\int_{-\infty}^{+\infty}dx\frac{\sin[2u(x,t)]}{\cosh z}, \tag{4.47}$$

and for the pair of slower ones,

$$\frac{d\lambda_n}{dt}=\frac{\epsilon}{4}\sigma_n\lambda_n\int_{-\infty}^{+\infty}dx\frac{\sin[2u(x,t)]}{\cosh[z_n+(-1)^n D/2\nu_n]}, \tag{4.48}$$

where $\lambda_n(t)$ are expressed in terms of the velocities $V_1(t)$ and $V_2(t)$ of the two kinks according to Eq. (2.60). The changes of these parameters due to the collision can be found by direct integration of Eq. (4.48):

$$\Delta\lambda_n=\int_{-\infty}^{+\infty}\frac{d\lambda_n}{dt}dt . \tag{4.49}$$

Equations (4.48) and (4.49) contain spatial and time integrations. To perform them explicitly, one should use Eq. (4.46a) and, after simple transformations, insert the perturbation $\sin(2u)$ into Eq. (4.48) in the form

$$\sin(2u)=\sin(2u_f)\cos(2u_{sl})+\sin(2u_{sl})-2\sin^2 u_f\sin(2u_{sl}) . \tag{4.50}$$

Using Eq. (4.46b), one can analogously simplify the second term from the right-hand side of Eq. (4.50):

$$\sin(2u_{sl})=\sin(2u_{sl})_2\cos(2u_{sl})_1+\sin(2u_{sl})_1-2\sin^2(u_{sl})_2\sin(2u_{sl})_1 . \tag{4.50'}$$

Substituting Eqs. (4.50) and (4.50') into Eqs. (4.48) and (4.49), we arrive at the final expression,

$$\Delta\lambda_1=\frac{128}{9\nu}\epsilon f(\delta), \tag{4.51}$$

where the odd function

$$f(\delta)=\tanh\delta\,\mathrm{sech}^2\delta(1-2\,\mathrm{sech}^2\delta), \tag{4.52}$$

and the parameter

$$\delta\equiv D/2\lambda_1 \tag{4.53}$$

characterize the degree of overlap between the slower kinks at the moment of their collision with the fast one. For the second kink, analogous consideration results in

$$\Delta\lambda_2=-\frac{8}{9\nu}\epsilon\lambda_1^{-4}f(\delta) . \tag{4.54}$$

Since, according to Eq. (4.45), $\lambda_1 \gg 1$, we see from comparing Eqs. (4.54) and (4.51) that

$$|\Delta\lambda_2| \ll |\Delta\lambda_1| \; . \tag{4.55}$$

This distinction is connected with the fact that the velocity of the first kink is in the same direction as that of the fast kink, while the velocity of the second kink is opposite to that of the fast kink; i.e., with regard to Eq. (4.45), the effective time of interaction with the fast kink is much larger for the first slow kink than for the second one.

The change in the fast kink's velocity can easily be found with the aid of energy and momentum conservation. Indeed, the energy E and momentum P transferred by the fast kink to the slower ones are

$$\Delta E = 8 \sum_{n=1}^{2} [1 - 1/(4\lambda_n^2)]\Delta\lambda_n \approx 8\Delta\lambda_1 \; , \tag{4.56}$$

$$P = 8 \sum_{n=1}^{2} [1 + 1/(4\lambda_n^2)]\Delta\lambda_n \approx 8\Delta\lambda_1 \approx \Delta E \tag{4.57}$$

[we have used Eqs. (4.51), (4.54), and (4.55) to neglect the second terms in Eqs. (4.56) and (4.57)]. The change $\Delta\nu$ of the "relativistic parameter" of the fast kink can be obtained from Eqs. (4.56) and (4.51): $\Delta\nu \approx -\Delta E/8$, or

$$\Delta\nu = -\frac{128}{9\nu}\epsilon f(\delta) \; . \tag{4.58}$$

As to the change in the velocity V itself, it can be easily expressed in terms of $\Delta\nu$ [recall $\nu \equiv (1-V^2)^{-1/2}$]: $\Delta V = \Delta\nu/\nu^3$. Analogously, the changes $\Delta V_{1,2}$ in the slow kinks' velocities can be expressed in terms of $\Delta\lambda_{1,2}$.

It is important that the function $f(\delta)$ defined by Eq. (4.52) vanishes when $\delta \to \infty$. This is a direct manifestation of the three-particle nature of the effect considered. It is also interesting to note that the function $f(\delta)$ is odd, i.e., depending on the sign of δ, the fast kink may either lose or acquire energy due to the collision with the pair of overlapped slow kinks. Finally, it is worth noting that, as can be seen from Eqs. (4.51)–(4.54) and (4.58), the effect does not depend on the polarities of the kinks involved.

b. Collision of a kink with a small-amplitude breather

Now let us consider inelastic adiabatic effects accompanying the collision of a kink with a quiescent small-amplitude breather (2.65) in the presence of the same conservative perturbation (1.25). We shall assume the kink to be fast, i.e., $\nu \gg 1$, which provides, as above, the splitting of the wave field:

$$u(x,t) = u_k + u_{\rm br} + O(\nu^{-1}) \; , \tag{4.59}$$

where u_k and $u_{\rm br}$ are the wave forms of the kink and the small-amplitude breather, respectively.

In terms of the inverse scattering transform, a small-amplitude breather corresponds to a pair of symmetric points $\lambda_{1,2} = \pm\lambda' + i\lambda''$ on the complex plane of the spectral parameter λ; see Eq. (2.62). The perturbation-induced evolution equations for the real parameters λ' and λ'' take the following form:

$$\frac{d\lambda'}{dt} = -(\epsilon\mu/8)\int_{-\infty}^{+\infty} dx \, \sin[2u(x,t)][\mu \, \mathrm{sech}(\mu x)\cos(t+\Psi_0) - \mathrm{tahn}(\mu x)\mathrm{sech}(\mu x)\sin(t+\Psi_0)] \; . \tag{4.60}$$

$$\frac{d\lambda''}{dt} = (\epsilon\mu/8)\int_{-\infty}^{+\infty} dx \, \sin[2u(x,t)][\mathrm{sech}(\mu x)\cos(t+\Psi_0) + \tanh(\mu x)\mathrm{sech}(\mu x)\sin(t+\Psi_0)] \tag{4.61}$$

[recall that Ψ_0 is the breather's internal phase at the moment of collision $t=0$; see Eq. (2.64b)]. Inserting Eq. (4.59) into Eqs. (4.60) and (4.61) yields

$$\Delta\nu = -\tfrac{64}{3}\pi\epsilon\nu^{-1}\sin(2\Psi_0)\mathrm{csch}(\pi/\mu) \; , \tag{4.62}$$

$$\Delta\lambda' = -\Delta\nu/4\mu, \quad \Delta\lambda'' = -\Delta\nu/2 \; . \tag{4.63}$$

The velocity W acquired by the breather due to the collision, and the change $\Delta\mu$ in the breather's amplitude are (Kivshar and Malomed, 1987a)

$$W = 2\Delta\lambda', \quad \Delta\mu = 2\Delta\lambda'' \; . \tag{4.64}$$

It is easy to verify that these results [Eqs. (4.62)–(4.64)] satisfy energy and momentum conservation. The energy ΔE and momentum ΔP transferred by the fast kink to the breather are

$$\Delta E = 32(\Delta\lambda'' - \mu\Delta\lambda') \approx 32\mu\Delta\lambda' \; ,$$

$$\Delta P = 32\mu(\Delta\lambda' + \mu\Delta\lambda'') \approx 32\mu\Delta\lambda' = \Delta E$$

(ΔE and ΔP must coincide approximately because the fast kink is relativistic).

Clearly, the results given by Eqs. (4.62)–(4.64) depend critically on the value of the breather's phase Ψ_0 at the moment of collision (we saw an analogous dependence in Sec. IV.A). A simple analysis of Eqs. (2.65) and (4.62) reveals that the fast soliton loses energy provided $\sin(2\Psi_0) > 0$, i.e., if at the moment of collision the kink and antikink inside the breather are moving to meet each other; the fast soliton gains energy in the opposite case. It is interesting to note that, as can be seen from Eq. (4.58), the same is true for the collision of a fast kink with a pair of slower kinks if the relative distance between them $|\delta|$ is less than $\ln(1+\sqrt{2})$.

Another noteworthy fact is the exponential smallness of the results (4.62)–(4.64) in the parameter μ^{-1}. As can be seen from Eqs. (4.60) and (4.61), the reason for this smallness is that the breather's size l is large: $l \sim \mu^{-1}$ [see Eq. (2.65)].

c. Collision of a kink with a low-frequency breather

We now proceed to the opposite case from that treated in the preceding section, i.e., collision of a kink with a low-frequency breather [Eq. (2.66)]. As was shown in Sec. IV.A, collision of a low-frequency breather with a fast kink in the presence of a dissipative perturbation may result (in first-order perturbation theory) in the breakup of the breather into a kink-antikink pair. With the same accuracy, this enelastic process cannot be generated by a conservative perturbation. Here we shall consider the same process for the perturbed sine-Gordon equation with the perturbation (1.25). We shall confine ourselves to first order, but we shall take into account the first order with respect to ν^{-1}, whereas Sec. IV.A.3 implied the zeroth order in this small parameter.

The evolution equations for the parameters of the low-frequency breather, $\lambda_{1,2}=\pm\lambda'+i\lambda''$ [see Eq. (2.62)], ensue from Eq. (4.9). When we make the familiar assumption $1-V^2 \ll 1$, the combined Jost functions split, as above, into those corresponding to the fast kink and to the breather, written in Eqs. (B3) and (B4). Inserting these "one-particle" Jost functions into Eq. (4.9), one can use Eqs. (4.50) and (4.50′) to cast the evolution equations into the form [cf. Eq. (4.12)]

$$\frac{d(\lambda')^2}{dt}=(16\epsilon/3\nu)T\cosh^2 t[\cosh^2 t-T^2(t)] \times[\cosh^4 t+T^4(t)-6T^2(t)\cosh^2 t] \times[\cosh^2 t+T^2(t)]^{-5}\ , \tag{4.65}$$

$$\frac{d\lambda''}{dt}=-2\lambda'[1+T(t)]\frac{d\lambda'}{dt}\ . \tag{4.66}$$

In this problem the most interesting three-particle effect is not the exchange of energy and momentum as in earlier problems, but the possibility of a breakup of the low-frequency breather (2.66) into a kink-antikink pair. However, one should keep in mind that the escaping kink and antikink emit radiation due to their overlapping (see Sec. VII). The total emitted energy is $E_{\rm rad}\sim\epsilon^2$ (Malomed, 1985), and if it is larger than the total kinetic energy $E_{\rm kin}$ of the escaping kink and antikink, they will again merge into a breather, i.e., the breakup will be unobservable. If the kink and antikink inside the breather overlap weakly at the moment of collision $t=0$, i.e., if $|\cos\Psi|\sim 1$, or, in terms of $T(t)$, $T_0^2\equiv T^2(0)\sim\zeta^{-2}$, the estimate for $E_{\rm kin}$, following from Eq. (4.65), is

$$E_{\rm kin}\sim\frac{\epsilon}{\nu}|T_0|^{-1}\sim\frac{\epsilon\zeta}{\nu}\ . \tag{4.67}$$

At the same time, for a "typical" initial condition that allows the breakup, $E_{\rm kin}\sim\zeta^2$, so that we obtain from Eq. (4.67) $\zeta\sim\epsilon/\nu$, or $E_{\rm kin}\sim\epsilon^2/\nu^2 \ll E_{\rm rad}\sim\epsilon^2$. Thus we infer that in the case $|T_0|\sim\zeta^{-1}$ the breakup is unobservable. If the overlap inside the breather is moderate at the moment of collision, i.e.,

$$1\lesssim T_0^2 \ll \zeta^{-2}\ , \tag{4.68}$$

a similar consideration enables us to rewrite the condition $E_{\rm em}\lesssim E_{\rm kin}$ in the form

$$\nu|T_0|\lesssim\epsilon^{-1}\ . \tag{4.69}$$

At last, if the overlap at the moment of collision is moderate, i.e.,

$$|T_0| \ll 1\ , \tag{4.70}$$

it follows from Eq. (4.65) that a condition analogous to (4.69) is

$$\nu\lesssim|T_0|\epsilon^{-1}\ . \tag{4.71}$$

Therefore breakup may take place under condition (4.68) or (4.70) supplemented by, respectively, Eq. (4.69) or (4.71).

Let us continue the analysis, assuming that the breakup is observable. As follows from the inequalities (4.68) or (4.70), during the collision Eq. (4.6′) reduces to Eq. (4.16). Substituting Eq. (4.16) into Eq. (4.65) yields the final result

$$\Delta(\lambda'^2)=(16\epsilon/3\nu)T_0\int_{-\infty}^{+\infty}dt\,\cosh^2 t[\cosh^2 t+(T_0+t)^2][\cosh^4 t+(T_0+t)^4-6(T_0+t)^2\cosh^2 t][\cosh^2 t+(T_0+t)^2]^{-5} \tag{4.72}$$

[cf. Eq. (4.17)]. Note that, according to Eq. (4.72), $\Delta(\lambda'^2)$ is an odd function of T_0.

The breakup takes place if $\Delta(\lambda'^2)\le-\zeta^2/4$ (see Sec. IV.A.3). The threshold value of the binding energy, i.e., the maximum value at which breakup is possible, is $E_{\rm thr}=32|\Delta(\lambda'^2)|$. If $|E_b|<|E_{\rm thr}|$, the collision breaks the breather into a kink-antikink pair described by the unperturbed solution (4.22a). The small relative velocity W of the kink and antikink, together with the constant δ, are determined by Eqs. (4.21) and (4.22b).

The sign of $\sin\Psi$ plays an important role in the above consideration. A qualitative picture of an inelastic kink-breather collision for the two opposite signs of $\sin\Psi$ is given in Figs. 17(a) and 17(b) (cf. Fig. 16).

If $\Delta(\lambda'^2)>0$, the above consideration may describe the inverse process of fusion of a kink-antikink pair (4.22a) into a breather [Eq. (2.66)] due to the collision with a fast kink. In the absence of a third kink, the conservative

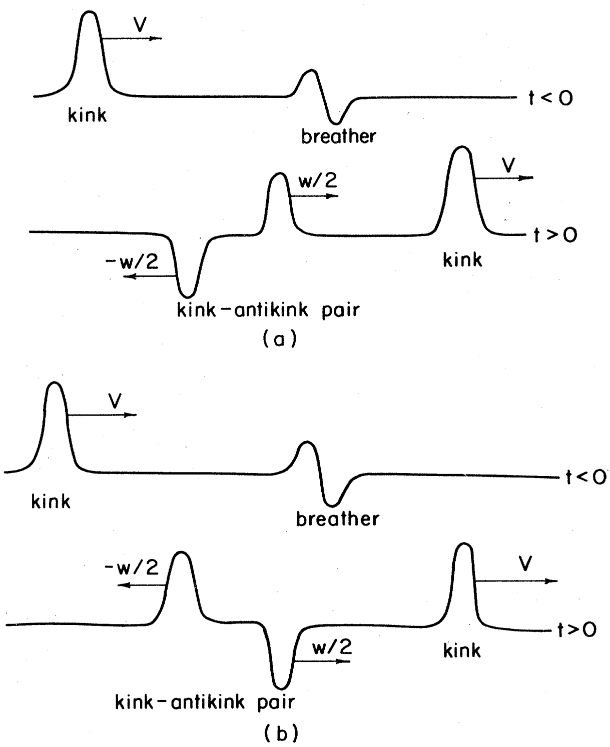

FIG. 17. The breakup of a low-frequency breather due to its collision with a fast kink in the presence of the conservative perturbation $P=\sin(2u)$: (a) $\mathrm{sgn}(\sin\Psi)=+1$; (b) $\mathrm{sgn}(\sin\Psi)=-1$. The figure represents u_x vs x.

perturbation considered may cause fusion of the pair into a breather on account of energy losses; see Sec. VII. In contrast to radiative fusion, the mechanism considered here is nonradiative: the surplus energy (which must be shed in order for the pair to merge into a breather) is accepted by the fast kink.

The result (4.72) essentially depends on T_0, i.e., on the value of Ψ. This crucial dependence on Ψ is also a feature of the results set forth for a dissipative perturbation in Sec. IV.A.3. However, the difference is that the conservative perturbation results are not sensitive to the polarity of the fast kink, while in the case of a dissipative perturbation it was crucial.

d. Two-breather collision

Now let us consider a collision between two fast SG breathers. Since each breather is a "two-particle" bound state, this collision may be interpreted as a four-particle interaction. Here we shall concentrate on the collision of two low-frequency breathers [Eq. (2.66)].

In the rest reference frame of one breather, the second breather has the form [cf. Eqs. (2.63) and (2.64)]

$$u_f^{(\mathrm{br})}(z)\approx 4\tan^{-1}[\tan\mu_2\sin(\cos\mu_2 z+\Psi_{02})\times\mathrm{sech}(\sin\mu_2 z)]\ , \tag{4.73}$$

where $z\equiv(x-t)/\sqrt{1-V^2}$, and V is the relative velocity of the breathers (we set $V=1$ in all expressions except for $1-V^2$). The parameters μ_2 and Ψ_{02} in Eq. (4.73) are the amplitude and initial phase of the second breather in its rest reference frame.

The most interesting case is that when the internal kinks are essentially overlapped inside both breathers at the moment of collision; then the interaction is four-particle indeed. In this case the change of parameter λ_1' of the first (quiescent) breather [cf. Eq. (4.72)] is

$$\Delta(\lambda_1'^2)=(\epsilon/\nu)[F_1(T_{02})G_1(T_{01})-\sigma F_2(T_{02})G_2(T_{01})]\ , \tag{4.74}$$

where the functions $F_{1,2}$ and $G_{1,2}$ are defined in Appendix C, and

$$\sigma\equiv\mathrm{sgn}(\sin\Psi_1)\mathrm{sgn}(\sin\Psi_2)\ . \tag{4.75}$$

The parameters T_{01} and T_{02} in Eq. (4.74) and Ψ_1 and Ψ_2 in Eq. (4.75) are the standard parameters defined for each breather in its rest reference frame. Note that, according to Eqs. (C1) and (C2), $F_2/F_1\to 0$ when $|T_{02}|\to\infty$, i.e., only the first term from the right-hand side of Eq. (4.74) survives in this limit to recover Eq. (4.72) with the doubled right-hand side. This is because a fast breather with small overlap of the internal kinks is equivalent to two fast independent kinks. If T_{02} is not large, the marked difference between Eqs. (4.74) and (4.72) is that F_2 is an even function of T_{01} in contrast with the odd function F_1.

The quantity $\Delta(\lambda_2'^2)$ for the second breather can be obtained from Eq. (4.74) by the obvious transposition $T_{01}\rightleftarrows T_{02}$. If

$$4\Delta(\lambda_j'^2)<-\zeta_j^2\quad(j=1,2)\ ,$$

the jth breather decays into a kink-antikink pair. Thus we see that, depending on the values of the breathers' parameters T_{01}, T_{02}, ζ_1, ζ_2, σ, three different scenarios are possible: both breathers survive; one survives and one decays; or both decay. The result of the inelastic collision depends essentially on $\mathrm{sgn}(\sin\Psi_n)$.

2. Three-soliton collisions in the nonlinear Schrödinger equation

Let us consider the NS equation perturbed by the conservative term (1.7a). Note that if we derive the perturbed NS equation from the SG equation with the perturbation (1.25) as an equation for small-amplitude breathers, we should substitute into Eqs. (1.15) and (1.25)

$$u(x,t)=\exp(-it)U(x,t)+\exp(it)U^*(x,t)$$

(Kaup and Newell, 1978b; Newell, 1978a, 1978b). The perturbing term (1.7a) arises as the third nonvanishing term of the expansion of $\sin u$ and $\sin(2u)$ in powers of U. At the same time, the expansions of the basic term $\sin u$ and that of the perturbing term $\sin(2n)$ differ irremovably beginning just from the coefficient in front of the third term, i.e., the NS perturbation (1.7) adequately models the SG perturbation (1.25). We shall consider a collision between a fast soliton (2.41) with velocity -4ξ and amplitude η and two overlapped slower solitons with velocities $-4\xi_{1,2}=\mp 4\kappa$ and amplitudes η_1 and η_2. We assume [cf. Eq. (4.44)]

$$\xi \gg \kappa, \quad \xi\eta \gg \kappa\eta_1, \kappa\eta_2 \tag{4.76}$$

and [cf. Eq. (4.45)]

$$\kappa \gg \eta_1, \eta_2 \ . \tag{4.77}$$

The conditions listed in Eq. (4.76) provide for splitting of the wave field analogous to Eq. (4.46a). Condition (4.77) provides for "secondary" splitting [analogous to Eq. (4.46b)] of $u_{\rm sl}(x,t)$ into two solitonic wave forms (2.41) with velocities $-4\xi_{1,2}=\mp 4\kappa$, amplitudes η_1, η_2, and phase constants $\phi_{1,2}, x_{1,2}^{(0)}$. As for the Jost functions, under conditions (4.76) and (4.77) they split directly into "one-particle" expressions (A4), in contrast with the case of the SG equation, where only the combinations $W(\psi_n, \sigma_1\phi_n)$ were split [see Eq. (2.73)].

Substituting Eqs. (A4) into Eqs. (2.48) and (2.49) yields the evolution equations for the parameters of the slow solitons ($j=1,2$) [cf. Eq. (4.48)]

$$\frac{d\xi_j}{dt}=\epsilon\eta_j \mathrm{Im}\left[\int_{-\infty}^{+\infty} dx\,|u(x,t)|^4 u(x,t) \times \frac{\sinh[z_j(x,t)]e^{i\zeta_j(x,t)}}{\cosh^2[z_j(x,t)]}\right], \tag{4.78}$$

$$\frac{d\eta_j}{dt}=-\epsilon\eta_j \mathrm{Re}\left[\int_{-\infty}^{+\infty} dx\,|u(x,t)|^4 u(x,t) \times \frac{e^{i\zeta_j(x,t)}}{\cosh[z_j(x,t)]}\right], \tag{4.79}$$

$$\frac{dx_j^{(0)}}{dt}=-\epsilon\, \mathrm{Re}\left[\int_{-\infty}^{+\infty} dx\,|u(x,t)|^4 u(x,t) \times \frac{x e^{i\zeta_j(x,t)}}{\cosh[z_j(x,t)]}\right], \tag{4.80}$$

$$\frac{d\phi_j}{dt}=\epsilon\, \mathrm{Im}\left[\int_{-\infty}^{+\infty} dx\,|u(x,t)|^4 u(x,t) \times \frac{1-2\eta_j x\tanh[z_j(x,t)]}{\cosh[z_j(x,t)]} e^{i\zeta_j(x,t)}\right], \tag{4.81}$$

and the same equations for parameters of the fast soliton ($j=3$), where

$$\xi_3\equiv\xi, \quad \eta_3\equiv\eta, \quad z_3(x,t)\equiv 2\eta(x+4\xi t-x^{(0)}) \ ,$$

$$\zeta_j(x,t)=2\xi_j x+4(\xi_j^2-\eta_j^2)t+\phi_j \ .$$

The full collision-induced changes in the solitons' parameters are $\Delta\xi_{1,2}\equiv\int_{-\infty}^{+\infty}dt(d\xi_{1,2}/dt)$, etc. [see Eq. (4.49)].

Inserting the split wave forms (4.46) into Eqs. (4.78) and (4.79), one can cast the final expressions into a simple form in two particular cases: if

$$\eta_2 \gg \eta_1 \ , \tag{4.82}$$

and if

$$\eta_2=\eta_1\equiv\theta \ . \tag{4.83}$$

In case (4.82) the results are

$$\Delta\xi_1=96\epsilon\eta_1^2\eta_2\eta\xi^{-1}g_1(\delta) \ , \tag{4.84a}$$

$$\Delta\xi_2=-96\epsilon\eta_1^3\eta\xi^{-1}g_1(\delta) \ , \tag{4.84b}$$

$$\Delta\xi=-192\epsilon\eta_1^3\eta_2\kappa\xi^{-2}g_1(\delta) \ , \tag{4.84c}$$

$$\Delta\eta_1=\Delta\eta_2=\Delta\eta=0 \ , \tag{4.85}$$

where the odd function is

$$g_1(\delta)=\tanh\delta\,\mathrm{sech}^2\delta \tag{4.86}$$

[cf. Eq. (4.52)]. As above [see Eq. (4.53)], the parameter δ characterizes the overlap between the two slow solitons at the moment of collision ($t=0$), with their centers located at the points $x_1^{(0)}$ and $x_2^{(0)}$:

$$\delta\equiv 2\eta_1(x_2^{(0)}-x_1^{(0)}) \ . \tag{4.87}$$

It can be readily verified that Eqs. (4.84) and (4.85) satisfy all the elementary conservation laws, the energy $\Delta E_{1,2}$ and momentum $\Delta P_{1,2}$ acquired by each of the slow solitons being

$$\Delta E_1=32\eta_1\kappa\Delta\xi_1=3.2^{10}\eta_1^3\eta_2\eta\kappa\epsilon\xi^{-1}g_1(\delta) \ , \tag{4.88a}$$

$$\Delta E_2=-32\eta_2\kappa\Delta\xi_2=\Delta E_1 \ , \tag{4.88b}$$

$$\Delta P_1=-8\eta_1\Delta\xi_1=-3\cdot 2^8\eta_1^3\eta_2\eta\epsilon\xi^{-1}g_1(\delta) \ , \tag{4.89a}$$

$$\Delta P_2=-8\eta_2\Delta\xi_2=-\Delta P_1 \ . \tag{4.89b}$$

The third elementary integral of motion of the unperturbed NS equation is the "number of plasmons" (Zakharov *et al.*, 1980). The number of plasmons bound inside a soliton is $N_j=4\eta_j$. According to Eq. (4.85), the quantities N_j do not change in the approximation considered. Moreover, as can be seen from Eq. (4.89b), the fast soliton does not exchange momentum with the slow ones.

Comparing Eq. (4.84a) to Eq. (4.84b), we see that the change of velocity is much larger for the first soliton, i.e., the one with the smaller amplitude [see Eq. (4.82)], than for the second one [cf. Eq. (4.55)]. This is quite natural because the soliton's width is $\sim\eta^{-1}$, hence the first soliton is much broader, and its interaction time with the fast soliton is much larger too. Nonetheless, the changes in the energies of both slow solitons are equal, as can be seen from Eq. (4.88b).

Another feature we should note is that, pursuant to

Eq. (4.84c), the change in the fast soliton's velocity is of second order in ξ^{-1}. Since this quantity is small, it seems reasonable to evaluate the perturbation-induced phase shifts of the fast soliton, using Eqs. (4.80) and (4.81):

$$\Delta x^{(0)}=0 \, ,$$
$$\Delta\phi=32\epsilon\xi^{-1}(\eta_2\eta^2+\eta_2^3/2+3\eta_1^2\eta_2\mathrm{sech}^2\delta) \, . \tag{4.90}$$

Equations (4.84)–(4.89), as well as Eqs. (4.51)–(4.58), are obviously three-particle results. At the same time, the phase shift (4.90) contains, on a level with the three-particle contribution $\Delta\phi_{123}$ vanishing at $\delta\to\infty$, a perturbation-induced two-particle contribution from the interaction of the fast soliton with the second slow soliton [it dominates over a contribution from the interaction with the first slow soliton due to Eq. (4.82)].

We now proceed to the case described by Eq. (4.83). As above, in this case the amplitudes do not change, while the velocities change as follows:

$$\Delta\xi=-192\epsilon\theta^4\kappa\xi^{-2}g_2(\delta) \, , \tag{4.91a}$$

$$\Delta\xi_1=-\Delta\xi_2=96\epsilon\theta^3\eta\xi^{-1}g_2(\delta) \, , \tag{4.91b}$$

where this time the parameter characterizing the overlap of the slow solitons at the moment of collision is $\delta\equiv 2\theta(x_2^{(0)}-x_1^{(0)})$ [cf. Eq. (4.87)], and the odd function

$$g_2(\delta)=[3(\delta-\tanh\delta)\coth^2\delta-\delta]\mathrm{csch}^2\delta \tag{4.92}$$

[cf. Eqs. (4.52) and (4.86).].

The energy exchanged between the fast soliton and the pair of slow ones can be easily calculated with the help of Eqs. (4.91) and (4.92) similarly to Eq. (4.88), while there is no momentum exchange, as above. There are likewise no changes in the number of plasmons.

Finally, the phase shifts for the fast soliton can be calculated in the present case by analogy with Eq. (4.90):

$$\Delta x^{(0)}=0 \, ,$$
$$\Delta\phi=64\epsilon\theta\xi^{-1}[\eta^2+\theta^2/2+3\theta^2 g_3(\delta)] \, , \tag{4.93}$$

where

$$g_3(\delta)=\cosh\delta\,\mathrm{csch}^3\delta(\delta-\tanh\delta) \, .$$

It seems natural to compare the results of Eqs. (4.91)–(4.93), obtained with condition (4.83), with those from Sec. IV.A.6. Indeed, in that section we actually considered three-particle effects in the collision of a fast soliton with a NS breather that might be regarded as a system of two quiescent solitons (i.e., $\kappa=0$ in present terms), with the close amplitudes

$$(\eta_1-\eta_2)^2\ll(\eta_1+\eta_2)^2 \, . \tag{4.94}$$

Condition (4.94) plainly is similar to Eq. (4.83), while the condition $\kappa=0$, in contrast to Eq. (4.77), points up the difference between the two problems (in particular, the breather configuration substantially differs from a superposition of two slow solitons, and the breather's Jost functions do not split into one-soliton expressions). According to Sec. IV.A.6, three-particle effects, except for the phase shift (4.93), are absent in the first approximation for a perturbation from Eq. (1.8b). This drastic difference between the two problems may be interpreted as a manifestation of the "coherent" nature of the NS breather.

Nontrivial results were described in Sec. IV.A for the dissipative perturbation corresponding to imaginary ϵ in Eq. (1.7). Comparing these results with Eqs. (4.91) and (4.92), we notice a substantial difference between three-particle effects caused by dissipative and conservative perturbations: the latter result in changes of velocities, while the former affect amplitudes.

Thus there is a strong difference between three-particle effects for the relativistic SG kinks, treated in the preceding section, and those for nonrelativistic small-amplitude breathers equivalent to NS solitons.

3. Two-soliton collisions in the sine-Gordon equation in the presence of an inhomogeneity

a. Collision of free kinks

Following Kivshar and Malomed (1986b), we shall consider the collision of two kinks described by the SG equation with the perturbation (1.19). The collision can generate three-particle effects, since the inhomogeneity plays the role of the third "particle." The results take a simple form in two particular cases. First let us consider the collision of a fast kink with a slower one. Following the approach set forth above, we obtain the collision-induced changes $\Delta\lambda_3,\Delta\lambda_1$ of the parameters (4.49) for the fast and slow kinks:

$$\Delta\lambda_3=-2\epsilon V_1\nu_1\nu_3^{-1}g_1(\tilde\delta) \, , \tag{4.95}$$

$$\Delta\lambda_1=2\epsilon\lambda_1\nu_3^{-1}g_1(\tilde\delta) \, , \tag{4.96}$$

where $g_1(\delta)$ is defined by Eq. (4.86). The parameter $\tilde\delta\equiv\nu_1(x_1^{(0)}-V_1x_3^{(0)})$ characterizes the overlap between the slow soliton and the inhomogeneity at the moment of collision $t=0$. It is easy to verify that, according to Eqs. (4.95) and (4.96), the total energy of the two kinks,

$$E\equiv 8\sum_{j=1,3}(\lambda_j+\tfrac{1}{4}\lambda_j) \, ,$$

is conserved, while the momentum ΔP absorbed by the inhomogeneity is

$$\Delta P=8\epsilon\nu_1^{-1}(1-V_3^2)^{1/2}g_1(\tilde\delta) \, . \tag{4.97}$$

Note that the expression (4.97) is odd with respect to $\tilde\delta$.

The second explicitly tractable case is that when the inhomogeneity is at rest in the center-of-mass reference frame of the two colliding kinks, i.e., their velocities are opposite: $V_1=-V_2=W$ [we do not require $\nu^{-1}=(1-W^2)^{1/2}\ll 1$]. The result is

$$\Delta\lambda_2 = -\Delta\lambda_1/4\lambda_1^2 ,$$
$$\Delta\lambda_1 = \frac{\epsilon\sigma W(1+W)D_+D_-}{2(2\sigma+b)^2}[6+(4\sigma-b)J_\sigma(W,\delta)] , \tag{4.98}$$

where

$$J_\sigma(W,\delta) = \begin{cases} (2/\sqrt{-\zeta})\ln[\frac{1}{2}(b+\sqrt{-\zeta})], & \zeta<0 , \\ (2/\sqrt{\zeta})\cot^{-1}(b/\sqrt{\zeta}), & \zeta>0 , \end{cases} \tag{4.99}$$

and

$$\zeta \equiv -W^2D_-^2(2\sigma+b), \quad b\equiv 2\sigma+W^2D_-^2 ,$$
$$D_\pm \equiv e^{\delta}\pm\sigma e^{-\delta} . \tag{4.100}$$

Here $\sigma\equiv\sigma_1\sigma_3$ is the relative polarity; $\delta\equiv x_0/(1-W^2)^{1/2}$, where x_0 is the distance of the inhomogeneity from the center of mass of the two-kink system. Note that in this case, as can be seen from Eq. (4.98), $\Delta P=0$, i.e., the inhomogeneity resting in the center-of-mass system cannot absorb momentum (momentum nonconservation is possible in higher approximations unless $x_0=0$).

The above results are equally applicable to the cases $\epsilon>0$ and $\epsilon<0$. At the same time, a kink may be pinned by the inhomogeneity only in the case $\epsilon>0$, when the inhomogeneity attracts kinks of both polarities, and the attraction potential is (McLaughlin and Scott, 1978)

$$U=-2\epsilon\,\mathrm{sech}^2\xi , \tag{4.101}$$

where ξ is the coordinate of the kink's center. It is easy to find the law of motion of a pinned kink:

$$\sinh\xi(t)=\sqrt{(2\epsilon-E)/E}\,\sin(\sqrt{E}\,t/2) , \tag{4.102}$$

where E is the pinning energy ($0<E\le 2\epsilon$). The collision of a fast kink with a pinned one described by Eq. (4.102) may result in a simple inelastic effect: the kicking out of the slow kink from the bound state. If the incident kink is not too slow, we may use an expression for the collision-induced phase shift $\Delta\xi$ of the slow (pinned) kink valid in the absence of perturbations (Zakharov *et al.*, 1980):

$$\Delta\xi \approx -\ln\left[\frac{1+(1-V_3^2)^{1/2}}{1-(1-V_3^2)^{1/2}}\right] , \tag{4.103}$$

where V_3 is, as above, the velocity of the fast kink. The duration of the collision is $\sim V_3^{-1}$, which is much smaller than the bound kink's oscillation period $4\pi/\sqrt{E}\sim\epsilon^{-1/2}$ [see Eq. (4.102)], hence one may neglect the change in the pinned kink's velocity during the collision. Thus the collision does not change the slow kink's kinetic energy, but, according to Eqs. (4.101) and (4.103), its potential energy does change by an amount $\Delta U=U(\xi_0+\Delta\xi)-U(\xi_0)$, where $U(\xi)$ and $\Delta\xi$ are defined by Eqs. (4.101) and (4.103), and where ξ_0 is the value of the slow kink's coordinate before the collision. So, it is evident that the bound kink escapes provided $\Delta U>E$, and it remains bound when $\Delta U<E$. This approach and the final result are valid provided $V_3 \gg \sqrt{\epsilon}$.

In the opposite case, when V_3 is small, another noticeable interaction may occur: reflection of the incident kink from the bound one (without releasing the latter kink), provided the kinks are unipolar. Note that the inhomogeneity itself is attractive, and it cannot reflect a kink. The maximum (threshold) value of V_3 for which reflection is possible can again be readily obtained by means of energetic arguments: $V_{\mathrm{thr}}^2=\frac{1}{4}E$.

b. Collision between a free fluxon and one pinned by a local inhomogeneity

The collision between a free kink (fluxon) and a pinned one, considered at the end of the preceding section, is of practical interest for fluxons in a long Josephson junction with a microshort or microresistor. Inelastic interfluxon interactions, such as release of a pinned fluxon, trapping of a fluxon, etc., can be employed to design logic elements of a Josephson computer. These interactions have been studied in detail by Malomed and Nepomnyashchy (1989a) with the use of both analytical and numerical methods. The full picture of the interactions obtained in that work is rather complicated. Here we shall describe several representative particular cases in which the analytical perturbative technique proves to be especially useful. We employ the standard model [Eq. (3.53)] (with $\beta=0$) of a damped dc-driven long Josephson junction with a microinhomogeneity installed.

First let us consider the case in which two fluxons are unipolar, and the pinned one is bound by a repulsive inhomogeneity ($\epsilon<0$). The full effective one-fluxon potential takes the form (3.55′), with the second term accounting for the driving force generated by the bias current. The potential (3.55′) is shown in Fig. 18. A quiescent pinned fluxon rests at the point $\xi_{\min}$ which is the minimum of the potential (3.55′). The distance l defined in Fig. 18 can be easily found under the assumption $l\gg 1$:

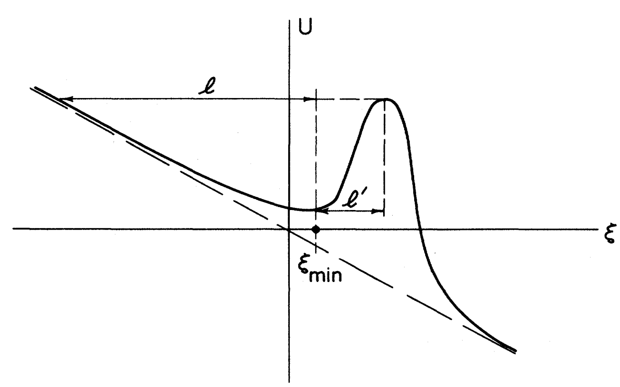

FIG. 18. The full effective one-fluxon potential (3.55′) with $\epsilon<0$. The dashed straight line is the potential $U_{\mathrm{dr}}=-2\pi f\xi$ of the driving force.

$$l \approx |\epsilon|/\pi f \ . \tag{4.104}$$

If the free fluxon is nonrelativistic, i.e., if its velocity far from the pinned fluxon is $V_0 \approx \pi f/4\gamma \ll 1$ [see Eq. (3.47)], and its kinetic energy $4V_0^2$ is larger than the height $2|\epsilon|$ of the effective potential barrier, the collision will result in a shift $\Delta\xi$ of the pinned fluxon which is determined by the well-known unperturbed expression [Zakharov *et al.*, 1980; cf. Eq. (4.103)]

$$\Delta\xi \approx -\ln(4/V_0^2) \ . \tag{4.105}$$

The minus sign in Eq. (4.105) implies that the pinned fluxon is displaced to the left, in terms of Fig. 18. Thus the displaced fluxon acquires additional potential energy $\Delta U = U(\xi_{\min}+\Delta\xi) - U(\xi_{\min})$. Neglecting dissipative losses, we conclude that the energy gain ΔU is sufficient to release the pinned fluxon if $\Delta U > 2|\epsilon|$, i.e., if $\Delta\xi > l$; see Fig. 18. Using Eqs. (4.104) and (4.105), one can obtain the condition under which the collision results in the release of the pinned fluxon:

$$f \geq f_{\rm thr}^{(1)} \equiv \epsilon[2\pi \ln(\gamma/|\epsilon|)]^{-1} \ . \tag{4.106}$$

The assumptions employed to derive Eq. (4.106) are summarized by the double inequality

$$\gamma^2 \ln(\gamma/\epsilon) \ll \epsilon \ll \gamma \ . \tag{4.107}$$

Let us emphasize that the inequalities (4.107) are needed solely to warrant the particular analytical formula (4.106); numerical investigation demonstrates that the release effect takes place over a much broader parametric range, and it is in qualitative agreement with the picture described above.

In the range $f < f_{\rm thr}^{(1)}$ the collision is elastic, i.e., after the collision one again observes a free fluxon and a pinned one. However, there exists another threshold value $f_{\rm thr}^{(2)} \sim \gamma\sqrt{|\epsilon|} \ll f_{\rm thr}^{(1)}$ [cf. Eq. (3.57)] such that in the range $f < f_{\rm thr}^{(2)}$ both fluxons will find themselves pinned after the collision. An exact value of $f_{\rm thr}^{(2)}$ can be found numerically.

In the case of an attractive inhomogeneity ($\epsilon > 0$), the pinned fluxon is again displaced to the left, but this time the distance l that provides the release of the pinned fluxon is much smaller (Fig. 19). Accordingly, in the present case (at least if $\epsilon \gtrsim \gamma^2$), there must be a threshold value $f_{\rm thr}^{(2)} \sim \gamma\sqrt{\epsilon}$ such that in the range $f > f_{\rm thr}^{(2)}$ both fluxons are free after the collision, and in the range $f < f_{\rm thr}^{(2)}$ both are pinned (although the elastic collision may take place in a narrow region of $f_{\rm thr}^{(2)}$).

Let us proceed to a collision between fluxons of opposite polarities. From the viewpoint of Figs. 18 and 19, the difference is that this time the free fluxon comes to the pinned one from the right; accordingly, the pinned fluxon displaces to the right. As can be seen in Figs. 18 and 19, the pinned fluxon will be released provided the shift $|\Delta\xi|$ exceeds the distance l' between the minimum and maximum of the effective potential. Straightforward analysis demonstrates that for either sign of ϵ the collision is elastic if

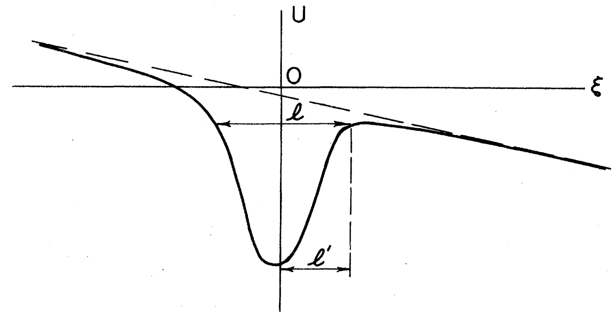

FIG. 19. The same as in Fig. 18, but with $\epsilon > 0$.

$$f \geq f_{\rm thr}^{(1)} \equiv (8\gamma/\pi)(4\gamma/|\epsilon|)^{1/3} \ , \tag{4.108}$$

and it results in release if $f \leq f_{\rm thr}^{(1)}$. In the case of opposite polarities a new inelastic interaction is possible, viz., fluxon-antifluxon annihilation. The threshold value $f_{\rm thr}^{(2)}$ of the bias current density that permits annihilation can be found from the energy balance as above [see Eq. (4.1′)], with the difference that now the total kinetic energy of the two fluxons is twice as small. So, one can find [cf. Eq. (4.3)]

$$f_{\rm thr}^{(2)} = 2(2\gamma)^{3/2} \ . \tag{4.109}$$

The annihilation takes place provided $f \leq f_{\rm thr}^{(2)}$.

It should be taken into account that a pinned state of one fluxon exists if $f < f_{\rm cr} = (4/3\pi\sqrt{3})|\epsilon|$ [for $f > f_{\rm cr}$ the effective potential (3.55′) has no local extrema]. Comparing $f_{\rm cr}$ to Eqs. (4.108) and (4.109), we conclude that both thresholds $f_{\rm thr}^{(1,2)}$ exist if $\gamma \leq \gamma_1 \equiv 2^{-5/4} \times 3^{-3/8}|\epsilon|$. If $\gamma_1 < \gamma < \gamma_2 \equiv \frac{1}{6}(|\epsilon|/\pi)^{2/3}$, the threshold $f_{\rm thr}^{(1)}$ is absent, i.e., a collision results in release for $f > f_{\rm thr}^{(2)}$, and in annihilation for $f < f_{\rm thr}^{(2)}$. Finally, the threshold $f_{\rm thr}^{(2)}$ disappears when γ_1 exceeds γ_2. In the latter case, a collision always results in annihilation.

A local inhomogeneity of another type in a long Josephson junction can be generated by an Abrikosov vortex crossing the junction. According to Aslamazov and Gurovich (1984), this inhomogeneity is described by the term $\epsilon\delta'(x)$, which gives rise to the effective potential [cf. Eq. (3.55)] $U(\xi) = 2\epsilon\sigma \sinh\xi$, where σ is the fluxon's polarity. It is possible to consider the interaction between a free fluxon and one pinned by an inhomogeneity of this type (Malomed and Nepomnyashchy, 1989a). Qualitatively, the results do not differ from those described above, except for the fact that in the case of opposite polarities the threshold value $f_{\rm thr}^{(1)}$ defined above proves to be less than $f_{\rm thr}^{(2)}$, provided $\gamma < (\pi^2/32)\epsilon^2$ [this was not possible in the model (3.53)]. In this situation, only the threshold $f_{\rm thr}^{(2)}$ actually exists, so that for $f > f_{\rm thr}^{(2)}$ the collision is elastic, and for $f < f_{\rm thr}^{(2)}$ it results in annihilation.

It was recently demonstrated by Gurovich and Mikhalev (1987) that the inhomogeneity $\epsilon\delta'(x)$, in contrast to $\epsilon\delta(x)\sin u$, admits a symmetric bound state of two unipolar fluxons attracted by the inhomogeneity. However, these authors overlooked the fact that the state is unstable against antisymmetric perturbations, so that it is physically meaningless.

4. Three-soliton collisions in perturbed Korteweg–de Vries and modified Korteweg–de Vries equations

Three-soliton collisions may also be considered in the framework of KdV and modified KdV equations with various conservative perpurbations. For instance, if the KdV equation arises as a result of an expansion in powers of a wave amplitude, it is natural to inquire into effects produced by higher terms of the expansion. In the simplest case this leads us to the following perturbed KdV equation:

$$u_t-6uu_x+u_{xxx}=\epsilon_1 u^2 u_x+\epsilon_2 u^3 u_x+\cdots \quad (4.110)$$

As above, it is natural to consider perturbation-induced three-soliton interactions. The "splitting conditions" analogous to those employed above this time take the form $\kappa_3 \gg \kappa_2 \gg \kappa_1$, where $\kappa_{1,2,3}$ are the amplitudes of the solitons involved. Using these conditions, one can verify that, in contrast with the situation for conservatively perturbed NS and SG equations, for Eq. (4.110) there are no soliton amplitude changes induced by three-particle interactions. This result, which directly follows from inspection of the perturbation equations, can be understood with the aid of a simple kinematic consideration. The perturbed Eq. (4.110) conserves three quantities: energy, momentum, and the "mass" $\int_{-\infty}^{+\infty} u\,dx$, so only a four-soliton collision may result in a change of soliton amplitudes in the adiabatic approximation. However, a three-soliton interaction may give a nontrivial contribution to the phase shifts. We have calculated the phase shift of the soliton possessing the greatest amplitude. If, for instance, a perturbation is represented by the second term on the right-hand side of Eq. (4.110), the three-particle contribution to the phase shift is

$$\Delta x_{123}=2\epsilon_2\frac{\kappa_1^3\kappa_2^2}{\kappa_3^2}\operatorname{sech}^3\delta\ , \quad (4.111)$$

where $\delta\equiv 2\kappa_1(x_{01}-x_{02})$ is the relative distance between the two slower solitons at the moment of their collision with the fast one. If one considers the first term on the right-hand side of Eq. (4.110) as a perturbation, analogous calculations yield

$$\Delta x_{123}=-4\epsilon_1\frac{\kappa_1^2\kappa_2}{\kappa_3^2}\operatorname{sech}^2\delta\ . \quad (4.112)$$

At the same time, it is well known that the KdV equation with this "perturbation" remains exactly integrable (see, for example, Zakharov *et al.*, 1980). Indeed, the linear transformation $u=av+b$, where $b=-3/\epsilon_1$, reduces Eq. (4.110) to the unperturbed modified KdV equation, which, in turn, is reduced to the KdV equation by the well-known Miura transformation (Miura, 1968). If we take as an initial condition for the "perturbed" equation a three-soliton state of the unperturbed KdV equation, this transformation will bring us to a perturbed three-soliton state, i.e., the three-soliton state plus a contribution from the continuous spectrum. On the other hand, it is well known that the interaction of a soliton with the continuous spectrum results in some phase shift of the soliton (see, for example, Martinez Alonso, 1985). It is natural to assume that the "perturbation-induced" phase shift (4.112) is a manifestation of that interaction.

Let us briefly adduce analogous results for the perturbed modified KdV equation

$$u_t-6u^2u_x+u_{xxx}=\epsilon u^4 u_x\ .$$

As in the case of the perturbed equation (4.110), we have three integrals of motion (energy, momentum, and "mass"), and the interaction of three solitons may not result in a change of soliton amplitudes in the adiabatic approximation. As for the phase shift of the soliton with the greatest amplitude (we again assume $\kappa_3\gg\kappa_2\gg\kappa_1$), it takes the form $\Delta x_{123}=\frac{4}{3}\epsilon(\kappa_1^4\kappa_2^3/\kappa_3^2)\operatorname{sech}^4\delta$, where $\delta\equiv 2\kappa_1(x_{01}-x_{02})$.

5. Stability of a nonlinear Schrödinger breather

Breathers have been observed in experiments with soliton-carrying optical fibers (Mollenauer and Stolen, 1982), and the stability of breathers described by various versions of the perturbed NS equation have been the subject of intensive studies (Pereira and Chu, 1979; Yajima *et al.*, 1979; Chu and Desem, 1985; Golovchenko *et al.*, 1985; Malomed, 1985; Wai *et al.*, 1986). Numerical experiments performed by Chu and Desem (1985), Golovchenko *et al.* (1985), and Wai *et al.* (1986) have demonstrated that a conservative perturbation may break a breather into a pair of free solitons. Here we shall consider this problem for the NS equation with the model perturbation (1.7a), which also occurs in nonlinear optics (although does not play a physically important role; Kumar *et al.*, 1986). The perturbation Hamiltonian corresponding to Eq. (1.7a) is

$$H_{\text{pert}}=\frac{\epsilon}{3}\int_{-\infty}^{+\infty}dx\,|u|^6\ . \quad (4.113)$$

For a pair of free solitons (2.41) with equal amplitudes η, the quantity (4.113) takes the value

$$(H_{\text{pert}})_{\text{sol}}=(2^{10}/15)\epsilon\eta^5\approx 68\epsilon\eta^5\ . \quad (4.114)$$

It is natural to compare Eq. (4.114) with the quantity $(H_{\text{pert}})_{\text{br}}$ ensuing from Eq. (4.113) on inserting there the expression for a breather with $\gamma_1(t)+\gamma_2(t)=0$, i.e., that corresponding to the moment of maximum overlap between the two constituent solitons:

$$(H_{\text{pert}})_{\text{br}} \approx 383\epsilon\eta^5 \ . \tag{4.115}$$

Comparison of Eqs. (4.115) and (4.114) reveals that the breather is stable provided $\epsilon<0$, and its binding energy is (Malomed, 1988d)

$$E_b \equiv (H_{\text{pert}})_{\text{sol}} - (H_{\text{pert}})_{\text{br}} \approx -315\epsilon\eta^5 \ . \tag{4.116}$$

As another particular case, one may take the breather in the weakly overlapping state (4.24). The corresponding binding energy averaged in the fast intrasoliton oscillations proves to be (Malomed, 1988d)

$$\langle E_b \rangle \approx -\tfrac{512}{5}\epsilon\zeta^4\eta^5 e^{-2\eta L} \ , \tag{4.117}$$

where L is the same as in Eq. (4.27) and it is assumed that $\eta L \gg 1$. Equation (4.117) corroborates that the breather's stability requires $\epsilon<0$.

In the theory of nonlinear optical fibers, an important role is played by perturbations described by the Hamiltonians

$$H_{\text{pert}} = i\epsilon\int_{-\infty}^{+\infty} dx\, u_{xx}u_x^* \ , \tag{4.118}$$

$$H_{\text{pert}} = i\epsilon\int_{-\infty}^{+\infty} dx\, u^2u^*u_x^* \ , \tag{4.119}$$

which account for, respectively, higher linear dispersion of the fiber and the group velocity dispersion [see Golovchenko *et al.* (1985), Kodama and Hasegawa (1986), and Bourkoff *et al.* (1987) and references therein; a brief survey can be found in the paper by Kodama (1985d)].

In contrast to Eq. (4.113), the perturbations (4.117) and (4.119) give zero binding energy in the same approximation in which we have obtained Eqs. (4.114)–(4.117). So, to investigate the stability of a breather in the presence of perturbations (4.118) and (4.119), one needs to take into account perturbation-induced corrections to the form of the breather.

Another approach to this problem has been put forward by Kodama and Hasegawa (1986) [see also Kodama and Nozaki (1987)]. They took an unperturbed wave configuration consisting of two solitons with equal velocities (which is just a breather), and calculated perturbation-induced shifts of the soliton velocities using first-order perturbation theory. One of their conclusions was that the perturbations (4.118) and (4.119) shifted the velocities of the solitons by different amounts, i.e., gave rise to some relative velocity ΔV, which had to explain the breather's breakup. However, this explanation does not seem reliable since, if one takes as an unperturbed state a two-soliton configuration with relative velocity $-\Delta V$, the perturbation may cancel it and thus create a new stable breather.

6. Solitons of the double sine-Gordon equation (wobblers)

Let us proceed to the perturbed double SG equation

$$u_{tt} - u_{xx} + \sin u = h\sin\frac{u}{2} + \epsilon P(u), \tag{4.120}$$

where $P(u)$ is some additional perturbation, and where both parameters ϵ and h are small. An example is a weak ferromagnet with a localized magnetic impurity (Bar'yakhtar *et al.*, 1985):

$$P(u) = \delta(x)\sin u \tag{4.121}$$

and dissipation $P(u) = -u_t$.

Though the double SG equation is not exactly integrable, it has an exact one-soliton solution (a 4π kink), which in the case $0<h\ll 1$ has the form (Bullough *et al.*, 1980)

$$u \approx 4\tan^{-1}(\sqrt{h/2}\sinh z) \ , \quad z \equiv (x-Vt)/(1-V^2)^{1/2} \ . \tag{4.122}$$

This may be interpreted as a bound state of two sine-Gordon 2π kinks (Newell, 1977). As was demonstrated by Newell (1977) [see also Burdick *et al.* (1987) and Willis *et al.* (1987)], the 4π kink can exist in an excited state, when the two internal 2π kinks oscillate relative to each other (Newell, 1977; Bullough *et al.*, 1980):

$$u \approx 4\tan^{-1}\left[\frac{W(t)\sinh x}{\cosh\chi(t)}\right] , \tag{4.123}$$

where

$$\chi(t) = a\sqrt{2/h}\,\sin(\Omega t) \ , \tag{4.124a}$$

$$W(t) = \sqrt{h/2} + a\cos(\Omega t), \quad \Omega = \sqrt{h} \ , \tag{4.124b}$$

and a is a small amplitude of the internal oscillations ($a^2 \ll h$). The excited 4π kink is sometimes called a wobbler (Bullough *et al.*, 1980). Equation (4.123) is written in the wobbler's rest reference frame ($V=0$). The solution (4.123) and (4.124) is approximate; since the double SG equation is not exactly integrable, the wobbler's internal oscillations slowly fade due to emission of radiation (see Sec. VII.E).

Proceeding to a study of adiabatic effects produced by the additional perturbation $\epsilon P[u]$ in Eq. (4.120), let us first consider the damping of the wobbler's internal oscillations under the action of the dissipation $\epsilon P = -\gamma u_t$. It is easy to find the energy of the internal oscillations, $E_{\text{osc}} = a^2 + O(a^4/h)$. Energy balance yields the damping law in the following form:

$$\frac{d}{dt}a^2 = -8\gamma a^2 + O(\gamma a^4/h) \ . \tag{4.125}$$

Oscillations of a finite amplitude and their dissipation-induced damping can be studied with the aid of general adiabatic equations developed by Bullough *et al.* (1980).

The dissipatively damped double SG equation admits stationary solitons of two types: quiescent unexcited 4π kinks and the usual 2π kinks moving with the equilibrium velocity (3.48). It is pertinent to note that, as follows from inspection of the form of these solitons, a collision between 2π and 4π kinks is not possible.

Now let us consider a collision between an excited 4π kink moving with velocity V and a local inhomogeneity described by the perturbation (4.121). A slight generalization of the approach developed by Kivshar and Malomed (1987b) yields the following result: after the collision, the 4π kink goes over into an excited state (4.123) and (4.124) with amplitude

$$a=\frac{32V\epsilon}{h}\sqrt{1-V^2}\ . \tag{4.126}$$

The size of a moving 4π kink is $\sim\sqrt{1-V^2}\ln h^{-1}$ according to Eqs. (4.123) and (4.124), and the derivation of Eq. (4.126) implies that the collision time $\tau\sim V^{-1}\sqrt{1-V^2}\ln h^{-1}$ is much less than the wobbler's internal oscillation period $T=2\pi/\Omega=2\pi\sqrt{2/h}$. [In the opposite case ($\tau\ll T$), the result will be exponentially small in τ/T.] Using the energy balance, it is easy to find the 4π kink's loss of velocity due to the collision:

$$\Delta(V^2)=\frac{a^2}{8}(1-V^2)\ .$$

Collision of two unexcited 4π kinks (in the absence of additional perturbations) moving with velocities $\pm V$ may result in excitation of the internal oscillations by the transfer of a part of their kinetic energy into internal degrees of freedom. This phenomenon has been revealed numerically by Campbell *et al.* (1986). However, the analytical calculation of Kivshar and Malomed (1987b) has demonstrated that for "relativistic" 4π kinks $[\nu\equiv(1-V^2)^{-1/2}\gg 1]$ there is no excitation of internal oscillations, at least in order ν^{-1}. In the same order, the effect of a collision is nontrivial if it takes place in the presence of the inhomogeneity (4.121). Let us consider the case when one 4π kink is "relativistic" in the inhomogeneity's rest reference frame [i.e., $\nu_1\equiv(1-V_1^2)^{-1/2}\gg 1$] and when the velocity V_2 of the second 4π kink is arbitrary. Then the change in the velocity of the second 4π kink is

$$\Delta V_2=-\frac{16\epsilon h}{\nu_1}(1-V_2^2)F(\delta;h)\ , \tag{4.127}$$

and the change of the parameter ν_1 of the first 4π kink is

$$\Delta\nu_1=\frac{16\epsilon h}{\nu_1}\frac{V_2}{(1-V_2^2)^{1/2}}F(\delta;h)\ , \tag{4.128}$$

where

$$F(\delta;h)=\frac{\sinh\delta\cosh\delta(2-h\sinh^2\delta)}{(2+h\sinh^2\delta)^3}$$

and δ is the distance between the collision point (i.e., a point at which the centers of the colliding 4π kinks overlap) and the point $x=0$ where the inhomogeneity is localized.

It is easy to verify that the change in total kinetic energy of the two wobblers corresponding to Eqs. (4.127) and (4.128) is zero, i.e.,

$$\Delta E_1+\Delta E_2\approx 16\left[\Delta\nu_1+\frac{V_2\Delta V_2}{(1-V_2^2)^{3/2}}\right]=0\ .$$

This means that, in the order considered ($\sim\nu_1^{-1}$), there is no excitation of the internal degrees of freedom. In the same order ν^{-1}, a collision of three fast 4π kinks has no adiabatic effect.

C. Interactions of solitons in systems of coupled equations

Systems of weakly coupled nearly integrable equations (which become exactly integrable in a decoupled form), such as coupled pairs of KdV, NS, and SG equations, occur in a number of important physical problems. These systems give rise to bisolitons, i.e., bound states of two solitons belonging to different subsystems. Interactions of bisolitons with each other and with ordinary solitons gives rise to a new and interesting class of dynamical phenomena.

1. Coupled Korteweg–de Vries equations

In this subsection we shall consider the system (1.3) and (1.4), which, as was mentioned in the Introduction, describes two resonantly interacting normal modes of internal-gravity-wave motion in a shallow stratified liquid. Gear and Grimshaw (1984; Gear, 1985) have demonstrated that two solitons belonging to different (u_1 and u_2) subsystems can form a leapfrogging bound state that moves with some common velocity and that has internal oscillations. The effect has been explained in the framework of perturbation theory by Malomed (1987g) and Kivshar and Malomed (1989d).

In the decoupled case ($\epsilon_1=\epsilon_2=\epsilon_3=0$), the two solitons take the form

$$u_j=-2\kappa_j^2\mathrm{sech}^2[\kappa_j(x-z_j)],\quad j=1,2, \tag{4.129}$$

$$\frac{dz_1}{dt}=4\kappa_1^2,\quad \frac{dz_2}{dt}=4\beta\kappa_2^2-V_0\ . \tag{4.130}$$

We shall confine ourselves to the case in which the amplitudes $\kappa_{1,2}$ of the unperturbed solitons are equal, $\kappa_{1,2}=\kappa+\lambda_{1,2}$, where $\kappa=$const, and the perturbation-induced variable parts $\lambda_{1,2}$ are small compared with κ. To form a bound state, we need the unperturbed velocities to coincide:

$$4\kappa_1^2=4\beta\kappa_2^2-V_0\ , \tag{4.131}$$

i.e., $\kappa^2=V_0/4(\beta-1)$; see Eq. (4.130). Internal oscillations of a bound two-soliton state are described by an equation for the relative coordinate $\zeta\equiv\kappa(z_1-z_2)$. The general perturbation-induced evolution equations for the parameters λ_1, λ_2, z_1, and z_2 are

$$\frac{d\lambda_1}{dt}=-\frac{d\lambda_2}{dt}=2\kappa^4[2\epsilon_3I_0(\zeta)-(\epsilon_1+\epsilon_2+6\epsilon_3)I_1(\zeta)]\ , \tag{4.132}$$

$$\frac{dz_1}{dt}=4\kappa^2+8\kappa\lambda_1+\epsilon F(\zeta)\ , \tag{4.133}$$

$$\frac{dz_2}{dt}=4\kappa^2\beta+8\kappa\beta\frac{d\lambda_2}{dt}-V_0+\alpha\epsilon F(\zeta)\ , \tag{4.134}$$

where

$$I_0(\zeta)\equiv\frac{2}{\cosh^2\zeta\tanh^4\zeta}[3\zeta-3\tanh\zeta-\zeta\tanh^2\zeta]\ , \tag{4.135}$$

$$I_1(\zeta)\equiv\frac{2}{\cosh^2\zeta\tanh^6\zeta}\left[-\tfrac{13}{3}\tanh^3\zeta+5\tanh\zeta-\frac{\zeta(5-\tanh^2\zeta)}{\cosh^2\zeta}\right], \tag{4.136}$$

and $F(\zeta)$ is a certain function that rapidly decreases at $|\zeta|\to\infty$. For the sake of simplicity, we shall present further results for the particular case $\alpha=1$. In this case the evolution equation for the coordinate ζ ensuing from Eqs. (4.132)–(4.136) reduces to an equation of motion for unit-mass nonrelativistic particle in a potential whose shape is shown in Figs. 20 and 21. In the case $\epsilon_1+\epsilon_2>-\frac{5}{2}\epsilon_3$ the potential has a minimum at $\zeta=0$ [Figs. 20(a) and 21(a); in the case $\epsilon_3>0$ it also has two maxima at $\zeta=\pm\zeta_m$; see Fig. 20(a)]. In the opposite case $\epsilon_1+\epsilon_2<-\frac{5}{2}\epsilon_3$ the point $\zeta=0$ is a maximum [Figs. 20(b) and 21(b); in the case $\epsilon_3<0$ the potential also has two minima at $\zeta=\pm\zeta_m$; see Fig. 21(b)].

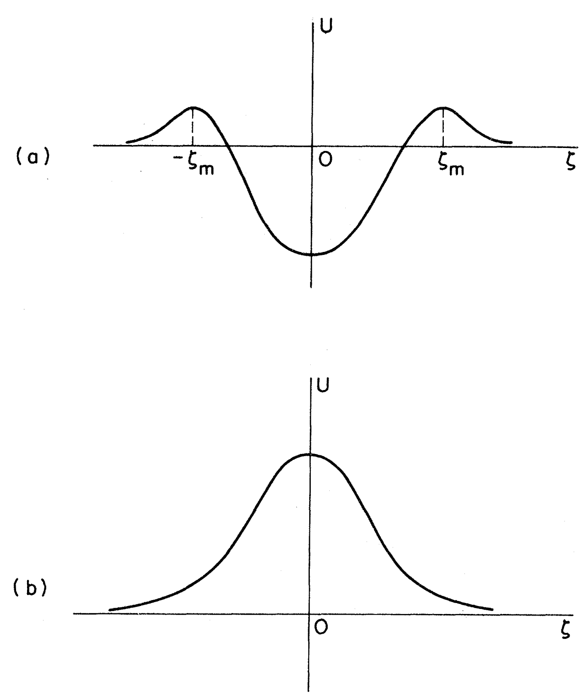

FIG. 20. The effective potential $U(\zeta)$ of the interaction between two solitons with nearly equal amplitudes described by the system of coupled kdV equations (1.3) and (1.4) with $\alpha=1$ and $\epsilon_3>0$: (a) $\epsilon_1+\epsilon_2>-\frac{5}{2}\epsilon_3$; (b) $\epsilon_1+\epsilon_2<-\frac{5}{2}\epsilon_3$.

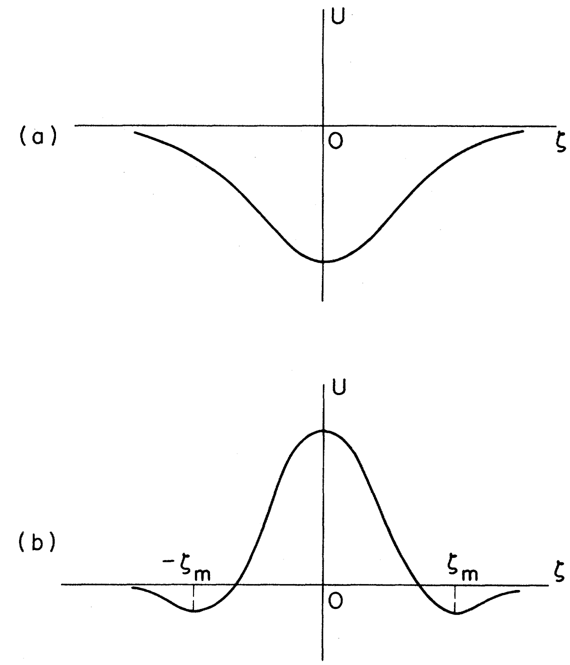

FIG. 21. The same as in Fig. 20, but with $\epsilon_3<0$: (a) $\epsilon_1+\epsilon_2>\frac{5}{2}|\epsilon_3|$; (b) $\epsilon_1+\epsilon_2<\frac{5}{2}|\epsilon_3|$.

In the particular case $\epsilon_1=\epsilon_2=0$ the potential takes the form shown in Figs. 20(a) ($\epsilon_3>0$) and 21(b) ($\epsilon_3<0$); in this case

$$U(\zeta)=32(1+\beta)\frac{\kappa^6}{\sinh^5\zeta}\left[\frac{\zeta}{2}(11\cosh\zeta+\cosh3\zeta)-3\sinh\zeta-\sinh3\zeta\right],$$

and $\zeta_m\approx2.1$ (Malomed, 1987g).

The presence of trapped trajectories in the potentials shown explains the existence of the stable leapfrogging bisolitons. In particular, if the potential has a minimum at $\zeta=0$, the frequency of small oscillations in its vicinity can be found in a general form (without assuming $\alpha=1$) (Kivshar and Malomed, 1989d):

$$\omega_0^2=\tfrac{256}{21}\kappa^6(1+\alpha\beta)[\epsilon_3+\tfrac{2}{5}(\epsilon_1+\epsilon_2)]\ . \tag{4.137}$$

Note that in the case shown in Fig. 20(a) the oscillation amplitude may not exceed the value ζ_m.

So far we have dealt with two solitons whose amplitudes κ_1 and κ_2 were nearly equal. The case $\kappa_1\gg\kappa_2$ (or $\kappa_2\gg\kappa_1$) is tractable too. According to Eq. (4.131), this case implies $\kappa_1^2\approx-V_0/4$ (or $\kappa_2^2\approx V_0/4\beta$). The final expression for the frequency of small oscillations of a corresponding bisoliton takes the form

$$\omega_0^2=-64\kappa_2^4\left[\epsilon_1\kappa_2^2-\frac{V_0}{4}(\tfrac{1}{3}\epsilon_2+4\epsilon_3)\right]\ . \tag{4.138}$$

If the solitons' velocities do not coincide, there is no bound state and, in fact, we are dealing with a collision problem. Phase shifts of colliding solitons belonging to different subsystems have been calculated by Malomed (1987g) and Kivshar and Malomed (1989d). In contrast with the case in which the solitons belong to one subsystem, the presence of these phase shifts is a nontrivial effect which is absent when the equations are decoupled.

Collision between a free soliton and a bisoliton may give rise to other interesting effects. For example, consider a free soliton with an amplitude κ' belonging to the first subsystem. Then, in the lowest approximation, its collision with a component of a bisoliton belonging to the same subsystem results in an unperturbed phase shift of the latter (Zakharov *et al.*, 1980),

$$\Delta\zeta_1=\kappa^{-1}\ln|(\kappa+\kappa')/(\kappa-\kappa')| \ . \tag{4.139}$$

Clearly, this is equivalent to a very quick shift of the relative coordinate ζ of the bisoliton from the equilibrium position $\zeta=\zeta_{eq}\equiv 0$ [Figs. 20(a) and 21(a)] or $\zeta=\zeta_{eq}\equiv\pm\zeta_m$ [Fig. 21(b)] to $\zeta=\zeta_{eq}+\kappa\Delta\zeta_1$. When $\epsilon_3>0$, $\epsilon_1+\epsilon_2>-\frac{5}{2}\epsilon_3$ [Fig. 20(a)], the shift causes the decay of the bisoliton into two free solitons provided $\Delta\zeta\equiv\kappa\Delta\zeta_1\geq\zeta_m$. According to Eq. (4.139), this inequality is equivalent to the following limitation on the amplitude κ':

$$\kappa\tanh(\zeta_m/2)\leq\kappa'\leq\kappa\coth(\zeta_m/2) \ .$$

2. Coupled nonlinear Schrödinger equations

In this subsection we shall look at the system of Eqs. (1.11) and (1.12) coupled by the terms (1.13) or (1.14). First let us consider the coupling (1.13) with $\epsilon_2=0$, $\epsilon_1\equiv\epsilon\neq 0$ being a real constant (linear nonderivative coupling). A natural problem is the collision between two solitons with equal amplitudes $\eta_1=\eta_2\equiv\eta$ and velocities $V_1=-V_2\equiv V$ belonging to different (u and v) subsystems. Straightforward application of the general equations yields the following changes in amplitude and velocity of the colliding solitons (Kivshar and Malomed, 1989e):

$$\Delta\eta_1=-\Delta\eta_2\equiv\Delta\eta=-\frac{\epsilon\pi^2\sin\chi_0}{4V\cosh^2(\pi V/8\eta)} \ , \tag{4.140}$$

$$\Delta V_1=\Delta V_2=\frac{V}{4\eta}\Delta\eta \ . \tag{4.141}$$

In Eqs. (4.140) and (4.141) $\chi_0=\phi_1^{(0)}-\phi_2^{(0)}$ stands for the difference between the initial phases of the two colliding solitons [see Eq. (2.41)]. The system considered conserves the total energy

$$E=\int_{-\infty}^{+\infty}dx\,[\,|u_x|^2-|u|^4+|v_x|^2-|v|^4 +\epsilon(vu^*+uv^*)] \ , \tag{4.142}$$

the total "number of plasmons"

$$N=\int_{-\infty}^{+\infty}(|u|^2+|v|^2)dx \ , \tag{4.143}$$

and the momentum of each subsystem, e.g.,

$$P_1=-i\int_{-\infty}^{+\infty}dx\,(u_xu^*-u_x^*u) \ . \tag{4.144}$$

In terms of the soliton parameters ($j=1,2$),

$$E=\sum_j(-\tfrac{16}{3}\eta_j^3+\eta_jV_j^2), \quad P_j=-V_j\eta_j, \quad N=4\sum_j\eta_j \ . \tag{4.145}$$

It is easy to see that, with regard to Eqs. (4.145), the changes (4.140) and (4.141) satisfy conservation of all the quantities (4.142)–(4.144).

Under the action of the same perturbation, two solitons belonging to different subsystems can form a bound state (bisoliton). Designating $\theta\equiv\eta_1-\eta_2$, $\Delta\equiv-\frac{1}{4}(V_1-V_2)$, $\chi\equiv\phi_1-\phi_2$, and $\delta\equiv 2\eta(x_{01}-x_{02})$, where x_{0j} ($j=1,2$) are the initial soliton coordinates [see Eq. (2.41)], we obtain the following evolution equations (Kivshar and Malomed, 1989f; see also Abdullaev *et al.*, 1989):

$$\frac{d\delta}{dt}=-8\eta\Delta \ , \tag{4.146}$$

$$\frac{d\Delta}{dt}=2\epsilon\eta\cos\chi I_1(\delta) \ , \tag{4.147}$$

$$\frac{d\chi}{dt}=-2V\Delta-8\eta\theta \ , \tag{4.148}$$

$$\frac{d\theta}{dt}=2\epsilon\eta\sin\chi I_2(\delta) \ , \tag{4.149}$$

where

$$I_1(\delta)\equiv\frac{2}{\cosh\delta\tanh^2\delta}(\tanh\delta-\delta) \ , \tag{4.150}$$

$$I_2(\delta)\equiv 2\delta/\sinh\delta \ . \tag{4.151}$$

We assume that the amplitudes of both solitons are close to a certain constant value η. Straightforward investigation of the dynamical system (4.146)–(4.149) demonstrates that it has a family of stationary solutions,

$$\delta=\Delta=\sin\chi=0, \quad \frac{d}{dt}\chi=-8\theta\eta \tag{4.152}$$

(in a stationary bisoliton state, the amplitude difference θ may be an arbitrary constant, while the phase difference χ grows linearly with time). Investigation of the stability of Eq. (4.152) demonstrates that small perturbations are characterized by four eigenfrequencies:

$$\omega=\pm 4\eta\sqrt{2\epsilon} \ , \tag{4.153}$$

$$\omega=\pm 4\eta[-\tfrac{2}{3}\epsilon\,\mathrm{sgn}(\cos\chi)]^{1/2} \ . \tag{4.154}$$

As immediately follows from Eq. (4.153) and (4.154), Eq. (4.152) is stable provided $\epsilon>0$ and $\cos\chi=-1$.

Let us proceed to the system of NS equations coupled by the derivative terms (1.14) with a real small coupling ϵ. In this case (in contrast with the preceding one), a collision between two solitons, in the adiabatic approximation, does not result in changes of their amplitudes and velocities. The only result is a change of the internal phases of both solitons by the amount (Kivshar and Malomed, 1986c)

$$\Delta\phi_1=\Delta\phi_2=\frac{\pi^3\epsilon V^2}{128\eta^2}\cos\chi\frac{\tanh(\pi V/8\eta)}{\cosh^2(\pi V/8\eta)} . \tag{4.155}$$

However, the derivative coupling may give rise to a bisoliton. The evolution equations similar to Eqs. (4.146)–(4.149) take the form (Kivshar and Malomed, 1988f)

$$\frac{d}{dt}\delta=-8\eta\Delta , \tag{4.156}$$

$$\frac{d}{dt}\Delta=8\epsilon\eta\left[\left[\frac{V^2}{16}-\eta^2\right]I_1(\delta)+2\eta^2 I_3(\delta)\right]\cos\chi , \tag{4.157}$$

$$\frac{d}{dt}\chi=-2V\Delta-8\eta\theta , \tag{4.158}$$

$$\frac{d}{dt}\theta=-4\epsilon\eta^2 I_1(\delta)\cos\chi , \tag{4.159}$$

where $I_1(\delta)$ is defined in Eq. (4.150), and

$$I_3(\delta)\equiv\frac{1}{2\sinh\delta\tanh\delta}[3(\sinh\delta\cosh\delta-\delta)-\tanh^2\delta(\sinh\delta\cosh\delta+\delta)] . \tag{4.160}$$

The system (4.156)–(4.159) has stationary solutions of two types:

$$\theta=\Delta=\delta=0 \tag{4.161}$$

with an arbitrary constant phase difference χ, and

$$\theta=\Delta=\cos\chi=0 \tag{4.162}$$

with an arbitrary constant relative distance δ between the centers of the two solitons.

The stability of the stationary solutions (4.161) against small perturbations is determined by Eqs. (4.156) and (4.157). The corresponding eigenfrequencies are

$$\omega=\pm 8\eta\left[\tfrac{2}{15}\epsilon\left[\frac{5V^2}{16}+3\eta^2\right]\cos\chi\right]^{1/2} . \tag{4.163}$$

As follows from Eq. (4.163), the bisoliton is stable provided $\epsilon\cos\chi>0$. If the bisoliton is stable, the centers of the two bound solitons may perform small oscillations relative to each other with frequency (4.163).

The stability of the stationary solutions (4.162) is determined by Eqs. (4.157)–(4.159), which yield three eigenfrequencies: $\omega=0$ and

$$\omega=\pm\left\{4\epsilon\sin\chi|V|\eta\left[\left[\frac{V^2}{16}-3\eta^2\right]I_1(\delta)+2\eta^2 I_3(\delta)\right]\right\}^{1/2} . \tag{4.164}$$

If $\omega^2>0$, the relative phase of the two solitons oscillates with the frequency (4.164), while the relative distance δ between their centers remains constant.

It is interesting to note that, in principle, a pair of solitons belonging to the same subsystem may form a bound state (breather) by the interaction of the solitons via their "images" in a paired subsystem; the binding energy of such a breather would be $\sim\epsilon^2$ (recall that the binding energy of a breather described by the unperturbed NS equation is exactly zero). However, this problem remains to be solved.

As an interesting example of a system of NS equations coupled by nonlinear terms, let us consider a pair of coupled nonlinear Schrödinger equations for Langmuir and dispersive ion acoustic waves, derived by Bhakta (1987) (see also Spatschek, 1978):

$$iu_{1t}+u_{1xx}+|u_1|^2u_1+b|u_2|^2u_1+i\gamma_1u_1=0 , \tag{4.165}$$

$$iu_{2t}+iVu_{2x}+pu_{2xx}+q|u_2|^2u_2+b|u_1|^2u_2+i\gamma_2u_2=0 , \tag{4.166}$$

where V, p, q, and b are, generally speaking, arbitrary parameters, p being always positive, and $\gamma_{1,2}$ are small damping coefficients. In the same paper, Bhakta found a solitonic solution of the form

$$\begin{aligned}u_1(x,t)&=A(x-\tilde V t)\exp(i\omega_1t-ik_1x) ,\\ u_2(x,t)&=\sqrt{\alpha}A(x-\tilde V t)\exp(i\omega_2t-ik_2x) ,\end{aligned} \tag{4.167}$$

where α is a constant, for the case $\gamma_1=\gamma_2=0$ and demonstrated that this solution minimized the energy of the system. The real function $A(z)$ is const$\times\eta\,\mathrm{sech}(\eta z)$. Bhakta (1987) also investigated the action of damping on the soliton. He arrived at the inference that the damped soliton had the form

$$|u_1|=a_1\eta\,\mathrm{sech}(\eta z),\quad |u_2|=a_2\eta\,\mathrm{sech}(\eta z) , \tag{4.168}$$

where $z=x-\tilde V t$, and the soliton's inverse width η remained constant, while the amplitudes $a_{1,2}$ decayed exponentially. Unfortunately, these assertions were wrong. Indeed, the derivation employed by Bhakta (1987) implied that Eq. (4.168) was a solution to Eqs. (4.165) and (4.166) if $\gamma_j=0$ and $a_j=$const. However, it follows from Eqs. (4.165) and (4.166) that this is true only in the case

$$\alpha\equiv(a_2/a_1)^2=(p-b)/(q-bp) . \tag{4.169}$$

In fact, Eqs. (4.167)–(4.169) are only a particular case of possible solitonic solutions to the system (4.165) and (4.166). If one inserts into Eqs. (4.165) and (4.166) (with $\gamma_j=0$)

$$u_j(x,t)=A_j(z)\exp[i\omega_jt+i\phi_j(z)] ,\quad j=1,2,\ z=x-\tilde V t , \tag{4.170}$$

where A_j and ϕ_j are real functions, one obtains a system of ordinary differential equations for the four functions $A_{1,2}(z)$ and $\phi_{1,2}(z)$ (Spatschek, 1978), straightforward qualitative analysis of which reveals that a full family of solitons must contain solutions in which $A_1(z)$ and $A_2(z)$ are not proportional to each other as in Eq. (4.167). If one takes an initial soliton in the form (4.167), under the action of the damping it will slowly evolve inside the full family and, in general, there is no reason for it to retain the form of Eq. (4.167) (Malomed, 1988c). Moreover, its inverse width η will inevitably decay along with the amplitudes, as in the case of a single damped nonlinear Schrödinger equation (Sec. III.B). To demonstrate this point, let us consider the special case $b=p=\sqrt{q}$. In this case Eq. (4.169) disappears, and Eqs. (4.165) and (4.166) with $\gamma_j=0$ have a family of solitonic solutions in the form (4.167) with

$$A(z)=\eta\sqrt{2/(1+\epsilon\alpha)}\,\mathrm{sech}(\eta z)\ , \tag{4.171}$$

$$k_1=-\tilde{V}/2,\quad k_2=(V-\tilde{V})/(2p)\ ,\qquad \omega_2=k_2V+p\omega_1\ , \tag{4.172}$$

where ω_1, α, and η are arbitrary constants; it is important that in this particular case, in contrast to the general one, α and η are both arbitrary; therefore the damped soliton will retain the form of Eq. (4.167). Insertion of Eq. (4.171) into the modified plasmon number conservation laws yields two damping-induced evolution equations (Malomed, 1988c):

$$\frac{d}{dt}\eta=-2\eta(1+\epsilon\alpha)^{-1}(\gamma_1+\gamma_2\epsilon\alpha)\ , \tag{4.173}$$

$$\frac{d}{dt}\alpha=2(\gamma_1-\gamma_2)\alpha\ . \tag{4.174}$$

Plainly, Eq. (4.173) tells us that η cannot remain constant.

Finally, the modified momentum conservation leads us, irrespective of the particular form of the soliton amplitude functions $A_{1,2}(z)$, to the conclusion that $\tilde{V}$ remains constant (Malomed, 1988c). Bhakta (1987) arrived at the same conclusion assuming that the damped soliton retains the form (4.167) and (4.168).

3. Coupled sine-Gordon equations

An important example of a weakly coupled system of two SG equations is

$$\varphi_{xx}-\varphi_{tt}-\gamma_1\varphi_t=\sin\varphi+f_1+\alpha\psi_{xx}\ , \tag{4.175}$$

$$\psi_{xx}-\psi_{tt}-\gamma_2\psi_t=\sin\psi+f_2+\alpha\varphi_{xx}\ . \tag{4.176}$$

According to Mineev, Mkrtchyan, and Shmidt (1981), the system (4.175) and (4.176) describes two weakly interacting parallel Josephson junctions, separated by a distance much larger than the Josephson penetration length. The small parameter α stands for a coupling constant.

This system was investigated by Kivshar and Malomed (1988c). In the absence of perturbations (i.e., at $f_j=\gamma_j=\alpha=0$, $j=1,2$), Eqs. (4.175) and (4.176) become uncoupled exactly integrable sine-Gordon equations. The solutions corresponding to fluxons (or antifluxons) are [cf. Eq. (2.61a)]

$$\varphi_j(x,t)=4\tan^{-1}\{\exp[\sigma_j(x-V_jt)/(1-V_j^2)^{1/2}]\}\ ,\quad j=1,2\ , \tag{4.177}$$

where V_j $(j=1,2)$ are the fluxon velocities and $\sigma_j=\pm1$ are their polarities. The interaction between junctions $(\alpha\neq0)$ distorts the form of the fluxon; most important is the "image" of the fluxon (4.177) in the paired junction. For instance, the "image" of the second fluxon $(j=2)$ in the first junction is

$$\varphi(z_2)=\frac{2\alpha\sigma_2\mathrm{sgn}z_2}{1-V_2^2}[z_2\cosh z_2-\sinh z_2\ln(2\cosh z_2)]\ , \tag{4.178}$$

where $z_2\equiv(x-V_2t)/(1-V_2^2)^{1/2}$.

Adiabatic equations that describe the interaction of two fluxons belonging to two different junctions can be obtained in a simple way if one employs the energetic approach. Inserting Eq. (4.177) into a term of the Hamiltonian that accounts for the interaction between the two junctions, i.e., $H_{\mathrm{int}}=\alpha\int_{-\infty}^{+\infty}dx\,\varphi_x\psi_x$, yields equations of motion for the centers of the two fluxons ξ_1 and ξ_2:

$$\frac{d^2\xi_1}{dt^2}=\frac{\pi f_1\sigma_1}{4}-\gamma_1\frac{d\xi_1}{dt}-\frac{\alpha\sigma_1\sigma_2}{\sinh\xi}(1-\xi/\tanh\xi)\ , \tag{4.179}$$

$$\frac{d^2\xi_2}{dt^2}=\frac{\pi f_2\sigma_2}{4}-\gamma_2\frac{d\xi_2}{dt}+\frac{\alpha\sigma_1\sigma_2}{\sinh\xi}(1-\xi/\tanh\xi)\ , \tag{4.180}$$

where $\xi\equiv\xi_1-\xi_2$. Equations (4.179) and (4.180) are written in the "nonrelativistic" approximation, i.e., for $(d\xi_j/dt)^2\ll1$ $(j=1,2)$. The simplest adiabatic effect that can be described by these equations is the binding of the two fluxons into a bifluxon (a bound state of fluxons) due to dissipative losses. We shall study this effect in the most interesting case, when the uniform motion of the free fluxons is "nonrelativistic," i.e., $f_j\ll\gamma_j$ [see Eq. (3.47)], and the coupling between the two equations is the strongest perturbation in Eq. (4.176), i.e., $|\alpha|\gg\gamma_j^2$. Our goal is to find a threshold condition admitting the binding of the two fluxons colliding with velocities $V_{1,2}$ defined according to Eq. (3.47). Using the conditions mentioned, we may consider the problem in the nearly inertial center-of-mass reference frame (the braking time of the center of mass will be much larger than the binding time). So, in the first approximation, we may neglect the terms $\sim f_j,\gamma_j$ in Eqs. (4.179) and (4.180) to arrive at the mechanical problem for a particle with reduced mass $m=4$ moving in the effective potential

$$U(\xi)=-8\alpha\sigma_1\sigma_2\xi/\sinh\xi\ . \tag{4.181}$$

Clearly, the law of motion is determined by the energy equation

$$2\left[\frac{d\xi}{dt}\right]^2+U(\xi)=2V_0^2\ , \quad (4.182)$$

where $V_0\equiv V_1-V_2$ is the particle's velocity at infinity.

The cases of attractive ($\alpha\sigma_1\sigma_2>0$) and repulsive ($\alpha\sigma_1\sigma_2<0$) potentials (4.181) are qualitatively different (as well as the interaction of a fluxon in a solitary junction with attractive or repulsive micro-inhomogeneities; see Sec. III.C.2). In the former case we assume $\gamma_j\sqrt{|\alpha|}\gg f_j$, which enables us to neglect the term $2V_0^2$ in Eq. (4.182). Then the law of motion of the two fluxons may be represented in the form

$$d\xi_{1,2}/dt=\pm 2\sqrt{|\alpha|\xi/\sinh\xi}\ . \quad (4.183)$$

To obtain the threshold condition for the binding process, it is necessary to calculate the total dissipative energy loss ΔE during the collision between fluxons. The threshold condition may be written as

$$\Delta E\geq T_1+T_2-T_b\ , \quad (4.184)$$

where $T_{1,2}\approx 4V_{1,2}^2\approx\pi^2 f_{1,2}^2/4\gamma_{1,2}^2$ are the kinetic energies of the two fluxons prior to the collision and $T_b\approx 8V_b^2$ is the kinetic energy of an eventual bound state (bifluxon), V_b being its equilibrium velocity analogous to $V_{1,2}$. To calculate V_b, let us note that the total driving force and the friction force acting upon it are, respectively, $2\pi(\sigma_1 f_1+\sigma_2 f_2)$ and $-8V(\gamma_1+\gamma_2)$, so that

$$V_b=\frac{\pi}{4}\frac{\sigma_1 f_1+\sigma_2 f_2}{\gamma_1+\gamma_2}\ . \quad (4.185)$$

The quantity ΔE can be calculated, with regard to Eq. (4.183), as follows:

$$\Delta E=\sum_{j=1}^{2}\gamma_j\int_{-\infty}^{+\infty}\left[\frac{d\xi_j}{dt}\right]^2 dt$$

$$\equiv(\gamma_1+\gamma_2)\int_{-\infty}^{+\infty}\frac{d\xi}{dt}d\xi=D(\gamma_1+\gamma_2)\sqrt{|\alpha|}\ , \quad (4.186)$$

where

$$D\equiv 4\int_0^{+\infty}dx\left[\frac{x}{\sinh x}\right]^{1/2}\approx 15.02\ .$$

Inserting Eqs. (4.185) and (4.186) into Eq. (4.184), we find that, for attraction, binding takes place provided

$$(\gamma_1+\gamma_2)\sqrt{|\alpha|}\geq\frac{\pi^2}{4D}\left[\frac{f_1^2}{\gamma_1^2}+\frac{f_2^2}{\gamma_2^2}-\frac{2(f_1\sigma_1+f_2\sigma_2)^2}{(\gamma_1+\gamma_2)^2}\right]. \quad (4.187)$$

Equation (4.187) is relevant if the potential force ($\sim\alpha$) acting upon the fluxons during their overlap is much larger than the friction force $\sim\gamma V\sim\gamma|\alpha|^{1/2}$, which explains the above assumption $\gamma_j^2\ll|\alpha|$.

For repulsion (i.e., $\alpha\sigma_1\sigma_2<0$), it is simpler to obtain a binding condition: the height $8|\alpha|$ of the potential barrier must be larger than the same combination $T_1+T_2-T_b$ of kinetic energies as appear on the right-hand side of Eq. (4.184) [cf. an analogous condition (3.57) for trapping of the fluxon by a repulsive micro-inhomogeneity],

$$|\alpha\sigma_1\sigma_2|\geq\frac{\pi^2}{32}\left[\frac{f_1^2}{\gamma_1^2}+\frac{f_2^2}{\gamma_2^2}-\frac{2(f_1\sigma_1+f_2\sigma_2)^2}{(\gamma_1+\gamma_2)^2}\right]\ , \quad (4.188)$$

provided the same assumption $|\alpha|\gg\gamma_{1,2}^2$ holds. The latter inequality demonstrates that condition (4.188) is less restrictive than (4.187).

The repulsive interaction of two fluxons ($\sigma_1=\sigma_2$, $\alpha<0$) in the particular case of two identical coupled junctions [$\gamma_1=\gamma_2$ in Eqs. (4.175) and (4.176)] with $f_1\neq f_2$ was considered by Volkov (1987).

A bifluxon may have damped internal oscillations; in particular, the frequency of small oscillations is $\omega^2\approx|\alpha|/3$ in the attraction case, and $\omega^2\approx\pi|(\sigma_1 f_1-\sigma_2 f_2)/8\alpha|$ in the repulsion case (the latter expression is valid as long as it gives $\omega^2\ll 1$). When there are no internal oscillations, the distance between the two fluxons bound into a bifluxon is $\xi_0\approx(3\pi/4\alpha)|f_1\sigma_1-f_2\sigma_2|$ for attraction and $\xi_0\approx\ln|\alpha/(f_1\sigma_1-f_2\sigma_2)|$ for repulsion [we again regard $(f_1\sigma_1-f_2\sigma_2)/\alpha$ as a small parameter].

There are other interesting adiabatic effects. One of them is the capture of a fluxon moving in one junction by a micro-inhomogeneity localized in another junction. If, for example, the micro-inhomogeneity is repulsive, an inequality providing for the capture can be readily obtained as in the case when both the fluxon and the inhomogeneity belong to the same junction (see Sec. III.C.2).

Collision of a bifluxon with a free fluxon may give rise to interesting inelastic effects (Kivshar and Malomed, 1988c). The simplest is the breakup of a bifluxon into a pair of free fluxons. Indeed, the full potential $U_{\rm tot}(\xi)$ of the interaction between two fluxons bound into a bifluxon is given by Eq. (4.181) plus a contribution from the bias currents (Fig. 22),

$$U_{\rm tot}(\xi)=-8\alpha\sigma\xi/\sinh\xi-\pi(\hat{f}_1-\hat{f}_2)\xi\ , \quad (4.189)$$

where $\sigma\equiv\sigma_1\sigma_2$, $\hat{f}_j\equiv\sigma_j f_j$ ($j=1,2$). It is straightforward to establish that, provided $|\alpha|\gg|\hat{f}_1-\hat{f}_2|$, the potential (4.189) has a local minimum at $\xi=\xi_{\rm min}$ and a local maximum at $\xi=\xi_{\rm max}$ separated by the large distance ξ_0 determined by

$$\exp(\xi_0)=\{16|\alpha|/[\pi(\hat{f}_1-\hat{f}_2)]\}\ln[16|\alpha|/\pi(\hat{f}_1-\hat{f}_2)] \quad (4.190)$$

(for definiteness, we assume $\hat{f}_1-\hat{f}_2>0$). On the other hand, a collision between two sine-Gordon kinks (fluxons) moving with nonrelativistic velocities V_j ($V_j^2\ll 1$) gives rise to unperturbed shifts $\Delta\xi_{1,2}$ of the centers of the kinks [cf. Eq. (4.103)],

$$\Delta\xi_{1,2}\approx{\rm sgn}(V_{1,2}-V_{2,1})\ln[4/(V_1-V_2)^2]\ . \quad (4.191)$$

For the sake of simplicity, in what follows we set $\gamma_1=\gamma_2\equiv\gamma$. Then a straightforward analysis based upon

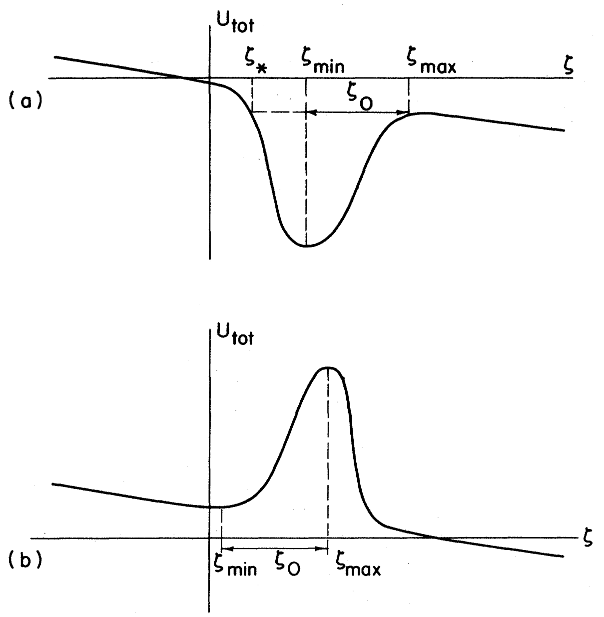

FIG. 22. The full potential of an interaction between kinks belonging to different junctions: (a) $\alpha\sigma_1\sigma_2>0$; (b) $\alpha\sigma_1\sigma_2<0$.

Eq. (4.191) demonstrates that collision between a bifluxon with $\alpha\sigma_1\sigma_2>0$ [Fig. 22(a)] and a free fluxon that belongs to a jth junction ($j=1,2$) and that has the same polarity σ_j as the corresponding component of the bifluxon results in a negative change $\Delta\xi$ in the relative distance ξ between the two fluxons bound into the bifluxon. As can be seen in Fig. 22(a), a negative shift of ξ from the equilibrium position ξ_{min} cannot directly cause breakup of the bifluxon. However, if

$$|\Delta\xi|\geq\xi_{min}-\xi_* \tag{4.192}$$

[see Fig. 22(a)], the bifluxon will eventually decay. As follows from Eqs. (4.190) and (4.191), the breakup condition (4.192) is equivalent to the inequality

$$\hat{f}_1-\hat{f}_2\leq 32\gamma^2/\pi|\alpha| \ . \tag{4.193a}$$

Equation (4.193a) neglects dissipative losses and assumes the free fluxon to be nonrelativistic. These assumptions require $|\alpha|^{3/2}\lesssim\gamma\ll|\alpha|$. Bifluxons with $\alpha\sigma_1\sigma_2>0$ are more stable against collision.

In the opposite case, when the polarity σ_3 of the free fluxon is $-\sigma_j$, the situation is more complicated. If, for instance, the free fluxon belongs to the first junction, and $\hat{f}_1>|\hat{f}_2|$, the quantity $\Delta\xi$ proves to be positive, hence breakup requires $\Delta\xi>\xi_0$ irrespective of the sign of $\alpha\sigma_1\sigma_2$. The latter inequality reduces to

$$(3\hat{f}_1+\hat{f}_2)^2/(\hat{f}_1-\hat{f}_2) \leq 16\gamma^2/\{\pi|\alpha|\ln[16|\alpha|/\pi(\hat{f}_1-\hat{f}_2)]\} \ . \tag{4.193b}$$

The inequality (4.193b) is relevant provided $\alpha^2\ll\gamma\ll|\alpha|$. However, in the present case ($\sigma_3=-\sigma_1$) two other types of inelastic interaction between the bifluxon and the free fluxon are possible. If $\alpha\sigma_1\sigma_2>0$, the fluxons with the polarities σ_1 and $\sigma_3=-\sigma_1$ colliding in the first junction annihilate (due to dissipative losses) into a breather under the condition [cf. Eq. (4.3)]

$$|3\hat{f}_1+\hat{f}_2|\leq 8\sqrt{3}\gamma^{3/2} \ , \tag{4.194}$$

so that eventually a free fluxon will remain in the second junction and nothing in the first one. If $\alpha\sigma_1\sigma_2<0$ and $|\alpha|\gg\gamma$, the collision leads, under the same condition (4.194), to another result ("recharge"): the two fluxons in the first junction switch roles, so that eventually there will be a free fluxon with polarity $\sigma_3=+\sigma_1$ in the first junction and a bifluxon with the reversed sign of $\alpha\sigma_1\sigma_2$ ($\alpha\sigma_1\sigma_2>0$).

Analogous consideration reveals that a collision between a bifluxon with $\alpha\sigma_1\sigma_2>0$ and a corresponding anti-bifluxon results in their dissipative annihilation provided

$$|\hat{f}_1+\hat{f}_2|\leq 8\gamma^{3/2} \ . \tag{4.195}$$

A colliding bifluxon and anti-bifluxon with $\alpha\sigma_1\sigma_2<0$ become, due to dissipative losses and under the same condition (4.195), a bifluxon–anti-bifluxon pair with $\alpha\sigma_1\sigma_2>0$. Finally, a collision between two bifluxons with polarities (σ_1,σ_2) and $(\sigma_1,-\sigma_2)$ gives rise to two free unipolar fluxons in the first junction and dissipative annihilation in the second junction, under the condition $|\hat{f}_2|\leq 4\sqrt{2}\gamma^{3/2}$.

Braun, Kivshar, and Kosevich (1988) have considered a system of two SG equations coupled by both linear and nonlinear terms [cf. Eqs. (4.175) and (4.176)]:

$$u_{tt}-u_{xx}+\sin u=-\beta\sin(u-v)+\alpha v_{xx}-\gamma u_t+f \ , \tag{4.196}$$

$$v_{tt}-v_{xx}+\sin v=-\beta\sin(v-u)-\alpha u_{xx}-\gamma v_t+f \ . \tag{4.197}$$

This system describes two weakly interacting linear chains of atoms adsorbed on a metallic surface. An effective potential of interaction between two kinks belonging to different subsystems has the form

$$U(\xi)=-8\alpha\sigma_1\sigma_2\frac{\xi}{\sinh\xi} +\frac{8\beta}{\sinh^3\xi}[2(\xi-\tanh\xi)\cosh\xi -\sigma_1\sigma_2[2\xi-\sinh(2\xi)+\xi\sinh^2\xi] \ ,$$

where ξ has the same sense as in Eq. (4.181). Though the system (4.196) and (4.197) seems more complicated than Eqs. (4.175) and (4.176), interactions between kinks and their bound states (bikinks) described by this system are qualitatively similar to those described by the system (4.175) and (4.176). In particular, a collision between a bikink and a free kink or between two bikinks may give rise to various inelastic interactions.

The system (4.196) and (4.197) with $f=\gamma=\alpha=0$ has also been proposed by Homma and Takeno (1983, 1984) as a model of two interacting chains which constitute a double DNA helix, u and v having the sense of torsion angles of the two chains.

4. Coupled double sine-Gordon equations

The system of equations

$$u_{1tt}-u_{1xx}+\sin u_1=\frac{\epsilon}{2}\sin\left[\frac{u_1}{2}\right]\cos\left[\frac{u_2}{2}\right], \qquad (4.198)$$

$$u_{2tt}-u_{2xx}+Q\sin u_2=\frac{\epsilon}{2}\sin\left[\frac{u_2}{2}\right]\cos\left[\frac{u_1}{2}\right], \qquad (4.199)$$

has been put forward by Zhang (1987) as a model of long-wave excitations in a double DNA helix [in the particular case $Q=1$ the same equations were proposed earlier as a model of DNA by Yomosa (1984, 1985)]. The variables u_1 and u_2 have the sense of the sum and the difference of two angles that determine local orientation of bases attached to the two helices. This model does not coincide with the model of Homma and Takeno (1983, 1984) mentioned at the end of the preceding subsection. As will be seen below, the difference between these systems is very important, and the dynamics of solitons described by Eqs. (4.198) and (4.199) differs drastically from that described in Sec. IV.C.3. The parameter Q is, in general, ~ 1, while $\epsilon \ll 1$ (Zhang, 1987).

The system (4.198) and (4.199) could be called a system of two coupled double SG equations; it possesses richer dynamics than many other nearly integrable systems. Here we shall present the first results (Malomed, 1987h) obtained from an investigation of the soliton dynamics governed by Eqs. (4.198) and (4.199) in the simplest particular case $Q=1$.

First of all, each of Eqs. (4.198) and (4.199) has 4π-kink solitonic solutions that can also exist in an excited (wobbler) form. In addition, the system possesses bikink solitonic solutions that consist of two ordinary 2π kinks belonging to different subsystems (u_1 and u_2). Like a 4π kink, a bikink may also exist in an excited state [an unexcited bikink was first described by Zhang (1987)]. Using the assumption $Q=1$, it is easy to find the effective interaction potential of the two 2π kinks inside the bikink:

$$U(\xi)=4\sigma\epsilon\xi\coth(2\xi/\sqrt{1-V^2}), \qquad (4.200)$$

where σ is the relative polarity of the kinks (we assume that the kinks' centers are located at the point $x=\pm\xi$ and that they move with velocities $\pm V$). Note that $U(\xi)\approx 4\epsilon\sigma|\xi|$ for $|\xi|\gg 1$, which coincides with the effective interaction potential for 2π kinks inside a wobbler (see Sec. IV.B.6).

First let us consider a collision between a quiescent bikink ($\epsilon\sigma<0$) and an unexcited 4π kink belonging to the u_1 subsystem and moving with velocity $W\gg\sqrt{\epsilon}$. Let the polarity of the 4π kink be opposite to that of the u_1 component of the bikink [Fig. 23(a)]. Using the well-known elementary interpretation of kink-kink and kink-antikink collisions in the unperturbed SG equation (see, for example, Zakharov *et al.*, 1980), we may say that in the lowest adiabatic approximation the collision causes a shift of the center of the bikink's u_1 component by $\Delta x=2|\Delta\xi_0|$, where $\Delta\xi_0$ is the well-known phase shift [Eq. (4.201)] resulting from a collision of two kinks in the unperturbed SG equation. This means

$$\Delta x=4\ln[(1+\sqrt{1-W^2})/W]\ . \qquad (4.201)$$

Thus the collision under consideration entails a quick (by virtue of the assumption $W\gg\sqrt{\epsilon}$) shift of the u_1 2π kink relative to its paired u_2 kink by the amount (4.201). This means that after the collision the bikink remains quiescent and, at the same time, it goes over into an excited state with an internal oscillation amplitude (defined as the maximum distance between centers of the oscillating constituent kinks) equal to the expression (4.201); in addition the bikink's center of mass shifts by $\frac{1}{2}\Delta x$. The corresponding period T of the internal oscillations of the excited bikink can easily be found on the basis of Eq. (4.200) under the condition $\Delta x\gg 1$, i.e., according to Eq. (4.201), $|\epsilon|\ll W^2\ll 1$:

$$T=2\sqrt{A(8+|\epsilon|A)/|\epsilon|}\ , \qquad (4.202)$$

where $A\equiv\Delta x$ is the oscillation amplitude. As for the

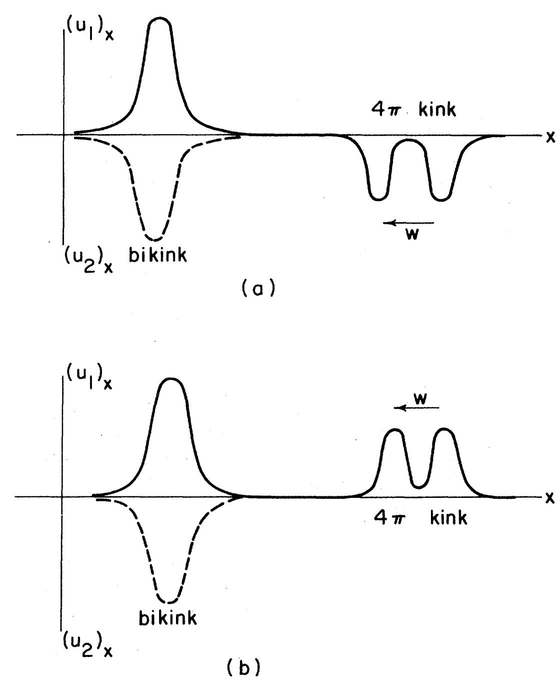

FIG. 23. A collision between a quiescent bikink ($\epsilon\sigma<0$) and an unexcited 4π kink belonging to the u_1 subsystem and moving with a velocity $W\gg\sqrt{\epsilon}$: (a) the polarity of the 4π kink is opposite to that of the u_1 component of the bikink; (b) the polarity of the 4π kink is the same as the polarity of the u_1 component.

traveling 4π kink, in the lowest approximation it undergoes no excitation, as the collision-induced shifts of both its constituent 2π kinks are equal. Since, as follows from Eqs. (4.200) and (4.201) ($\xi\equiv\frac{1}{2}\Delta x$), the energy

$$\Delta E=8|\epsilon|\frac{(1+\sqrt{1-W^2})^8+W^8}{(1+\sqrt{1-W^2})^8-W^8}\ln\left[\frac{1+\sqrt{1-W^2}}{W}\right]$$

is expended on excitation of the bikink's internal oscillations, the kinetic energy of the moving 4π kink, $E_{4\pi}\approx 16/\sqrt{1-W^2}$, decreases by the same ΔE.

The situation is altogether different when the 4π kink's polarity coincides with that of the bikink's u_1 component [Fig. 23(b)]. In the same approximation as above, the collision of the nearest 2π kink belonging to the 4π kink with the bikink's u_1 component results in the stopping of this 2π kink, its whole momentum $8W/\sqrt{1-W^2}$ being transferred to the bikink's u_1 component. Further kinematic analysis yields, with regard to Eq. (4.200), that after the collision the centers of the bikink and wobbler (the 4π kink excited due to the collision) move with the common velocity

$$W'=W(1+\sqrt{1-W^2})\ . \tag{4.203}$$

In the reference frame moving with the same velocity (4.203), both compound solitons have internal oscillations of equal amplitude,

$$A=\frac{8}{|\epsilon|}\{[(1+\sqrt{1-W^2})/2\sqrt{1-W^2}]^{1/2}-1\}\ , \tag{4.204}$$

and both have the period (4.202). It is interesting to note that in the nonrelativistic limit $W^2\ll 1$ the original kinetic energy $8W^2$ of the moving 4π kink is divided, according to Eqs. (4.200), (4.203), and (4.204), into four equal shares, which go over into (1) the bikink's kinetic energy, (2) the bikink's internal oscillation energy, (3) the residual wobbler's kinetic energy, and (4) the wobbler's internal energy.

In the more general case $Q>1$ [see Eq. (4.198)], the mass of the bikink's u_2 component is $8/\sqrt{Q}$; the wobbler's velocity and internal oscillation amplitude after the collision turn out to be equal to the same expressions (4.203) and (4.204), while the bikink's velocity is

$$W'_{\rm bi}=\sqrt{Q}\,W/(\sqrt{1+W^2}+W)>W'\ . \tag{4.205}$$

In the case $Q<1$ the situation is more complicated. The velocity $W'_{\rm bi}$ [see Eq. (4.205)] proves to be smaller than W', and so must result in a repeated collision.

Finally, a collision between two unexcited bikinks results (at $Q=1$) in only trivial phase shifts. This assertion follows from the fact that one may represent such a collision by setting $u=\pm V$, and it is obvious that the substitution $u=\pm v$ transforms the system (4.198) and (4.199) (with $Q=1$) into a single unperturbed SG equation.

D. Dynamics and kinetics of solitons in the driven damped double sine-Gordon model

1. Introductory remarks

As was mentioned in I.B.3.a, a fairly general situation that gives rise to the SG model in solid-state physics is that when a chain of interacting atoms is placed against a spatially periodic sinusoidal substrate potential. In certain cases, an ionic lattice that gives rise to the substrate potential may be subject to a period-doubling instability. An example is the Peierls instability of one-dimensional metals with the commensurability index $M=2$ (see, e.g., Kivelson, 1986). One would expect that in this situation the substrate potential $\Pi(u)$, where u is a dynamic variable of the atomic displacement type, would contain a basic strong component $\Pi_0(u)=1-\cos u$ and a weak subharmonic component $\Pi_1(u)=2\epsilon(1-\cos u/2)$, where ϵ is a small parameter. Adding the dissipative term and an external drive, we obtain the perturbed SG equation (Malomed, 1989)

$$u_{tt}-u_{xx}+\sin u=-\epsilon\sin\frac{u}{2}-f-\gamma u_t \tag{4.206}$$

[cf. Eqs. (1.16a), (1.17), and (4.120)]. For definiteness, in what follows we shall assume ϵ and f to be positive.

In the absence of a driving force ($f=0$), Eq. (4.206) is the double SG equation (4.120) with a damping term. As was explained in Sec. IV.B.6, that equation supports soliton solutions in the form of quiescent 4π kinks (4.122) and 2π kinks (2.61) moving with the equilibrium velocity (3.48). For the time being, we shall rewrite the 2π kink's wave form as

$$u_k=4\tan^{-1}\{\exp[\sigma(x-V_0t)(1-V_0^2)^{-1/2}]\}+2\pi m\ , \tag{4.207}$$

where m is an arbitrary integer. The results given below were obtained by Malomed (1989b).

2. Two-soliton collisions

In the presence of a driving term, the 4π kink acquires an equilibrium velocity V_0 which, in the first approximation, coincides with Eq. (3.47). The 2π kinks can be subdivided into two sorts, fast and slow, which correspond, respectively, to even and odd m in Eq. (4.207). The equilibrium velocities of the fast and slow 2π kinks are $\sigma V_{\rm f}$ and $\sigma V_{\rm sl}$, where σ is the polarity, and

$$\begin{aligned}V_{\rm f}&=\{1+[4\gamma/(\pi f+2\epsilon)]^2\}^{-1/2}\ ,\\ V_{\rm sl}&=\{1+[4\gamma/(\pi f-2\epsilon)]^2\}^{-1/2}{\rm sgn}(\pi f-2\epsilon)\ .\end{aligned} \tag{4.208}$$

At $f=0$, the only possible collisions between kinks moving with the equilibrium velocities are (1) a collision between two unipolar 2π kinks with values of m differing by one [see Eq. (4.207)], and (2) a collision between 2π

kinks with opposite polarities σ. The eventual result of the latter process is annihilation of the kink-antikink pair into a breather, which is then damped by dissipation, while the result of the former collision is fusion of the two unipolar 2π kinks into a 4π kink (it will also have damped internal oscillations). A collision between a moving 2π kink and a quiescent 4π kink is not possible. In the presence of the drive ($f>0$), the following collisions are possible: (i) fast 2π kink, fast 2π antikink; (ii) slow 2π kink, slow 2π antikink; (iii) fast 2π kink, slow 2π kink (it is implied that the polarities of the two kinks are like); (iv) 4π kink, slow 2π antikink (the opposite polarities are implied); (v) 4π kink, 4π antikink. Other collisions (e.g., between a 4π kink and a 2π kink with like polarities) are impossible. In what follows, the abbreviation k_f will stand for fast 2π kinks, k_{sl} will stand for slow 2π kinks, and K will stand for 4π kinks. The same abbreviations with overbars will denote the corresponding antikinks.

The simplest collision is that of type (iii): As in the case $f=0$, it always results in fusion of the two unipolar 2π kinks into a 4π kink of the same polarity,

$$k_f+k_{sl}\rightarrow K\ . \tag{4.209}$$

The results of collisions of types (i) and (ii) crucially depend on the sign of V_{sl} [see Eq. (4.208)]. In the case

$$\pi f<2\epsilon \tag{4.210}$$

the colliding kinks always annihilate into a breather. In the opposite case, annihilation takes place if the net kinetic energy of the kinks before collision is less than the collision-induced dissipative energy loss (4.1′) with $\beta=0$:

$$E_{diss}=8\pi^2\gamma\ . \tag{4.211}$$

It is easy to see that in case (i) this condition amounts to

$$\pi f+2\epsilon<4\pi\gamma^{3/2}\ , \tag{4.212a}$$

while in case (ii) it amounts to

$$\pi f-2\epsilon<4\pi\gamma^{3/2}\ . \tag{4.212b}$$

If the inequalities (4.210a) and, respectively, (4.212a) or (4.212b) do not hold, collisions of type (i) and (ii) result in the conversion of a fast kink-antikink pair into slow one and vice versa:

$$k_f+\bar{k}_f\rightleftarrows k_{sl}+\bar{k}_{sl}\ . \tag{4.213}$$

A collision of type (iv) takes place under the condition $V+V_{sl}>0$ or

$$\pi f>\epsilon\ .$$

In principle, this collision may give either of two results:

$$K+\bar{k}_{sl}\rightarrow k_f\ , \tag{4.214a}$$

$$K+\bar{k}_{sl}\rightarrow k_f+k_{sl}+\bar{k}_{sl} \tag{4.214b}$$

(see Fig. 24). To analyze the collision in detail, let us first consider the interaction between a 2π kink and a 2π antikink moving with velocities V_1 and V_2. The collision-induced loss of the kinks' kinetic energy is given by Eq. (4.211). It is easy to see that in the lowest approximation [in which Eq. (4.211) is relevant] the net collision-induced dissipative loss of the kinks' momentum is zero. So, using energy and momentum balance, one can find the velocities of the kinks $V'_{1,2}$ after the collision:

$$V'_{1,2}=\tfrac{1}{2}\{(V_1+V_2)\pm[(V_1-V_2)^2-\tfrac{1}{2}E_{diss}]^{1/2}\}\ . \tag{4.215}$$

This expression is valid in the nonrelativistic case $V_{1,2}^2\ll 1$, which will be of basic interest below.

Now let us return to the type-(iv) collision. According to Eq. (4.215), after the collision between the 2π antikink A moving with velocity $-V_{sl}$ and the 2π kink B moving with velocity V_0 [Fig. 24(a)], the former acquires the velocity

$$-V'_{sl}=-\tfrac{1}{2}(\{[(\pi f-\epsilon)/(2\gamma)]^2-\tfrac{1}{2}E_{diss}\}^{1/2}-\epsilon/(2\gamma))\ . \tag{4.216}$$

Analogous consideration of the collision between a 2π antikink A moving with velocity (4.216) and a second 2π kink C moving with velocity V [Fig. 24(a)] leads us to infer that *partial annihilation* (4.214a) takes place under the condition $(V-V'_{sl})^2<\frac{1}{2}E_{diss}$, or

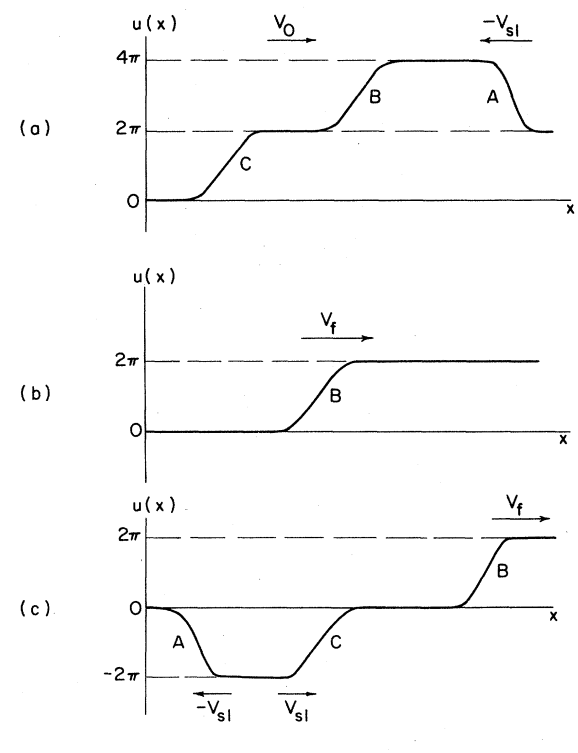

FIG. 24. A collision between a 2π antikink A and a 4π kink consisting of two 2π kinks B and C: (a) the field configuration before the collision; (b) configuration after the collision in the case (4.214a) (partial annihilation); (c) configuration after collision in the case (4.214b) (the breakup of the 4π kink).

$$\pi f+\epsilon<10\pi\gamma^{3/2} \; . \tag{4.217}$$

In the opposite case the *breakup* (2.214b) of the 4π kink must take place. As a matter of fact, type-(iv) collisions also result in partial annihilation when the inequality (4.217) does not hold but when condition (4.210a) is valid: in this case, the emerging slow 2π kinks A and C [Fig. 24(b)] will eventually annihilate.

Finally, let us proceed to the type-(v) collision [see Fig. 25(a)]. According to Eq. (4.215), the 2π antikink A acquires the velocity

$$-V'=-(V_0^2-\tfrac{1}{8}E_{\rm diss})^{1/2} \tag{4.218}$$

after collision with the 2π kink C [recall that V_0 is the equilibrium velocity of the 4π kink approximately equal to (3.47)]. The subsequent collision of the 2π antikink A with the 2π kink D results in their annihilation under the condition $(V_0-V')^2<\frac{1}{2}E_{\rm diss}$; see Eq. (4.215). Quite analogously, the 2π antikink B annihilates with the 2π kink C. Thus, using Eq. (4.218), one can find a condition for *complete annihilation* of the colliding 4π kinks:

$$f\gamma^{-3/2}<5 \; . \tag{4.219}$$

When $f\gamma^{-3/2}<4$, the annihilation of the 4π kinks goes another way: first the 2π kinks A and C are annihilated, then B and D [see Fig. 25(a)].

If condition (4.219) does not hold, a type-(v) collision may give rise to an *incomplete annihilation*,

$$K+\bar{K}\to k_{\rm sl}+\bar{k}_{\rm sl} \tag{4.220}$$

[Fig. 25(b)], or be elastic: $K+\bar{K}\to K+\bar{K}$. In the former case, the kink-antikink pair (B,D) is annihilated. Using Eq. (4.215), it is easy to find a velocity V'' of the 2π kink D after its collision with the 2π antikink A moving with velocity (4.218):

$$V''=\tfrac{1}{2}([V_0-(V_0^2-\tfrac{1}{8}E_{\rm diss})^{1/2}] +\{[V_0+(V_0^2-\tfrac{1}{8}E_{\rm diss})^{1/2}]^2-\tfrac{1}{2}E_{\rm diss}\}^{1/2}) \; . \tag{4.221}$$

Quite analogously, the 2π antikink B acquires the velocity $-V''$ after collision with the 2π kink C. Then it is straightforward to obtain a condition for incomplete annihilation: $8(V'')^2<E_{\rm diss}$, or, with regard to Eq. (4.221),

$$f\gamma^{-3/2}<8\zeta \; , \tag{4.222}$$

where $\zeta\approx0.73$ is determined as a positive real root of the equation $16\zeta^4-16\zeta^3-4\zeta^2+12\zeta-5=0$. Thus incomplete annihilation takes place in the interval $5<f\gamma^{-3/2}<8\zeta\approx5.84$. However, the annihilation is, in fact, complete in the whole range $f\gamma^{-3/2}<8\zeta$ if inequality (4.210a) holds. In this case, the emerging slow 2π kinks A and C will eventually annihilate [cf. the analogous situation analyzed above for a type-(iv) collision]. When $f\gamma^{-3/2}>8\zeta$, a type-(v) collision is elastic.

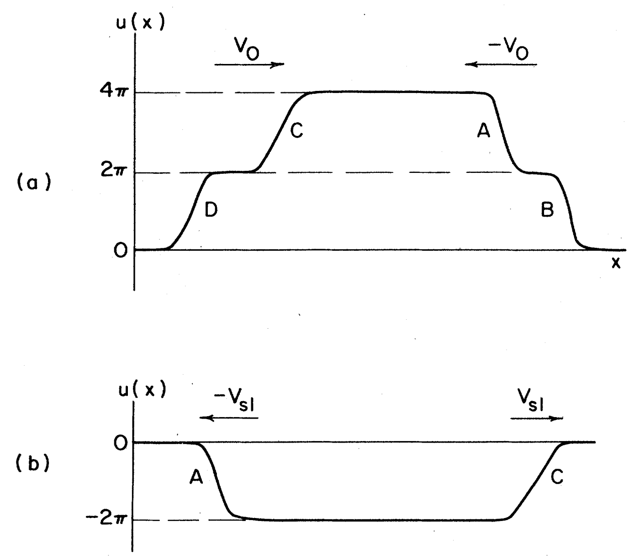

FIG. 25. A collision between a 4π kink and a 4π antikink: (a) the field configuration before the collision; (b) the configuration after the collision in the case (4.220) (incomplete annihilation).

3. Kinetics of a rarefied soliton gas

A full picture of the inelastic interactions between kinks is given in Fig. 26. In the parametric sector 1 in Fig. 26, the inelastic nonannihilation processes (4.209), (4.213), and (4.214b) take place simultaneously. In this case, it is interesting to analyze the kinetics of a rarefied soliton gas [investigation of the rarefied gas of the SG kinks was initiated by Bishop *et al.* (1980) and Currie *et al.* (1980); for the ϕ^4 model, it was begun still earlier by Krumhansl and Schrieffer (1975)]. Assuming the gas to be neutral on average, let us designate N, $n_{\rm f}$, and $n_{\rm sl}$ as the densities of the 4π kinks, fast 2π kinks, and slow 2π kinks, respectively. The densities of the corresponding antikinks are the same. A system of kinetic equations that take account of the inelastic collisions (4.209), (4.213), and (4.214b) is obvious:

$$\frac{dN}{dt}=-(V_0+V_{\rm sl})Nn_{\rm sl}+(V_{\rm f}-V_{\rm sl})n_{\rm f}n_{\rm sl} \; , \tag{4.223a}$$

$$\frac{dn_{\rm f}}{dt}=(V_0+V_{\rm sl})Nn_{\rm sl}+2V_{\rm sl}n_{\rm sl}^2 -(V_{\rm f}-V_{\rm sl})n_{\rm f}n_{\rm sl}-2V_{\rm f}n_{\rm f}^2 \; , \tag{4.223b}$$

$$\frac{dn_{\rm sl}}{dt}=(V_0+V_{\rm sl})Nn_{\rm sl}-2V_{\rm sl}n_{\rm sl}^2 -(V_{\rm f}-V_{\rm sl})n_{\rm f}n_{\rm sl}+2V_{\rm f}n_{\rm f}^2 \; . \tag{4.223c}$$

The system (4.223) has the evident integral of motion $n=2N+n_{\rm f}+n_{\rm sl}$, which may be called a net density of the 2π kinks. Excluding the variable N by means of this conservation law, it is straightforward to obtain a dynamical system for the variables $n_{\rm f}$ and $n_{\rm sl}$. That system has the unstable stationary point $n_f = n_{sl} = 0$, $N = n$, and the stable one

$$n_{sl} = n\sqrt{V_f}(V_0 + V_{sl})[V_0(\sqrt{V_f} + \sqrt{V_{sl}}) + V_{sl}(\sqrt{V_f} - \sqrt{V_{sl}}) + 2V_f\sqrt{V_{sl}}]^{-1} , \quad (4.224a)$$

$$n_f = \sqrt{V_{sl}/V_f}\, n_{sl} . \quad (4.224b)$$

In other sectors of the parametric plane (Fig. 26), the kinetics is trivial: All the kinks annihilate, except for sector 2, where the 4π kinks survive.

The situation changes if one includes in the analysis processes by which 2π kink-antikink pairs are produced by the external drive on account of quantum fluctuations (Coleman, 1977; Maki, 1977; Katz, 1978) and/or thermal functions (Wonneberger, 1980; Büttiker and Landauer, 1981a, 1981b; Marchesoni, 1986b). In this case, dynamic equilibrium between pair production and annihilation must set in.

According to Sec. IV.A.3, collision between a 2π kink and a breather may result in breakup of the latter. However, it is possible to show that this process may be neglected when analyzing the kinetics of a rarefied soliton gas.

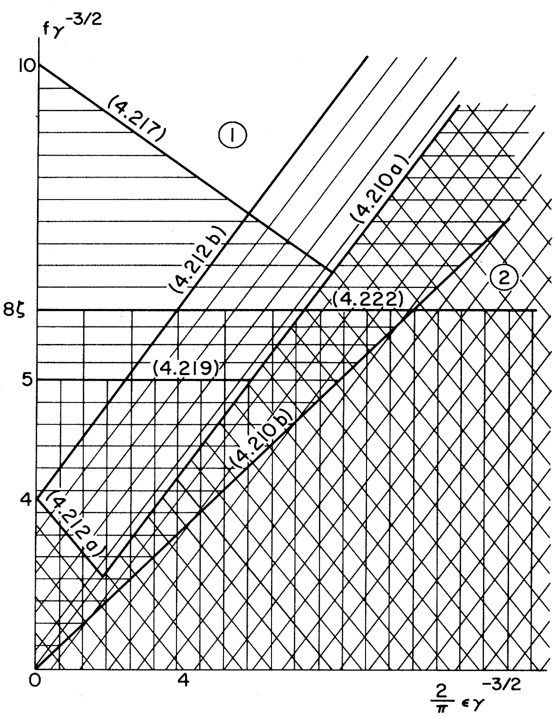

FIG. 26. A parametric diagram of the inelastic interactions between kinks. The parametric ranges where the annihilation $k_f + \bar{k}_f \to 0$, annihilation $k_{sl} + \bar{k}_{sl} \to 0$, partial annihilation $\bar{k}_{sl} + K \to k_f$ [Eq. (4.214a)], complete annihilation $K + \bar{K} \to 0$, and incomplete annihilation $K + \bar{K} \to k_{sl} + \bar{k}_{sl}$ [Eq. (4.220)] take place are shaded, respectively, by \\\, ///, ≡, |||, and | | |. The figures in parentheses over straight-line boundaries between different parametric ranges refer to equations that determine these boundaries.

V. STOCHASTIC DYNAMICS OF A SOLITON

A. Dynamics of sine-Gordon solitons in the presence of thermal fluctuations

In the present section we survey only selected problems of the stochastic dynamics of solitons (those which seem to us most interesting). A more comprehensive review of this topic has been recently published by Abdullaev (1989).

1. Motion of a kink in the presence of thermal fluctuations

In this subsection we shall deal with the SG equation perturbed by the term

$$\epsilon P[u] = -\gamma u_t - n(x,t) , \quad (5.1)$$

where $n(x,t)$ is a random driving force determined by the Gaussian correlators

$$\langle n(x,t) \rangle = 0 , \qquad \langle n(x,t) n(x',t') \rangle = \Gamma \delta(x-x')\delta(t-t') , \quad (5.2)$$

the angular brackets standing for averaging in the fluctuations. The random driving force describes the coupling of a corresponding physical system (e.g., a long Josephson junction) to a heat reservoir. The loss term γu_t in Eq. (5.1) is intrinsically related to the thermal noise term by the fluctuation-dissipation theorem (see, for example, Trullinger *et al.*, 1978; Büttiker and Landauer, 1980, 1981a, 1981b): $\Gamma = 2\gamma k_B T$, where T is the reservoir temperature and k_B is the Boltzmann constant.

The fluctuation dynamics of a kink has been studied, in the framework of different approaches, by Salerno *et al.* (1984) and Kaup and Osman (1986). They have calculated the mean kinetic energy of a kink moving in the presence of thermal fluctuations,

$$\langle E_k \rangle = \tfrac{1}{2} k_B T , \quad (5.3)$$

the mean energy per one-phonon mode,

$$\langle E_{ph} \rangle = k_B T , \quad (5.4)$$

and the kink's diffusion coefficient

$$D = \frac{k_B T}{8\gamma} . \quad (5.5)$$

Note that the diffusion coefficient (5.5) is connected with the kink's mobility in a viscous medium $\mu = 1/m\gamma$, where $m = 8$ is the kink's mass, by the Einstein's relation $D = \mu k_B T$ (Salerno *et al.*, 1984). The same expressions (5.3) and (5.4) were obtained earlier on the basis of a classical statistical-mechanics analysis of a dilute gas of unperturbed SG kinks (Bishop *et al.*, 1980).

All these results can also be obtained within the framework of perturbation theory. In particular, Eqs. (5.3) and (5.5) can be obtained with the aid of the general adiabatic equations of motion (3.43a) and (3.43b). To this end, one should insert the perturbation (5.1) into Eqs. (3.43a) and (3.43b) and, using Eq. (5.2), derive a corresponding Fokker-Planck equation. A stationary solution of that equation can be readily found, and, in the nonrelativistic limit $k_B T \ll m \equiv 8$, it generates Eq. (5.3). To recover Eq. (5.5) in a similar way, one may add an infinitely small constant force f [see Eq. (1.17)] to perturbation (5.1) and express the soliton's mobility in terms of a stationary solution of the conformably modified Fokker-Planck equation. Finally, resorting to the general evolution equation (2.72) for the continuous spectrum, one can also recover Eq. (5.4). The application of perturbation theory to these problems is reviewed by Bass *et al.* (1988).

It is pertinent to note that Kaup and Osman (1986) have also found correlation functions for fluctuation corrections to the kink's form.

Sometimes, it is stated that a kink of the unperturbed SG equation, placed in contact with a thermal reservoir, cannot undergo Brownian motion because the SG system is completely integrable (Ogata and Wada, 1986). However, it has been demonstrated (e.g., Marchesoni and Willis, 1987) that the unperturbed phase shifts of the kink scattered by thermal phonons (quasilinear wave packets) gives rise to Brownian motion of the kink's center governed by a corresponding Fokker-Planck equation, at least in the limit $k_B T \ll m$, where m is the kink's mass (in our notation, $m=8$). Of course, this inference relies upon the assumption that the reservoir of phonons has already been brought into a state of thermal equilibrium. In this sense, one may regard the *perturbed* SG equation (5.1) as a phenomenological model of the unperturbed thermal SG system, the perturbing terms modeling the interaction with the *thermal* phonons.

2. Fluctuation-induced nucleation of solitons

Analogy with the kinetic theory of first-order phase transitions suggests that thermal fluctuations may generate new solitons, viz., breathers or kink-antikink pairs. If one regards the first-order (adiabatic) evolution equations for the soliton parameters corresponding to perturbation (5.1) as Langevin equations, one can derive a conformable Fokker-Planck equation for a distribution function of those parameters. The Fokker-Planck equation makes it possible to find a probability density (per unit of time and length) for the production of a breather or a kink-antikink pair. The activation energy E_a for such production is equal to the solitons' rest energy [i.e., $E_a = 16\sin\mu$ for a breather (2.63), and $E_a = 16$ for a pair]. If the thermal energy $k_B T$ is small, the probability density contains a small exponent $\exp(-E_a/k_b T)$ (Wonneberger, 1980; see also Marchesoni, 1986b). Wonneberger (1980) also calculated a preexponential factor. To estimate the probability density for the birth of a kink-antikink pair with a finite relative velocity, one should add the pair's kinetic energy to E_a.

For an overdamped SG equation (i.e., one without the term u_{tt}) perturbed by the terms (5.1), the same problem was considered by Büttiker and Landauer (1980, 1981a; see also Büttiker and Landauer, 1981b, 1982). Their approach was also based on the Fokker-Planck equation, and the same exponential factor $\exp(-E_a/k_B T)$ was obtained.

B. Decay of a low-frequency breather under the action of a random force

A low-frequency breather [Eq. (2.66)] is an object highly sensitive to small perturbations, since they readily break it into a kink-antikink pair. In this section we shall consider the decay of a low-frequency breather under the action of a random force, following Malomed (1987d, 1987e). First we consider the case in which the force (1.17) is a random function of time only:

$$\langle f(t)\rangle = 0, \quad \langle f(t)f(t')\rangle = \delta(t-t') \tag{5.6}$$

(randomly varying bias current in a long Josephson junction). The evolution equation for the breather's parameter ζ^2 [see the definition in Eq. (2.66)] corresponding to the considered perturbation with an arbitrary function $f(t)$ has been obtained by Nozaki (1982):

$$\frac{d}{dt}\zeta^2 = -\frac{\pi\epsilon f(t)}{2}\sqrt{(1-\zeta^2T^2)(1+T^2)}\ , \tag{5.7}$$

where the quantity T is defined in Eqs. (4.4b) and (4.6′) [recall that negative ζ^2 corresponds to a kink-antikink pair (4.4) with $V^2 = -\zeta^2$]. Considering Eq. (5.7) as the Langevin stochastic equation, and taking into account Eqs. (5.6) and (4.6′), we arrive at the Fokker-Planck equation for the distribution function $F(z,T;t)$, where $z \equiv \zeta^2$:

$$\begin{aligned}\frac{\partial F}{\partial t} &= \mp\frac{\partial}{\partial T}[(1-zT^2)^{1/2}F] \\ &\quad + (\pi^2\epsilon^2/16)T^2(1+T^2)\frac{\partial F}{\partial z} \\ &\quad + (\pi^2\epsilon^2/8)\frac{\partial^2}{\partial z^2}[(1-zT^2)(1+T^2)^{-1}F]\ .\end{aligned} \tag{5.8}$$

The natural initial condition to Eq. (5.8) is

$$F(z,T;t=0) = (\zeta_0/\pi)(1-\zeta_0^2T^2)^{-1/2}\delta(z-\zeta_0^2)\ , \quad \zeta_0^2 \ll 1\ , \tag{5.9}$$

which describes an initial state with the definite value ζ_0 of ζ and with uniform distribution over the phase variable $\sin^{-1}(\zeta T)$. The characteristic time t_D of decay of the breather can be estimated as the time necessary for the diffusing distribution function $F(z,T;t)$ to reach the point $z=0$:

$$t_D \sim \epsilon^{-2}\zeta_0^4(1+T^2) . \quad (5.10)$$

Though we cannot analytically solve the Fokker-Planck equation (5.8) in general, an approximate solution can be found for the case

$$t_D \ll \zeta_0^{-1}, \text{ i.e., } \zeta_0^3 \ll \epsilon^2 . \quad (5.11)$$

Indeed, condition (5.11) means that t_D is much smaller than the period of the breather's internal oscillations $\tau \gtrsim 2\pi/\zeta_0$. So, if one inquires into the evolution of the distribution function over the times $t \sim t_D$ that are of basic interest for the study of the breather's decay, the unperturbed evolution described by the first term on the right-hand side of Eq. (5.8) may be neglected. Then the Fokker-Planck equation takes the form

$$\frac{\partial F}{\partial \theta} = \frac{\partial}{\partial y}\left[y\frac{\partial F}{\partial y} + \tfrac{1}{2}F \right] , \quad (5.12)$$

where $\theta \equiv \{\pi^2\epsilon^2 T^4/[8(1+T^2)]\}t$; $y \equiv 1-zT^2$. As can be seen from Eq. (5.7), at the point $y=0$, $d(\zeta^2)/dt=0$. This means that $\int_0^\infty F(y,t)dy$ should be an integral of motion of the Fokker-Planck equation, i.e., Eq. (5.12) should be supplemented by the boundary condition

$$\left[y\frac{\partial F}{\partial y} + \tfrac{1}{2}y \right]\Bigg|_{y=0} = 0 . \quad (5.13)$$

The eigenfunctions of Eqs. (5.12) and (5.13) are

$$F_\omega(\theta,y) = e^{-\omega\theta}y^{-1/2}\cos(2\sqrt{\omega y}) , \quad (5.14)$$

where ω is a continuous positive parameter. Decomposing the initial condition (5.9) over the eigenfunctions (5.14) at $\theta=0$ and then performing the inverse transformation at $\theta>0$ are straightforward. The eventual result is

$$F(z,T;t) = (\zeta_0/\epsilon)[(2/\pi^5)(1+T^2)(1-\zeta_0^2T^2)^{-1}(1-zT^2)^{-1}]^{1/2}t^{-1/2} \times \exp\{-(8/\pi^2\epsilon^2T^4t)[2-T^2(z+\zeta_0^2)]\}\cosh\{(16/\pi^2\epsilon^2T^4t)(1+T^2)[(1-zT^2)(1-\zeta_0^2T^2)]^{1/2}\} . \quad (5.15)$$

Equation (5.15) is valid at times $t \ll \zeta_0^{-1}$. In the limit $T \to 0$, Eq. (5.15) goes over into the well-known solution of the diffusion equation:

$$F(z;t) = (\zeta_0/\epsilon)\sqrt{2/\pi^5 t}\,\exp[-2(z-\zeta_0^2)^2/(\pi^2\epsilon^2 t)] . \quad (5.16)$$

The same problem can be considered for a perturbation (1.21) with random $f(t)$ (a ferromagnet in a randomly varying external magnetic field). In this case, the solution can be obtained from Eq. (5.15) by replacing ϵ^2 by

$$\epsilon_{\text{eff}}^2 \equiv \epsilon^2[4T^2/\pi^2(1+T^2)] \times [1+T^{-1}(1+T^2)^{-1/2}\,\text{arcsinh}T]^2 .$$

It is also worthwhile to note that the problem [in the case of the perturbation (1.17)] can be generalized for the situation when f is a random function of both x and t with the correlator

$$\langle f(t,x)f(t',x')\rangle = \delta(t-t')\delta(x-x') . \quad (5.17)$$

In this case the evolution equation for ζ^2 takes the form

$$\frac{d}{dt}\zeta^2 = \sqrt{1-\zeta^2T^2}\int_{-\infty}^{+\infty} dx\, f(t,x)\cosh x[\cosh(2x) + 1 + 2T^2]^{-1} . \quad (5.18)$$

The solution of the corresponding Fokker-Planck equation reduces to Eq. (5.15) with ϵ^2 replaced by

$$\epsilon_{\text{eff}}^2 = \epsilon^2(2/\pi^2)[1+T^{-1}(1+T^2)^{-1/2}\,\text{arcsinh}T] .$$

The solution of the Fokker-Planck equation for a perturbation (1.17) with $f(t,x)$ determined by Eq. (5.17) can also be expressed in terms of Eq. (5.15).

If one wants to take the dissipation (1.16a) into account, the right-hand side of Eq. (5.7) should be supplemented by the dissipative term, which transforms Eq. (5.12) into

$$\frac{\partial F}{\partial \theta} = \frac{\partial}{\partial y}\left[y\frac{\partial F}{\partial y} + \tfrac{1}{2}F + ayF \right] , \quad (5.19)$$

where

$$a \equiv (16\gamma/\pi^2\epsilon^2)T^{-2}[1+T^{-1}(1+T^2)^{-1/2}\,\text{arcsinh}T] ,$$

with γ the dissipative coefficient. The boundary condition (5.13) is transformed into

$$\left[y\frac{\partial F}{\partial y} + (\tfrac{1}{2}+ay)F \right]\Bigg|_{y=0} = 0 . \quad (5.20)$$

The eigenfunctions of Eqs. (5.19) and (5.20) are [cf. Eq. (5.14)]

$$F_\omega(\theta,y) = y^{-1/2}e^{-ay}\,{}_1F_1(-\omega/a,\tfrac{1}{2},ay)e^{-\omega\theta} ,$$

where ${}_1F_1$ is the Pochgammer function. The final expression for $F(z,T;t)$ is very ponderous in this case.

C. Stochastic decay of a breather under the action of a time-periodic external force

A time-periodic external field acting upon a low frequency breather may give rise to dynamical chaos (Nozaki, 1982; Nozaki and Bekki, 1983). In this section we give some results on this problem obtained by Nozaki

(1982) and Malomed (1987d, 1987e).

First let us consider the perturbation (1.21) with $f(t)=\sin(\omega t)$. The perturbation-induced evolution equation for the parameter ζ^2 takes the form

$$\frac{d}{dt}\zeta^2=\epsilon\sin(\omega t)\sqrt{1-\zeta^2T^2}T \times[(1+T^2)^{-1}+T^{-1}(1+T^2)^{-3/2}\,\mathrm{arcsinh}T]\ , \tag{5.21}$$

where T is the same as in Eqs. (4.4b) and (4.6′). The contribution from the perturbation to Eq. (4.6′) for T may be neglected. The evolution of ζ^2 is dominated by the overlap stage, i.e, the relatively small part of the period $\tau=2\pi/\zeta$ of the breather's internal oscillations singled out by the condition $T^2\lesssim 1$, which means that the kink and antikink inside the breather are strongly overlapped. During the overlap stage $\zeta^2T^2\ll 1$, $T\approx t$ [see Eq. (4.6′)], and Eq. (5.21) takes the form

$$\frac{d}{dT}\zeta^2=\epsilon[\sin(\omega T+\delta)]T \times[(1+T^2)^{-1}+T^{-1}(1+T^2)^{-3/2}\,\mathrm{arcsinh}T]\ , \tag{5.22}$$

where δ is the phase difference between the breather's internal oscillations and the external variable field (1.17) at the moment $T=0$ of maximum overlap. So, taking into account $dT/dt\approx 1$, one finds from Eq. (5.22) the change of ζ^2 generated by one overlap:

$$\Delta(\zeta^2)\approx\int_{-\infty}^{+\infty}\frac{d(\zeta^2)}{dT}dT=\epsilon\cos\delta\, g(\omega)\ , \tag{5.23}$$

where

$$g(\omega)\equiv 2\int_{-\infty}^{+\infty}dT[\sin(\omega T)](1+T^2)^{-1} \times[T+(1+T^2)^{-1/2}\,\mathrm{arcsinh}T]\ . \tag{5.24}$$

In particular, for $\omega\ll 1$, $g(\omega)=\pi+(\pi^2/2)\omega^2\,\mathrm{sgn}\omega+O(\omega^3)$, and for $\omega\gg 1$, $g(\omega)\sim\exp(-\omega)$.

In contrast with ζ^2, the evolution of the phase difference δ is uniformly contributed to over the whole period τ, so that the change of δ per half-period between two overlappings is

$$\Delta(\delta)=\pi(\omega/\zeta-1)\ . \tag{5.25}$$

Thus Eqs. (5.23) and (5.25) reduce the perturbation-induced breather's dynamics to the discrete map

$$\zeta_{n+1}^2=\zeta_n^2+\epsilon g(\omega)\cos\delta_n\ , \tag{5.26}$$

$$\delta_{n+1}=\delta_n+\pi(\omega/\zeta_{n+1}-1)\ , \tag{5.27}$$

where n is the number of overlaps. This map occurs in plasma physics problems, and in that connection it has been minutely studied by Zaslavskii (1985) and Zaslavskii and Chernikov (1985). Using their results, one can immediately obtain a qualitative estimate for the range of the parameter ζ where the breather's oscillations are stochastic (chaotic):

$$|\zeta^3|\lesssim\omega g(\omega)|\epsilon|\ . \tag{5.28}$$

Stochastic oscillations of a low-frequency breather imply the possibility of its stochastic decay (Nozaki, 1982).

The estimate (5.28) can also be obtained in another way. Indeed, in the absence of perturbations the breather solution with $\zeta=0$ plays the role of separatrix on the $(T,\dot{T})$ plane: it separates locked trajectories from free ones corresponding to kink-antikink pairs (Fig. 13). As is known, a small perturbation generates a narrow stochastic layer in a vicinity of the separatrix. A standard estimate for the width of the layer coincides with Eq. (5.28).

Now let us proceed to the perturbation (1.17) with $f(t)=\sin(\omega t)$. As was mentioned above, this problem was considered by Nozaki (1982), who obtained an estimate for the stochastic range by means of the resonance overlapping criterion (Chirikov, 1979). Here we shall give some additional results. As above, the evolution of ζ^2 is dominated by the overlapping stage. It is easy to obtain from Eq. (5.7) with $f(t)=\sin(\omega t)$ an expression analogous to (5.23):

$$\Delta(\zeta^2)=-\pi\epsilon K_0(\omega)\sin\delta\ , \tag{5.29}$$

where K_0 is the modified Hankel function [Eq. (5.29) is valid provided $\omega\gtrsim|\zeta|$]. Thus Eq. (5.29) brings us to the map

$$\zeta_{n+1}^2=\zeta_n^2-\pi\epsilon K_0(\omega)\sin\delta_n\ , \tag{5.30}$$

$$\delta_{n+1}=\delta_n+\pi(\omega/\zeta_{n+1}-1)\ . \tag{5.31}$$

Equations (5.30) and (5.31) are a variant of the same stochasticity-generating map that we have already encountered in the form (5.26) and (5.27). The estimate for the stochasticity range is

$$|\zeta|^3\lesssim(\pi^2/2)|\epsilon|\omega K_0(\omega)\ . \tag{5.32}$$

In particular, at small ω, Eq. (5.32) takes the form $|\zeta|^3\lesssim(\pi^2/2)\omega(\ln\omega^{-1})|\epsilon|$. Comparing this expression with the estimate obtained by Nozaki (1982), we infer that the estimation based on reducing the breather's dynamics to the map (5.30) and (5.31) enables us to catch the logarithmic factor, the presence of which is a rather subtle effect [the logarithmic factor, found by Malomed (1987d, 1987e) was missing in the paper of Nozaki (1982)].

If we take into account the dissipation (1.16a) on a level with the term (1.17) or (1.21), the right-hand side of Eq. (5.22) is supplemented by the term

$$\left[\frac{d}{dt}\zeta^2\right]_{\mathrm{diss}}=2\gamma(1+T^2)^{-1}(1-\zeta^2T^2) \times[1+T^{-1}(1+T^2)^{-1/2}\,\mathrm{arcsinh}T]\ , \tag{5.33}$$

which generates the term

$$\Delta_{\mathrm{diss}}(\zeta^2)=-\pi\gamma(\pi-\zeta_n)+O(\gamma\zeta_n^2) \tag{5.33′}$$

on the right-hand side of Eq. (5.26) or Eq. (5.30). If γ is sufficiently small, we may expect the dissipative term to turn the Hamiltonian stochastic structure generated by Eqs. (5.26) and (5.27) or (5.30) and (5.31) into a strange attractor (see Izrailev *et al.*, 1981).

It is also pertinent here to mention the paper by Abdullaev *et al.* (1985), in which the stochasticity range for a breather subjected to the action of the perturbation

$$P=\sin(\omega t)\sin u \tag{5.34}$$

was estimated by means of the resonance overlapping criterion. The same result can be obtained in another way (Malomed, 1987d, 1987e). Indeed, in the new variables $x'=x[1-(\epsilon/2)\sin(\omega t)]$, $t'=t+\epsilon/2\omega\cos(\omega t)$, Eq. (1.15) with the perturbation (5.34) becomes, in the first approximation,

$$u_{t't'}-u_{x'x'}+\sin u=(\omega\epsilon/2)\cos(\omega t')u_{t'} . \tag{5.35}$$

The right-hand side of Eq. (5.35) may be regarded as a "dissipative" perturbation with a variable coefficient. Using Eq. (5.33′), we immediately obtain the map (provided $\omega \ll 1$)

$$\zeta_{n+1}^2=\zeta_n^2+(\pi^2/2)\epsilon\omega\cos\delta_n , \tag{5.36}$$

$$\delta_{n+1}=\delta_n+\pi(\omega/\zeta_{n+1}-1) . \tag{5.37}$$

Applying to Eqs. (5.36) and (5.37) the method employed above for the investigation of Eqs. (5.26) and (5.27) and (5.30) and (5.31), we obtain an estimate for the width of the stochastic range generated by this map: $|\zeta|^3 \lesssim (\pi^3/4)\epsilon\omega^2$. This estimate coincides with that obtained by Abdullaev *et al.* (1985).

One more problem of this kind arises when one considers a semi-infinite Josephson junction in an ac magnetic field described by the unperturbed SG equation with the additional constraint $u(-x)=u(x)$ and the boundary condition $u_x(t,x=0)=\epsilon\sin(\omega t)$, where ϵ is the field amplitude. In this case a breather solution describes a fluxon pinned by the edge of the junction ($x=0$) (Olsen and Samuelsen, 1986b). If the fluxon's binding energy is small, i.e., if it is described by a low-frequency-breather solution, the oscillations of the fluxon may become stochastic under the action of the time-periodic perturbation. An estimate for a corresponding stochasticity range has been obtained by Malomed (1987i): $|\zeta|^3 \lesssim \omega|\epsilon|$ (for $\omega \lesssim 1$).

A more complicated (but, also more practically important) problem allied to the one mentioned is to analyze oscillations of a fluxon in a finite-length junction ac driven at the edges $[u_x(x=0)=\pm u_x(x=L)=\epsilon\sin(\omega t)]$. The presence of the terms (1.16a) and (1.17) is assumed too. By direct numerical simulations, Salerno *et al.* (1989) have demonstrated that oscillations of the fluxon become chaotic via a sequence of Feigenbaum's period doublings (Feigenbaum, 1978, 1979). Independently, Malomed (1989d) has found analytically the underlying regime of periodic oscillations and its first period-doubling bifurcation.

D. Stochastization of a nonlinear Schrödinger soliton by a periodic external force

As shown in Sec. III.B, a NS soliton subjected to the action of an external time-periodic perturbation is phase locked to the external frequency, provided the perturbation amplitude exceeds a threshold value $\sim\gamma$, γ being the dissipation coefficient. Nozaki and Bekki (1984) have considered the dynamics of a quiescent soliton described by the equation

$$iu_t+u_{xx}+2|u|^2u=i\epsilon_1 e^{i\omega t}+i\epsilon_2 e^{2i\omega t}+i\gamma u_{xx} . \tag{5.38}$$

Taking the perturbed NS soliton in the form

$$u(x,t)=2\eta\,\mathrm{sech}(2\eta x)\exp[i(\psi-\pi/2)] , \tag{5.39}$$

they derived the first-order evolution equations

$$\frac{d\psi}{dt}=4\eta^2 , \tag{5.40}$$

$$\frac{d\eta}{dt}=-\tfrac{8}{3}\gamma\eta^3-\frac{\pi}{2}[\epsilon_1\sin(\omega t-\psi)+\epsilon_2\sin(2\omega t-\psi)] . \tag{5.41}$$

The system (5.40) and (5.41) has two stable limit cycles:

$$\eta\approx\eta_1+\tfrac{1}{2}\pi\epsilon_2\cos(\omega t+\chi_1) , \tag{5.42a}$$

$$\psi=\omega t-\chi_1+4\pi\epsilon_2\eta_1\sin(\omega t+\chi_1) \tag{5.42b}$$

and

$$\eta\approx\eta_2-\tfrac{1}{2}\pi\epsilon_1\cos(\omega t-\chi_2) , \tag{5.43a}$$

$$\psi=2\omega t-\chi_2+4\pi\epsilon_1\eta_2\sin(\omega t-\chi_2) , \tag{5.44b}$$

where $\eta_j=\frac{1}{2}\sqrt{j\omega}$, and $\sin\chi_j=-(16\gamma/3\pi\epsilon_j)\eta_j^3$ $(j=1,2)$. The limit cycles (5.42) and (5.43) exist under the conditions $|\epsilon_j|>(2\gamma/3\pi)(j\omega)^{3/2}$ and $|\epsilon_1|+|\epsilon_2|<(\sqrt{\omega}/\pi)(\sqrt{2}-1)$. If either ϵ_1 or ϵ_2 vanishes, one of the cycles disappears and another shrinks into the fixed point (3.31). Nozaki and Bekki (1984) have studied numerically the evolution of the two limit cycles with an increase of the parameters ϵ_j. In particular, they observed a period-doubling bifurcation for the limit cycle (5.43) at $\epsilon_1=\epsilon_2=0.037$ in the case $\gamma=0.05$, $\omega=1$. After a series of subsequent period-doubling bifurcations, they observed the appearance of a strange attractor at $\epsilon_1=\epsilon_2=0.0401$.

In another paper, Nozaki and Bekki (1983) studied numerically the stochastization of a soliton described by the equation

$$iu_t+u_{xx}+2|u|^2u=i(\epsilon_1-\epsilon_2|u|^2)u+i\epsilon_3u_{xx} - i\epsilon_0\sum_{-\infty}^{+\infty}\exp(in\omega_0 t) . \tag{5.45}$$

Note that the last term on the right-hand side of Eq. (5.45) is equivalent to a periodic array of delta-function pulses. In the case of Eq. (5.45), as well as in the case (5.38), stochastization takes place when the perturbation parameter exceeds a certain small but finite critical value.

A similar problem was considered by Nozaki and Bekki (1985) for the perturbation (1.9): They assumed that the wave field was a superposition of the soliton (5.39) and a long-wave component of radiation. Using second-order evolution equations of perturbation theory (Maslov, 1980), they derived a system of equations for the parameters of the soliton and for the radiation amplitude, taking account of the influence of radiation on the soliton. Further numerical simulation revealed a sequence of period doublings and chaos.

E. Stochastic oscillations of a pinned soliton

1. Stochastic escape of a pinned kink

Motion of a kink pinned by the attractive inhomogeneity (1.19) (with positive sign) can also become stochastic under the action of an external time-periodic field. To demonstrate this, let us consider the dynamics of a kink in the presence of the combined perturbation

$$\epsilon P=\epsilon\delta(x)\sin u+(4/\pi)\rho\sin(\omega t),\quad \epsilon\gg|\rho|\ , \tag{5.46}$$

(a Josephson junction fluxon pinned by a microresistor with superimposed ac bias current).

When $\rho=0$, the kink oscillates near the point $x=0$ according to the law of motion

$$\sinh\xi(t)=\sqrt{(2\epsilon-E)/E}\,\sin(\Omega t),\quad \Omega\equiv\sqrt{E}\,/2\ , \tag{5.47}$$

where E is the binding energy ($0<E\le 2\epsilon$). The kink's motion may become stochastic under the action of the forcing term in Eq. (5.46). The problem can be reduced to the study of a map of the type considered above (Malomed, 1987d, 1987e). Indeed, under the action of the forcing term the kink's energy varies as

$$dE/dt=-8\rho\frac{d\xi}{dt}\sin(\omega t)\ . \tag{5.48}$$

Proceeding from the analogy with Eq. (5.21), it is easy to establish that the evolution of E is dominated by a relatively small part of the kink's oscillation period τ when the kink is sufficiently close to the point $x=0$: $\xi^2(t)\lesssim 1$. The duration t of that part (which is analogous to the strong overlap stage in the breather problem) is $t\sim 1/\sqrt{\epsilon}\ll\tau/2=2\pi/\sqrt{E}$. Thus, by inserting the kink's law of motion (5.47) into (5.48) and using the condition $E\ll\epsilon$, we can calculate the change of E per one overlap [cf. Eq. (5.23)]:

$$\Delta E\approx-\sqrt{8\epsilon E}\,\rho\int_{-\infty}^{+\infty}t\sin(\omega t)/\sqrt{1+\epsilon t^2/2}\,dt\ . \tag{5.49}$$

Calculating the integral in Eq. (5.49), one arrives at the map [cf. Eqs. (5.30) and (5.31)]

$$E_{n+1}=E_n-8\sqrt{2E/\epsilon}\,\rho K_1(\omega\sqrt{2/\epsilon})\cos\delta_n\ , \tag{5.50}$$

$$\delta_{n+1}=\delta_n+\pi(2\omega/\sqrt{E_{n+1}}-1)\ , \tag{5.51}$$

where δ is the phase difference between the kink's oscillations and the external variable force, while K_1 is the modified Hankel function. Standard methods (Chirikov, 1979) yield the following estimate for the stochastic range generated by the map (5.50) and (5.51):

$$E\le 8\pi\sqrt{2/\epsilon}\,\rho K_1(\omega/\sqrt{2/\epsilon})\ . \tag{5.52}$$

In particular, when $\omega\ll\sqrt{\epsilon}$, Eq. (5.52) takes the form $E\lesssim 8\pi\rho/\omega$, and in the opposite case $\omega\gg\sqrt{\epsilon}$ it takes the form $E\lesssim 40\rho(\omega\sqrt{\epsilon})^{-1/2}\exp(-\sqrt{2}\omega/\sqrt{\epsilon})$. Note that, when $\rho=0$, the phase portrait of the kink's motion in the coordinates $\xi,\dot{\xi}$ coincides qualitatively with that shown in Fig. 13, so that the stochastic layer can again be interpreted as arising from a former separatrix. Stochastic motion of a weakly bound kink implies its stochastic escape, just as stochastic oscillations of a weakly bound breather imply its stochastic decay.

It is pertinent to note that stochastic oscillations and escape of a bound kink under the action of a time-periodic force have been observed in numerical experiments by Fukushima and Yamada (1986, 1987).

2. Stochastic dynamics of a breather interacting with a localized inhomogeneity

In contrast with the above problems, where stochastization was stipulated by the presence of external periodic or random forces, a low-frequency breather subjected to the action of the perturbation (1.19) furnishes an example of an autonomous system that may demonstrate stochasticity generated solely by its internal dynamics. At the same time, estimating a range of stochasticity is an easier problem in this case, since the perturbation is characterized by the single parameter ϵ, while above we dealt with two parameters, ϵ and ω.

Assuming, for the time being, that the period $\tau=2\pi/\cos\mu$ of the breather's unperturbed internal oscillations is much smaller than the characteristic time scale $t\sim|\epsilon|^{-1/2}$ of perturbation-induced motions, one may represent the full Hamiltonian of the system, averaged in the breather's internal oscillations, in the following form [cf. Eq. (3.153)]:

$$\begin{aligned}\langle H\rangle+H_1=&\,16\sin\mu(1+V^2/2)-4\epsilon\cot\mu\cosh\phi(1+\cot^2\mu\cosh^2\phi)^{-3/2}\\&-8\epsilon\cot\mu\cosh\phi(1+\cot^2\mu\cosh^2\phi)^{-3/2}[1-4\cot\mu\cosh\phi(1+\cot^2\mu\cosh^2\phi)^{3/2}\\&+2\cot^2\mu\cosh^2\phi(3+2\cot^2\mu\cosh^2\phi)]\\&\times\cos[2(t\cos\mu+\Psi_0)],\quad \phi\equiv\xi\sin\mu\ .\end{aligned} \tag{5.53}$$

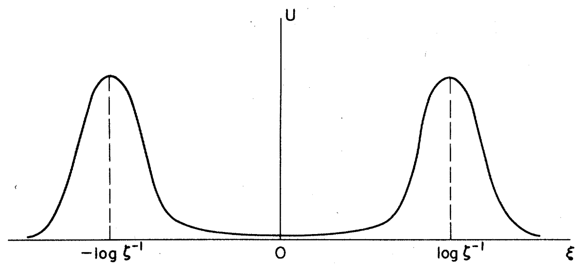

FIG. 27. The shape of the potential (5.54) when $\epsilon<0$, $\zeta^2 \ll 1$.

The first two terms on the right-hand side of Eq. (5.53) give the averaged Hamiltonian $\langle H \rangle$ proper, and the third term H_1 is the first oscillating correction to it. The equations of motion corresponding to $\langle H \rangle$ describe a particle of mass $m=16\sin\mu$ moving in the potential (see Fig. 27)

$$U=-8\epsilon\cot\mu\cosh(\xi\sin\mu)\times[1+\cot^2\mu\cosh^2(\xi\sin\mu)]^{-3/2} . \qquad (5.54)$$

When $\zeta\sim\sqrt{|\epsilon|}$, one may expect the internal and external oscillations of the breather around the point $x=0$ to resonate. To estimate the range of the resulting stochasticity, one may employ the time-dependent Hamiltonian (5.53) (Malomed, 1987d).

Estimating is straightforward if $\epsilon>0$. As can be seen from Fig. 11(b) (Sec. III.C.6), in this case the effective particle oscillates inside either of two potential wells $U\approx-4\epsilon e^z(4+e^{2z})^{-3/2}$, $z\equiv|\xi|-\ln\zeta^{-1}$, which are separated by the large distance $L=2\ln\zeta^{-1}$. In this situation, the resonance overlapping criterion (Chirikov, 1979) demonstrates the naive estimate $\zeta\sim\sqrt{\epsilon}$ to be quite accurate.

In the opposite case $\epsilon<0$ the particle travels in the valley between two potential hills (see Fig. 27) separated by the same distance L. Hence the external oscillation frequency is smaller than in the above case: $\omega\sim\sqrt{|\epsilon|}/L$. Therefore a naive estimate of the range of stochasticity is $\zeta\lesssim\sqrt{|\epsilon|}/L$, i.e., $\zeta\lesssim\sqrt{|\epsilon|}/\ln(|\epsilon|^{-1})$. However, detailed analysis of the resonance overlap yields the more accurate estimate $\zeta\lesssim\sqrt{|\epsilon|}$, the stochasticity for $\zeta\sim\sqrt{|\epsilon|}$ being stipulated by overlaps of higher resonances with the numbers $m\sim\ln|\epsilon|^{-1}$, while, in the case of $\epsilon>0$, $m\sim1$ were dominating.

VI. ONE-SOLITON RADIATIVE EFFECTS

A. Preliminary remarks

Perturbation-induced emission of radiation gives rise to qualitatively new effects in one-soliton dynamics. As has already been mentioned above, in the case of the SG equation radiative effects are exponentially small when a characteristic frequency Ω of the soliton's motion in the presence of perturbations is small compared with the plasma frequency (the minimum frequency of linear waves) $\omega_0=1$. However, when $\Omega\gtrsim1$, emission plays an important role. In the NS equation the frequencies of linear waves occupy the spectral half-axis $\omega=k^2\geq0$, where k is the wave number. This means that emission may be generated by motions of the soliton that have positive frequencies $\Omega>0$. As for the perturbed KdV equation, emission may be generated by periodic motion of a soliton with an arbitrary frequency.

Dealing with radiation processes in one-soliton dynamics, we encounter physical problems of three different types: emission of radiation proper, localized corrections to a soliton's form ("tails"), and generation of new (secondary) solitons. For problems of the first type, the main point is to find a wave number of the emitted radiation and the emission power, i.e., a rate of emission of energy. If the emission is not monochromatic, it is also interesting to investigate its spectral composition, i.e., distribution of the emission power over the spectrum of the emitted waves. For problems of the second type, especially interesting is the near-resonant case, when (for the SG equation) the external frequency Ω is close to the plasma frequency $\omega_0=1$. In this case, the length of the "tail" generated by the perturbation is much larger than the size of the soliton.

This section is devoted to problems of the first two types, which can be effectively solved by means of the inverse scattering transform (or alternative versions of perturbation theory for solitons). Consideration of problems of the third type (generation of new solitons), which are more peculiar and, in fact, are not amenable to solution by consistent analytical methods, is deferred to Sec. VIII.

B. General formalism of the perturbation theory for emission (sine-Gordon equation)

Prior to discussion of particular emission problems, let us review the IST-based approach to those problems, following the paper by Malomed (1987d).

We recall that, within the framework of the IST technique, the radiation wave function is described by the complex Jost coefficient (the field amplitude) $b(\lambda)$, where λ is a real spectral parameter; see Sec. II.E. Let us write the perturbed evolution equation (2.72) for $B(\lambda)\equiv b(\lambda)\exp[i(\lambda+1/4\lambda)t]$ in the form

$$\frac{dB}{dt}=(i\epsilon/4)\exp(i\omega t)\int_{-\infty}^{+\infty}dx\,P[u(x)]Q(x,\lambda)\equiv\epsilon F(t,\lambda)\exp(i\omega t)\quad(\omega\equiv\lambda+1/4\lambda) , \qquad (6.1)$$

$$Q(x,\lambda)\equiv a(\lambda)\{[\Psi^{(1)*}(x,\lambda)]^2-[\Psi^{(2)*}(x,\lambda)]^2\} . \qquad (6.2)$$

The important case is that in which the function $F(t,\lambda)$ is periodic in t with a period $2\pi/\Omega$:

$$F(t,\lambda)=\sum_{n=-\infty}^{+\infty}F_n(\lambda)\exp(-in\Omega t) . \qquad (6.3)$$

Since the emitted waves have the plasma dispersion law $\omega^2=1+k^2$, emission is generated by the terms of the Fourier series (2.63) with $n>1/\Omega$. Let us consider the contribution to the emission from one of those terms. According to Eq. (2.69b), the spectral density of the radiation energy may be expanded as follows:

$$\mathcal{E}(k)=(4/\pi)|B(\lambda)|^2+O(|B(\lambda)|^4)\ , \tag{6.4}$$

provided $|B(\lambda)|^2 \ll 1$ [λ is related to k according to Eq. (2.54)]. The emission intensity is characterized by its power, i.e., the energy emission rate. The emission power spectral density $\mathcal{W}(k)$ is [with regard to Eq. (6.4)]

$$\mathcal{W}(k)=d\mathcal{E}(k)/dt$$
$$=(8/\pi)\mathrm{Re}\{B^*(\lambda)[dB(\lambda)/dt]\}\ . \tag{6.5}$$

According to the above, we keep only one term of the series (6.3) in Eq. (6.1):

$$\frac{d}{dt}B(\lambda)=-\frac{\gamma}{2}B(\lambda)+\epsilon F_n(\lambda)\exp[i(\omega-n\Omega)t]\ , \tag{6.6}$$

where we have also added a term generated by the dissipation (1.16a). The solution to Eq. (2.7) is evident:

$$B(\lambda,t)=\epsilon F_n(\lambda)[(\gamma/2)+i(\omega-n\Omega)]^{-1}$$
$$\times\exp[i(\omega-n\Omega)t]\ . \tag{6.7}$$

Multiplying Eq. (6.6) by the complex-conjugate expression (6.7) yields

$$B^*(\lambda)[dB(\lambda)/dt]=-(\gamma/2)|B(\lambda)|^2$$
$$+(\gamma/2)\epsilon^2[(\gamma/2)^2+(\omega-n\Omega)^2]^{-1}$$
$$\times|F_n(\lambda)|^2\ . \tag{6.8}$$

The first term in Eq. (6.8) describes dissipative absorption of the emitted energy, while the second gives the emission power proper. Inserting this term into Eq. (6.5), we obtain

$$\mathcal{W}(k)=(8\epsilon^2/\pi)|F_n(\lambda)|^2(\gamma/2)[(\gamma/2)^2+(\omega-n\Omega)^2]^{-1}\ . \tag{6.9}$$

In particular, when $\gamma\to 0$, Eq. (6.9) goes over into

$$\mathcal{W}(k)=8\epsilon^2|F_n(\lambda)|^2\delta(\omega-n\Omega)\ . \tag{6.10}$$

As one sees comparing Eq. (6.9) to Eq. (6.10), dissipation results in the "Lorenz broadening" of the emission spectral line. The physical sense of this phenomenon is that, due to dissipative absorption, the emitted wave is not strictly monochromatic.

The radiation frequency $\omega=n\Omega$ is related to the two values of the radiation wave number $k_\pm=\pm\sqrt{(n\Omega)^2-1}$ corresponding to waves emitted to the right (k_+) and to the left (k_-). When $n\Omega-1\gtrsim 1$, both spectral densities (6.9) and (6.10) result in the same expression for the total power:

$$W=\int_{-\infty}^{+\infty}\mathcal{W}(k)dk=W_+ + W_-\ ,$$
$$W_\pm\equiv 8\epsilon^2 n\Omega[(n\Omega)^2-1]^{-1/2}|F_n(\lambda_\pm)|^2\ , \tag{6.11}$$

where, according to Eq. (2.54), $\lambda_\pm=\frac{1}{2}[n\Omega\pm\sqrt{(n\Omega)^2-1}]$ and $W_\pm$ stands for the powers emitted to the right and to the left. In the resonant case $n\Omega=1+\delta\Omega$, where $0<|\delta\Omega|\ll 1$, integrating the spectral density (6.11) yields

$$(W_\pm)_{\mathrm{res}}=4\sqrt{2}\epsilon^2|F_n(\lambda_\pm)|^2\{[2\delta\Omega+\sqrt{4(\delta\Omega)^2+\gamma^2}]$$
$$\times[4(\delta\Omega)^2+\gamma^2]^{-1}\}^{1/2}\ . \tag{6.12}$$

At $\gamma=0$ Eq. (6.12) takes the form

$$(W_\pm)_{\mathrm{res}}=4\sqrt{2}\epsilon^2|F_n(\lambda_\pm)|^2\theta(\delta\Omega)/\sqrt{\delta\Omega} \tag{6.13}$$

[the same expression ensues from Eq. (6.11)]. The difference between Eqs. (6.12) and (6.13) is significant when $\delta\Omega\lesssim\gamma$. However, in this case the absorption length of the emitted wave $L\sim\gamma^{-1}\sqrt{\delta\Omega}$ becomes of order or smaller than its wavelength $l\sim(\delta\Omega)^{-1/2}$, so that the underlying Eq. (6.1) describes, in fact, not emission but oscillating corrections to the soliton's shape. This question will be addressed in more detail in the next section.

In the absence of dissipation, the emitted energy is transferred to infinity. Far from the emitting soliton, the radiation field looks like a traveling monochromatic wave. Calculating the energy flux transferred by the wave and equating it to $W_\pm$, one readily finds the amplitudes $A_\pm$ of the emitted waves:

$$A_\pm^2=2W_\pm/\omega\sqrt{\omega^2-1}\ . \tag{6.14}$$

C. Energy emission from a sine-Gordon kink in external fields

The results set forth in this section were obtained by Malomed (1987d, 1987e).

1. The resonant case

Let us start with the perturbation (1.17). Inserting it (as well as any perturbation that is nonvanishing at $|x|\to\infty$) into Eq. (6.1), one will encounter a divergent integral. To circumvent this difficulty, we substitute into Eqs. (1.15) and (1.17) [cf. Eq. (3.49)]

$$u(x,t)=U(x,t)+u^{(0)}(t)\ , \tag{6.15}$$

where $u^{(0)}(t)$ is a solution of the perturbed SG equation far from the kink. In the nonresonant situation, $u^{(0)}(t)=\epsilon(1-\Omega^2)^{-1}\cos(\Omega t)$. Insertion of Eq. (6.15) with this $u^{(0)}(t)$ into the SG equation with the perturbation (1.17) transforms it into the SG equation for the function $U(x,t)$ with the renormalized perturbation [cf. Eq. (3.50)]

$$\epsilon P_R = \tilde{\epsilon}\cos(\Omega t)(1-\cos U), \quad \tilde{\epsilon} \equiv \frac{\epsilon}{1-\Omega^2} \tag{6.16}$$

[for the particular case $\Omega=0$, the transformation of perturbation (1.17) into (6.16) was first performed by Olsen and Samuelsen (1982)]. In the resonant case $\Omega=1+\delta\Omega$, $|\delta\Omega| \ll 1$, the function $u^{(0)}(t)$ takes the form

$$u^{(0)}(t) \approx a\cos(\Omega t+\delta\phi), \tag{6.17}$$

where $\delta\phi$ is some phase shift and a is determined by the equation ensuing from the nonlinear resonance theory (Landau and Lifshitz, 1973),

$$a^2[(\delta\Omega + a^2/16)^2+\gamma^2/4] = \epsilon^2/4 . \tag{6.18}$$

The renormalized perturbation corresponding to Eq. (6.17) is

$$P_R = a\cos[(1+\delta\Omega)t+\delta\phi](1-\cos U) + (a^2/2)\cos^2[(1+\delta\Omega)t+\delta\phi]\sin U - \gamma U_t + O(a^3) . \tag{6.19}$$

On the right-hand side of Eq. (6.19), we see two terms of second order in a: $(a^2/4)\cos[2(1+\delta\Omega)t+2\delta]\sin U$ and $(a^2/4)\sin U$. The former term might result in a parametric resonance (see below), but it proves that this term does not meet particular conditions necessary to give rise to a parametric resonance, i.e., in the first approximation it may be omitted. As to the latter term, it may not be ignored. Indeed, to exclude it accurately, we must rewrite the equation in the rescaled variables

$$x' = x(1-a^2/8), \quad t' = t(1-a^2/8) , \tag{6.20}$$

i.e., we must replace the frequency detuning $\delta\Omega$ by

$$\delta\Omega' = \delta\Omega + a^2/8 . \tag{6.21}$$

From the other side, higher-order corrections to resonant emission are negligible provided $a^2 \ll \sqrt{\delta\Omega'}$. Thus, in the range $a^4 \ll \delta\Omega' \ll a^2$, these corrections are still immaterial, while the difference between $\delta\Omega$ and $\delta\Omega'$ should be taken into account.

In contrast to the original perturbation (1.17), the renormalized ones (6.16) and (6.19) do not generate divergencies. In particular, inserting Eq. (6.16) into Eq. (6.1) yields (for one kink)

$$\frac{d}{dt}B(\lambda) = (i\pi\tilde{\epsilon}/4)k(\lambda^2+\nu^2)^{-1}(1+V) \times[(1+V)^2/4-\lambda^2(1-V)^2] \times \operatorname{csch}(\pi k\sqrt{1+V^2}/2)\cos(\Omega t)\exp(i\omega t - ik\xi) , \tag{6.22}$$

where $\nu \equiv \frac{1}{2}\sqrt{(1+V)/(1-V)}$, $V(t)$ and $\xi(t)$ are the kink's velocity and coordinate, and k and ω are related to λ according to Eqs. (2.54) and (2.56).

The expression for dB/dt corresponding to the resonant renormalized perturbation (6.19) can be obtained from Eq. (6.22) in an evident way: ϵ is replaced by a, and Ωt is replaced by $(1+\delta\Omega')t'+\delta\phi$, where t' and $\delta\Omega'$ are defined in Eqs. (6.20) and (6.21). In the resonant case, the first order of perturbation theory is sufficient, which implies setting $\xi=V=0$ in Eq. (6.22). Then application of Eqs. (6.3) and (6.12)–(6.14) yields the following results. The kink emits energy symmetrically to both sides, the total emission power being

$$W_{\text{res}} = \sqrt{2}a^2\delta\Omega'\{[2\delta\Omega'+\sqrt{4(\delta\Omega')^2+\gamma^2}] \times[4(\delta\Omega')^2+\gamma^2]^{-1}\}^{1/2} . \tag{6.23}$$

The emitted waves have the frequency (defined with respect to the unrescaled time t) $\omega=\Omega$, the wave numbers

$$k = \pm\sqrt{2\delta\Omega'} , \tag{6.24}$$

and the amplitude (at $\gamma=0$) $A_\pm = a$. Note that emission takes place under the condition $\delta\Omega' > 0$, while $\delta\Omega$ may itself be negative.

Let us describe the dependence of the emission power (6.23) on the frequency detuning $\delta\Omega$. First of all, in the range $\delta\Omega \gg \max\{\epsilon^{2/3};\gamma\}$ the solution of Eq. (6.18) is $a^2 \approx (\epsilon/2\delta\Omega)^2$, and Eq. (6.23) takes the form $W_{\text{res}} \approx \epsilon^2/\sqrt{8(\delta\Omega)^3}$. At $\delta\Omega \sim 1$ this equation agrees with the nonresonant solution $W \sim \epsilon^2$. Further, the influence of dissipation may be neglected provided $\gamma^3 \ll \epsilon^2$. Then Eq. (6.23) simplifies to

$$W_{\text{res}} = a^2\sqrt{2\delta\Omega'} . \tag{6.25}$$

This expression is valid as long as $a^4 \ll \delta\Omega'$; in the opposite case a second-order contribution should be taken into account.

The emission power determined by Eqs. (6.18) (with $\gamma=0$), (6.21), and (6.25) grows monotonically with the decrease of $\delta\Omega$, attaining the value $W_{\max} = 4\sqrt{10}\epsilon$ at $\delta\Omega = \delta\Omega_{\min} \equiv -3(\epsilon/16)^{2/3}$. In the range $\delta\Omega < \delta\Omega_{\min}$ the dependence $a^2(\delta\Omega)$ determined by Eq. (6.18) becomes two valued (Landau and Lifshitz, 1973). Accordingly, the dependence $W(\delta\Omega)$ is hysteretic in this range [Fig. 28(a)]. The "natural" branch of the dependence $a^2(\delta\Omega)$, corresponding to our slowly turning on the perturbation (1.17) at a fixed value of $\delta\Omega < \delta\Omega_{\min}$, results in values of a^2 such that $\delta\Omega'$ is negative [see Eq. (6.21)], so that the "natural" branch of the dependence $W_{\text{res}}(\delta\Omega)$ is $W_{\text{res}}=0$. The second branch, corresponding to our slowly changing $\delta\Omega$ from $\delta\Omega > \delta\Omega_{\min}$ to $\delta\Omega < \delta\Omega_{\min}$ at a fixed value of ϵ, remains monotonically growing. Some hysteretic phenomena in the resonant range $|\delta\Omega| \ll 1$ (at $\delta\Omega < 0$) have been observed in numerical experiments (If *et al.*, 1985; Christiansen, 1986; Fordsman *et al.*, 1986).

The radiative energy dissipation rate W_{res} is physically interesting if it exceeds the rate W_{diss} of direct energy dissipation. To calculate W_{diss}, we need the kink's adiabatic law of motion corresponding to the perturbation (6.16):

$$\xi(t) = (\pi a/4)\cos[(1+\delta\Omega)t+\delta\phi] . \tag{6.26}$$

We can see that in the resonant situation the kink's oscil-

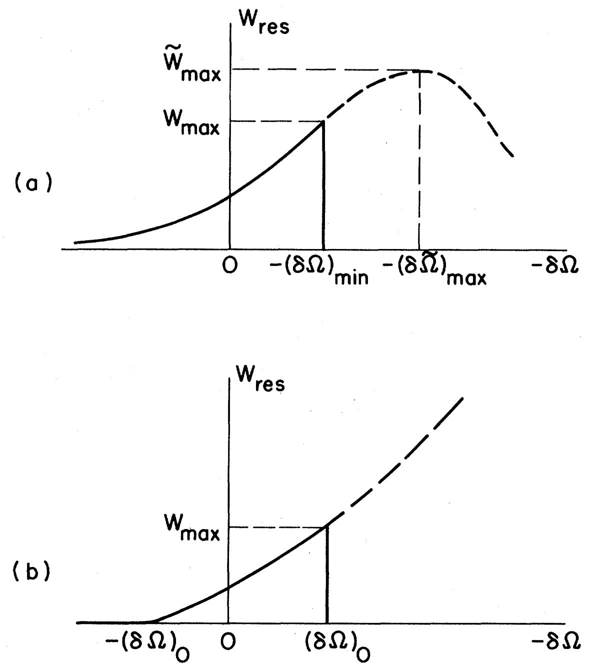

FIG. 28. (a) The dependence of the resonant emission power on the frequency detuning $\delta\Omega$; (b) the same dependence for the parametric resonance. In the hysteretic range (a) $\delta\Omega<(\delta\Omega)_{\min}$ and (b) $\delta\Omega<-(\delta\Omega)_0$, the solid line $W_{\rm res}=0$ stands for the "natural" branch of the dependence $W(\delta\Omega)$, and the dashed line stands for the second stable branch corresponding to slowly varying $\delta\Omega$ (a) from $\delta\Omega>(\delta\Omega)_{\min}$ to $\delta\Omega<(\delta\Omega)_{\min}$ or (b) from $\delta\Omega>-(\delta\Omega)_0$ to $d\Omega<-(\delta\Omega)_0$ at fixed ϵ.

lation amplitude is much larger than the nonresonant amplitude, which is $\sim\epsilon/\Omega^2$ for $\Omega>1$. The energy dissipation rate corresponding to Eqs. (6.19) and (6.26) can be readily calculated: $W_{\rm diss}=4\pi^2a^2\gamma$. Comparing this with Eqs. (6.23) and (6.25), we infer that $W_{\rm diss}\ll W_{\rm res}$ for all values of $\delta\Omega$ if $\epsilon\gg\gamma^2$, and for $\delta\Omega\gg\gamma^2$ if $\epsilon\lesssim\gamma^2$.

These results may be employed to interpret the numerical experiment performed by Olsen and Samuelsen (1986a), in which the dynamics of a kink on an oscillating background was studied in the framework of the unperturbed SG equation. The numerical data of Olsen and Samuelsen (1986a) show that the kink emits radiation. To find the corresponding emission power within the framework of our approach, we consider small oscillations of the background $u_0(t)=a\sin[(1-a^2/16)t]+O(a^3)$, where $a\ll 1$ is an arbitrary small amplitude, and insert into the unperturbed SG equation $u=u_0(t)+U(t,x)$, which brings us to the effective perturbation (6.19) with $\gamma=0$, $\delta\Omega=-a^2/16$. Plainly, $\delta\Omega'=a^2/16>0$, and the emission does indeed take place. Equations (6.25) and (6.24) give the emission power $W_{\rm res}=a^3/2\sqrt{2}$ and the radiation wave numbers $k=\pm a/2\sqrt{2}$. Finally, the amplitude A of the emitted waves is equal to a according to Eq. (6.14).

With the increase of the dissipation coefficient γ, the hysteresis in the dependence $W_{\rm res}(\delta\Omega)$ relaxes, and the location $\delta\tilde{\Omega}_{\max}$ of the absolute maximum of $W_{\rm res}(\delta\Omega)$ shifts to larger $\delta\Omega$. At $\gamma^3=(3\sqrt{3}/64)\epsilon^2$ the hysteresis disappears. However, at $\gamma^3\gtrsim\epsilon^2$ we have $\delta\Omega'\lesssim\gamma$, so that the notion of emission becomes irrelevant, and the real effect of a resonant perturbation is to form oscillating corrections to the kink's shape. For the nonresonant case the corrections have been investigated by Kivshar (1984). However, the resonant case requires special consideration.

The resonant variant of Eq. (6.22) yields

$$b(\lambda)=-ia\Lambda[\gamma-2i(2\Lambda^2-\delta\Omega')]^{-1}\times\exp[-i(1+\delta\Omega')t'-i\delta\phi]\ , \tag{6.27}$$

where $\Lambda\equiv\lambda-\frac{1}{2}$, and we employ the rescaled time (6.22) [$\delta\phi$ is the same as in Eq. (6.17)]. Next, inserting Eq. (6.27) into the IST equations (Zakharov *et al.*, 1980), one can find corrections to the Jost functions, and then the corrections $\delta U(x,t)$ to the kink's shape. The resulting expression for $\delta U(x,t)$ contains terms of two types: some that are localized in the same range (of width $L\sim 1$) as the kink's "core," and others ("tails") that have an anomalously large length $L\gg 1$. The "tails," which appear only in the resonant case, are of primary interest. In the range $|x|\gg 1$ (far from the kink's "core"), they take a relatively simple form [we set $\sigma=+1$ in Eq. (2.61)]:

$$\delta U(x,t)=a\exp\{-\gamma[\sqrt{\gamma^2+4(\delta\Omega')^2}+2\delta\Omega']^{-1/2}|x|/\sqrt{2}\}\cos\{[\sqrt{\gamma^2+4(\delta\Omega')^2}+2\delta\Omega']^{1/2}|x|/\sqrt{2}-(1+\delta\Omega')t'-\delta\phi\}\ , \tag{6.28}$$

As can be seen from Eq. (6.28), at $\gamma\sim\delta\Omega'$ the length of the tail $L\sim\gamma^{-1/2}\gg 1$ is indeed of the same order as the radiation wavelength $l\sim(\delta\Omega')^{-1/2}$ [see Eq. (6.24)].

A more general formulation of the emission problem is related to the perturbation (1.17) with $f(t)$ taken in the form

$$\epsilon f(t)=\epsilon_0+\epsilon_1\cos(\Omega t)\ . \tag{6.29}$$

Under the combined action of the constant bias current and dissipation, a kink moves with velocity (3.47). If the ac component (the second term) is present in Eq. (6.29), the moving kink emits radiation. For the nonresonant case the corresponding emission problem was considered by Mineev and Schmidt (1980). Here we shall briefly consider the resonant case, following Malomed (1987d).

To renormalize the perturbation, we insert into the perturbed SG equation

$$u=\epsilon_0+a\sin[(1+\delta\Omega)t]+U(t,x)\ , \tag{6.30}$$

where a is determined by Eq. (6.18) with ϵ replaced by ϵ_1

and $\delta\Omega$ replaced by $\delta\Omega+\epsilon_0^2/4$. The "renormalized" perturbation takes the form (6.19) with the additional term $-(\epsilon_0^2/2)\sin U$, i.e., this time $\delta\Omega'=\delta\Omega+a^2/8+\epsilon_0^2/4$. Using Eqs. (6.1) and (6.11), we conclude that, unlike the above problem ($V_0=0$), where two waves with the wave numbers $\pm k$ were emitted symmetrically to both sides, here, in the first approximation, resonant emission is represented by one wave with wave number $k=-\delta\Omega'/V_0$, the corresponding emission power being

$$W_{\text{res}}=2a^2|V_0|(1-V_0^2)\ . \tag{6.31}$$

Note that here we do not require the condition $\delta\Omega'>0$. Neglecting dissipative absorption of the emitted wave, it is easy to find its amplitude [cf. Eq. (6.14)],

$$A^2=2W_{\text{res}}/|V_0|=4a^2(1-V_0^2)\ . \tag{6.32}$$

2. The parametric resonance

Here we consider the emission of radiation by a kink under the action of perturbation (1.21) with $f(t)=\sin(\Omega t)$. This problem exhibits the following peculiarity: in the high-frequency case $\Omega\gtrsim 1$, analysis in the first order of perturbation theory, analogous to that developed for perturbation (1.17) by Mineev and Schmidt (1980), yields $dB/dt=0$. The emission appears if one takes into account the perturbation-induced small oscillations of the kink: $\xi(t)=-(\epsilon/2\Omega^2)\sin(\Omega t)$. Then the emission power is (Malomed, 1987d, 1987e)

$$W=(\pi^2\epsilon^4/8\Omega\sqrt{4\Omega^2-1})\,\text{sech}^2(\pi\sqrt{4\Omega^2-1}/2)\ , \tag{6.33}$$

and the radiation frequency is 2Ω (we assume $\Omega>\frac{1}{2}$), while in the case of perturbation (1.17) $W\sim\epsilon^2$ and the radiation frequency is Ω.

Perturbation (1.21) gives rise to parametric resonance at the frequencies $\Omega=2(1+\delta\Omega)/(2n+1)$, where $n=0,1,2,\ldots$, and $|\delta\Omega|\ll 1$. In the simplest case, $n=0$, we insert into Eqs. (1.1) and (1.21)

$$u(x,t)=U(x,t)+u^{(0)}(x,t)\ , \tag{6.34}$$

where the background solution $u^{(0)}(x,t)$ satisfies the equation

$$u_{tt}^{(0)}+\gamma u_t^{(0)}+\sin u^{(0)}=-\frac{\epsilon}{2}(\text{sgn}x)\sin[2(1+\delta\Omega)t]u^{(0)}. \tag{6.35}$$

The sign multiplier on the right-hand side of Eq. (6.35) is stipulated by the fact that at $x\to\pm\infty$, $\sin(u/2)\approx\cos(U/2)u^{(0)}/2$, and at $|x|=\infty$, $\cos(U/2)=-\text{sgn}(\sigma x)$ (for definiteness, we set $\sigma=+1$). The solution to Eq. (6.30) is

$$u^{(0)}(x,t)\approx a\sin[(1+\delta\Omega)t+(\pi/4)\text{sgn}x+\delta\phi]\ , \tag{6.36}$$

where $\delta\phi$ is some phase shift, and the amplitude a is determined by the equation of the nonlinear parametric resonance theory, which has two roots (see Landau and Lifshitz, 1973): $a=0$ and

$$a^2=2(\sqrt{\epsilon^2-16\gamma^2}-8\delta\Omega)\ . \tag{6.37}$$

With a decrease in $\delta\Omega$, the zero root becomes unstable just at $\delta\Omega=(\delta\Omega)_0\equiv\sqrt{\epsilon^2-16\gamma^2}/8$, when the root (6.37) arises (Landau and Lifshitz, 1973).

Inserting Eqs. (6.34) and (6.36) into Eq. (1.21) results, with regard to the term $U_{xx}^{(0)}$, in the renormalized perturbation [cf. Eq. (6.19)]

$$\begin{aligned}P_R=&a\sin[(1+\delta\Omega)t+(\pi/4)\text{sgn}x+\delta\phi](1-\cos U)\\&+\sqrt{2}a\cos[(1+\delta\Omega)t+\delta\phi]\delta'(x)\\&+(a^2/2)\sin^2[(1+\delta\Omega)t+(\pi/4)\text{sgn}x\\&+\delta\phi]\sin U-\gamma U_t\ .\end{aligned} \tag{6.38}$$

As follows from Eqs. (6.38) and (6.37),

$$\delta\Omega'=\delta\Omega+a^2/8=\tfrac{1}{4}\sqrt{\epsilon^2-16\gamma^2}-\delta\Omega \tag{6.39}$$

[see Eq. (6.21)]. Equation (6.39) determines the wave numbers of the emitted radiation according to Eq. (6.24). Proceeding to evaluation of the emission power, we set $\gamma=0$. Substitution of Eq. (6.38) into Eq. (6.1) results eventually in an equation differing from Eq. (6.25) by the multiplier $\frac{1}{2}$. Thus, using Eqs. (6.38) and (6.39), one finds the emission power

$$W=(\epsilon-8\delta\Omega)\sqrt{(\epsilon-4\delta\Omega)/2}\ . \tag{6.40}$$

When $\delta\Omega$ decreases from $(\delta\Omega)_0$ to $-(\delta\Omega)_0$, the emission power (6.40) grows monotonically from zero to $W_{\max}=2\sqrt{2}\epsilon^{3/2}$ [Fig. 28(b)]. In the range $\delta\Omega<-\delta\Omega_0$ the root $a=0$ again becomes stable (Landau and Lifshitz, 1973), and the dependence $W_{\text{res}}(\delta\Omega)$ becomes hysteretic; the "natural" branch, corresponding to our gradually turning on the external field at a fixed $\delta\Omega<-\delta\Omega_0$, is $W_{\text{res}}=0$. Another branch, that corresponds to our slowly changing $\delta\Omega$ from $\delta\Omega>-\delta\Omega_0$ to $\delta\Omega<-\delta\Omega_0$ at a fixed value of ϵ, is depicted by the dashed line in Fig. 28(b).

Let us proceed to a more general problem, in which the external field contains both variable and constant components. It is convenient to write $f(t)$ in the following form:

$$f(t)=\pi\epsilon_0/2+\epsilon_1\cos(\Omega t) \tag{6.41}$$

[cf. Eq. (6.29)]. The constant field, acting in combination with the dissipation (1.16a), causes the kink to move with mean velocity (6.30). In the nonresonant situation, the ac field [the second term on the right-hand side of Eq. (6.41)] generates emission at the wave numbers

$$k_{1,2}=(\Omega V_0\pm\sqrt{\Omega^2+V_0^2-1})/(1-V_0^2)\ . \tag{6.42}$$

It is straightforward to evaluate the emission powers $W_{1,2}$ corresponding to the wave numbers (6.42) (Malomed, 1987d, 1987e):

$$W_{1,2}=(\pi^2\epsilon_1^2/8)V_0^2[\Omega\pm V_0(\Omega^2+V_0^2-1)^{1/2}]\times(\Omega^2+V_0^2-1)^{-1/2}\times\text{sech}^2\{(\pi/2)[\Omega V_0\pm(\Omega^2+V_0^2-1)^{1/2}]\times(1-V_0^2)^{-1/2}\}\ , \qquad (6.43)$$

provided $|V_0|\gg|\epsilon_1|$.

Now let us consider the perturbation (1.21) and (6.41) in the resonant case $\Omega=2(1+\delta\Omega)$, $|\delta\Omega|\ll 1$. The amplitude of the effective perturbation and the effective detuning $\delta\Omega'$ are determined by Eqs. (6.37) and (6.39) with ϵ replaced by ϵ_1 and $\delta\Omega$ replaced by $\delta\Omega+\pi\epsilon_0/8$:

$$a^2=2[(\epsilon_1^2-16\gamma^2)^{1/2}-8(\delta\Omega+\pi\epsilon_0/8)]\ ,$$

$$\delta\Omega'=\delta\Omega+a^2/8=\tfrac{1}{4}(\epsilon_1^2-16\gamma^2)^{1/2}-\pi\epsilon_0/4-\delta\Omega\ .$$

The radiation wave number is the same as above: $k=-\delta\Omega'/V_0$. As to the emission power and squared amplitude of the emitted wave, the expressions for them, written in terms of a, V_0, and $\delta\Omega'$, differ from Eqs. (6.31) and (6.32) by a factor of $\frac{1}{2}$, just as in the problem with $V_0=0$ (see Sec. IV.C.1).

3. The low-frequency case

Though equations of the type (6.22) were obtained in the first order of the perturbation theory with respect to the radiation, for low-frequency adiabatic motion these equations make it possible to calculate emission intensities that are exponentially small in $\sqrt{\epsilon}$ or ϵ due to the presence of the multiplier

$$\exp[-ik\xi(t)]\ , \qquad (6.44)$$

provided the adiabatic law of motion $\xi(t)$ takes the perturbation into account exactly. For perturbation (1.17) with $f(t)=\cos(\Omega t)$, the adiabatic law of motion is (setting $\gamma=0$)

$$\xi(t)=-\Omega^{-1}\sin^{-1}\{(\pi\epsilon/4\Omega)\cos(\Omega t)\times[1+(\pi\epsilon/4\Omega)^2]^{-1/2}\}\ . \qquad (6.45)$$

After substituting Eq. (6.45) into Eq. (6.22), we need to evaluate the Fourier coefficients F_n of the time-periodic function $F(\lambda,t)$ defined in Eq. (6.1). In the low-frequency case $\Omega\ll 1$, we are interested in large $n>1/\Omega$. Inserting Eq. (6.45) into Eq. (6.44), which is a multiplier in $F(\lambda,t)$, one notes that, provided $\Omega\ll|\epsilon|$, an integral determining $F_n(\lambda)$ has the saddle point at $\cos(\Omega t)=\pm 4in\Omega^2/\pi\epsilon$, distinguished by the condition $k\,d\xi/dt+n\Omega=0$. Evaluating the integral by means of the steepest-descent method, one can find

$$W=(\sqrt{3}\pi^2)|\epsilon|^{3/2}\exp(-8/\pi|\epsilon|)\ . \qquad (6.46)$$

Then, for the perturbation (1.21), which we find convenient to rewrite here as

$$\epsilon P=(\pi\epsilon/2)\cos(\Omega t)\sin\left[\frac{u}{2}\right]\ , \qquad (6.47)$$

the same final expression (6.46) is valid in the low-frequency case $\Omega\ll\epsilon^{3/2}$.

4. Emission from a kink accelerated by a constant external force

Equation (6.46) above does not depend on the frequency Ω. Therefore it is natural to consider the case $\Omega=0$, i.e., emission from a kink moving under the action of the constant driving force $2\pi\epsilon$. In the adiabatic approximation, the corresponding law of motion for the kink as $t\to\infty$ takes the asymptotic form

$$V(t)\approx 1-8(\pi\epsilon t)^{-2}\ . \qquad (6.48)$$

Insertion of Eq. (6.48) into the general equation (6.22) and subsequent evaluation of the final amplitudes of the emitted radiation

$$B_f\equiv\int_{-\infty}^{+\infty}dt\frac{dB(k)}{dt} \qquad (6.49)$$

yield the following expression for the spectral density of emitted radiation (2.69b) (Malomed, 1987h):

$$\mathcal{E}(k)=16\pi\exp(-8/\pi|\epsilon|)\ . \qquad (6.50)$$

Note that Eq. (6.50) features the same exponential smallness as Eq. (6.46). As to the instantaneous value of the energy emission rate, it can be estimated as follows:

$$W\sim|\epsilon|\exp(-8/\pi|\epsilon|)\ . \qquad (6.51)$$

For the perturbation (6.47) with $\Omega=0$, the same formulas [(6.50) and (6.51)] are valid.

Another problem of the same kind may be found in the equation

$$u_{\tau\chi}=\sin u-\epsilon u_\chi,\quad \epsilon>0\ , \qquad (6.52)$$

which, according to Drühl and Alsing (1986), is a particular model of stimulated Raman scattering in a dissipative medium, where ϵ is a small dissipative coefficient. The change of variables $x=\tau+\chi$, $t=\pm(\tau-\chi)$ transforms Eq. (6.52) into the perturbed SG equation

$$u_{tt}-u_{xx}+\sin u=\epsilon(u_x\mp u_t)\ , \qquad (6.53)$$

where the lower and upper signs correspond to the so-called scattering and superradiance cases, respectively (Drühl and Alsing, 1986). The perturbation on the right-hand side of Eq. (6.53) acts on a kink as an external force, and the kink's corresponding equation of motion takes the form

$$\frac{d}{dt}V=-\epsilon(1\pm V)(1-V^2)\ . \qquad (6.54)$$

The associated emission problem was considered by Malomed (1987h). First of all, in the superradiance case [the upper signs in Eqs. (6.53) and (6.54)], the perturbation on the right-hand side of Eq. (6.53) becomes, when we insert the kink's wave form,

$$u_x - u_t = 2\sqrt{(1+V)/(1-V)}\operatorname{sech}\left[\frac{x-\xi(t)}{\sqrt{1-V^2}}\right], \quad (6.55)$$

and the kink's asymptotic law of motion ensuing from Eq. (6.54) is

$$1+V(t) \approx (2\epsilon t)^{-1} . \quad (6.56)$$

With the use of Eqs. (6.55) and (6.56), one can obtain a final result analogous to Eq. (6.50):

$$\mathcal{E}(k) = \text{const} \times \exp(-|k|/\epsilon) \quad (6.57)$$

(the radiation is emitted backwards, which means $k<0$), where const is independent of k and ϵ.

The scattering case [the lower signs in Eqs. (6.53) and (6.54)] gives more interesting results. Here the perturbation takes the form [cf. Eq. (6.55)]

$$u_x + u_t = 2\sqrt{(1-V)/(1+V)}\operatorname{sech}\left[\frac{x-\xi(t)}{\sqrt{1-V^2}}\right], \quad (6.58)$$

and the asymptotic law of motion is [cf. Eq. (6.56)]

$$1+V(t) \approx \tfrac{1}{2}\exp(-4\epsilon t) . \quad (6.59)$$

The important difference between the scattering and superradiance cases is the fact that for scattering the emitted waves are themselves unstable within the framework of the corresponding variant of Eq. (6.53), the instability growth rate being equal to ϵ for $k \to -\infty$. Taking this fact into account and using Eqs. (6.58) and (6.59), one can obtain the following expression for the emitted energy spectral density as a function of time and wave number ($k<0$):

$$\mathcal{E}(k,t) = \frac{\text{const}}{|k|} e^{2\epsilon t}\theta\left[t - \frac{1}{2\epsilon}\right]\ln|k| , \quad (6.60)$$

where $\theta(z)$ is the standard Heaviside step function. Equation (6.60) is valid even at the stage $\mathcal{E}(k,t) \gtrsim 1$, when the waves emitted by the kink and enhanced by instability cannot be regarded as linear.

Finally, let us consider the equation

$$u_{tt} - u_{xx} + \sin u = -\alpha u_t \cos u, \quad 0<\alpha \ll 1 , \quad (6.61)$$

which describes the evolution of transverse phase modulations of parallel rolls in a low-Prandtl-number convective layer subject to resonant spatially periodic forcing (Coullet and Huerre, 1986). The equation of motion for a kink takes the form $dV/dt = (\alpha/3)(1-V^2)V$; as $t \to \infty$ it actually coincides with Eq. (6.59): $1-V(t) \approx \frac{1}{2}\exp(-2\alpha t/3)$. However, substituting the right-hand side of Eq. (6.61) into Eq. (6.59) yields $dB/dt = 0$, i.e., radiative effects may arise only in higher orders of the perturbation theory (Malomed, 1987h).

D. Energy emission from a kink scattered by inhomogeneities

Before consideration of particular physical problems, it is pertinent to discuss a general methodological issue common for perturbations P that do not vanish at $u=0$, e.g.,

$$P = \delta(x), \quad P = \delta'(x) ,$$

$$P = \sin(u/M + \theta)\delta(x)$$

$$\equiv (\cos\theta \sin u/M - 2\sin\theta \sin^2 u/M)\delta(x)$$

$$+ \sin\theta\delta(x), \quad P = f'(x)$$

with a random function $f(x)$ (see Secs. VI.D.1–VI.D.3), and so on. In all these cases, Eq. (6.1) takes the form $dB/dt = e^{i\omega t}F(t,\lambda)$, where the function $F(t,\lambda)$ has nonzero limits as $t \to \pm\infty$, and $|F(t=+\infty)| = |F(t=-\infty)|$. Clearly, direct integration of this equation is not possible due to divergence. This difficulty can be surmounted as proposed by Malomed (1987d): if the perturbation has the form $P = g(x)$, one must look for a solution to the perturbed SG equation in the form $u(x,t) = U(x,t) + \epsilon u_0(x)$, where $u_0(x)$ is a background solution [cf. Eq. (6.16)] defined by the linearized equation

$$u_0'' + u_0 = g(x) .$$

In first order in ϵ, this substitution transforms the original equation for $u(x,t)$ into the renormalized one for $U(x,t)$ [cf. Eq. (6.17)]:

$$U_{tt} - U_{xx} + \sin U = 2\epsilon U_0(x)\sin^2\left[\frac{U}{2}\right] .$$

Unlike the original equation, the renormalized one does not result in divergencies

1. A local inhomogeneity

Let us consider radiative effects accompanying the interaction of a kink with a local inhomogeneity described by the perturbation (3.89) with $M=1$,

$$P = -\delta(x)\sin(u+\theta) .$$

First we consider the case when the kink's velocity V is sufficiently large, $V^2 \gg \epsilon$. Due to this assumption, we may neglect a perturbation-induced disturbance of the kink's law of motion. The spectral density of the total energy emitted by the kink during its interaction with the inhomogeneity has been calculated by Malomed (1988e) [in the particular cases $\theta=0$ and π the result was obtained earlier by Kivshar (1984)]:

$$\mathcal{E}(k) = \frac{\pi\epsilon(1-V^2)^2[\omega(k)-kV]^2}{4V^6} \times \left[\sin^2\theta \operatorname{csch}^2[\pi\sqrt{1-V^2}\omega(k)/2V] + \cos^2\theta \operatorname{sech}^2\left[\frac{\pi\sqrt{1-V^2}\omega(k)}{2V}\right]\right], \quad (6.62)$$

where $\omega(k)\equiv\sqrt{1+k^2}$. Note that, according to Eq. (6.62), when $1-V^2\ll 1$ almost all the energy is emitted backwards, and $\mathcal{E}(k)$ takes a maximum value $\mathcal{E}_{max}\sim\epsilon^2$ at $k\sim-(1-V^2)^{-1/2}$. The corresponding total emitted energy $E_{em}\equiv\int_{-\infty}^{+\infty}\mathcal{E}(k)dk$ can be explicitly evaluated in the two limiting cases: $V^2\ll 1$ and $1-V^2\ll 1$. In the former case, E_{em} is exponentially small,

$$E_{em}\sim\epsilon^2\exp(-\pi/|V|) \tag{6.63}$$

(Kivshar, 1984), and in the latter case it takes the form (Malomed, 1988e)

$$E_{em}=\tfrac{2}{3}\epsilon^2\sqrt{1-V^2}(1+\sin^2\theta)\ . \tag{6.64}$$

Though E_{em} cannot be explicitly found for arbitrary V^2, it is clear that it, as a function of V^2, takes a maximum value

$$(E_{em})_{max}=\alpha(\theta)\epsilon^2 \tag{6.65}$$

at some $V^2=V^2_{max}(\theta)$, where the functions $\alpha(\theta)$ and $V^2_{max}(\theta)$ take values of order one ($V^2_{max}<1$) and are independent of the parameters of the problem.[6]

Equations (6.63)–(6.65) can be applied to the description of a long Josephson junction ($\theta=0$ or π) and of a long CDW system (arbitrary θ) with periodically or randomly installed local impurities of the type (3.76). Indeed, if the mean distance between neighboring impurities is l (we assume $l\gg 1$), the effective averaged force that brakes the motion of the kink is

$$F_{br}=\langle E_{em}\rangle/l+8\gamma V/\sqrt{1-V^2}\ , \tag{6.66}$$

where $\langle E_{em}\rangle$ stands for the averaged energy emitted by the kink during its interaction with an individual impurity, and the second term in Eq. (6.66) is a contribution from the direct (nonradiative) friction, γ being the dissipative constant. Under the condition $8\gamma l\ll\epsilon^2$, the dependence $F_{br}(V^2)$ ensuing from Eqs. (6.63)–(6.66) takes the form depicted in Fig. 29. For a given driving force $2\pi f$, the kink's equilibrium velocity V_0 is determined by an intersection of the curve in Fig. 29 by the horizontal line $F_{br}=2\pi f$. Plainly, the intersections with the segments of the curve that have a positive slope (solid line in Fig. 29) give rise to stable equilibrium motions, while intersection with the negative-slope segment (dashed line in Fig. 29) gives rise to an unstable motion. A current-voltage characteristic of the system, i.e., the dependence $f(V_0)$ in the case of a long Josephson junction, or the dependence $V_0(f)$ in the case of the CDW system, generated by the dependence $F_{br}(V^2)$ shown in Fig. 29, is hysteretic in some range $F_{min}<f<f_{max}$, $(V_0^2)_{min}<V_0^2<(V_0^2)_{max}$; see Fig. 30 (Mineev, Feigel'man, and Shmidt, 1981; Malomed, 1988e).

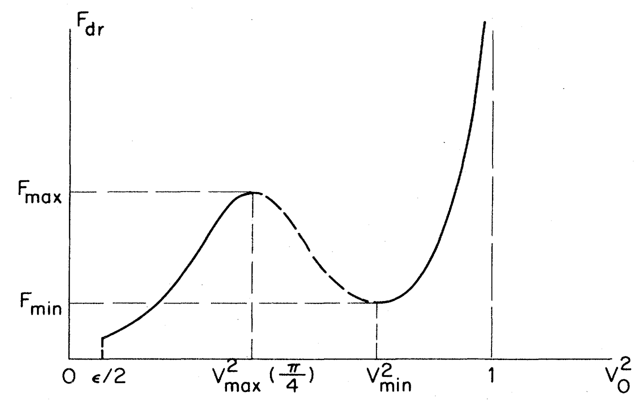

FIG. 29. The effective force (6.66) braking the kink's motion vs V^2. The graph terminates at $V^2=\epsilon/2$, since for smaller V^2 the kink will be captured by an inhomogeneity.

Now let us briefly consider the case $V^2\ll\epsilon$. Here radiative losses are exponentially small in $\sqrt{\epsilon}$. An interesting methodological problem is to find a maximum (threshold) velocity V_{thr} for a kink that admits its capture by the inhomogeneity (in the absence of direct dissipation, i.e., at $\gamma=0$). For the particular case $\theta=\pi$ this quantity has been found by Malomed (1985):

$$V^2_{thr}=2^{21/4}\sqrt{\pi}\epsilon^{3/4}\exp(-2\sqrt{2/\epsilon})\ . \tag{6.67}$$

For a more general case (arbitrary θ), the exponential factor in Eq. (6.67) changes to $\exp[-2\sqrt{2/\epsilon}/|\sin(\theta/2)|]$ (Malomed, 1988e).

As was mentioned in Sec. III.C.2.c, in the theory of CDW systems a local perturbation (3.89) also naturally arises with an arbitrary $M\geq 2$:

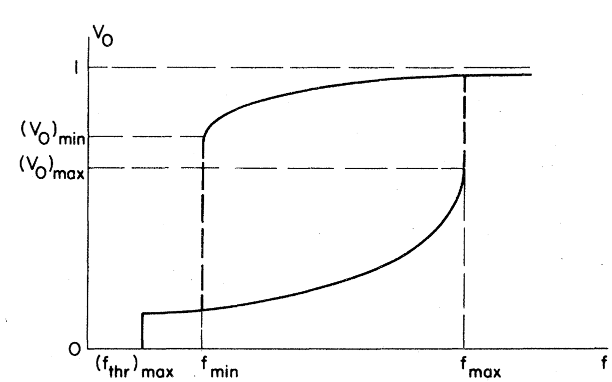

FIG. 30. The current-voltage characteristic following from Fig. 29.

[6]A small part of the energy emitted may be expended on excitation of a small-amplitude breatherlike (oscillating) localized mode pinned by attractive ($\epsilon>0$) inhomogeneity (the mode is described in detail in VI.H.2 below). This problem was considered in different contexts by Nakamura (1978), Yoshida and Sakuma (1978, 1982), Watanabe and Toda (1981), Klinker and Lauterborn (1983), and others. We shall not dwell on it here.

$$P=-\delta(x)\sin\left[\frac{u}{M}+\theta\right].$$

Malomed and Nepomnyashchy (1989b) have shown that when $\epsilon \ll V^2 \ll 1$ the total emitted energy exhibits the same exponential smallness (6.63). The case of the opposite limit $1-V^2 \ll 1$ is more interesting. The corresponding spectral density of the energy emitted by a kink moving across an inhomogeneity has the form

$$\mathcal{E}(k)=\frac{1}{\pi}\frac{\epsilon^2}{k^2+1}\sin^2\left[\frac{\pi}{M}\right]\cos^2\left[\theta+\frac{\pi(2n+1)}{M}\right] + O(\sqrt{1-V^2}) , \tag{6.62$'$}$$

where it is implied that the kink's wave field at $x \to \pm\infty$ takes the asymptotic values $2\pi n$ and $2\pi(n+1)$; see Eq. (4.207). Note that Eq. (6.62′), in contrast with (6.62), does not depend on the sign of k, i.e., the radiation is emitted symmetrically to the left and to the right. Integration of the spectral density (6.62′) yields the total emitted energy

$$E_{\text{em}}=\epsilon^2\sin^2\left[\frac{\pi}{M}\right]\cos^2\left[\theta+\frac{\pi(2n+1)}{M}\right]+O(\sqrt{1-V^2}) . \tag{6.64$'$}$$

Equation (6.64′), in contrast with (6.64), does not vanish at $\sqrt{1-V^2}\to 0$. The main terms of Eqs. (6.62′) and (6.64′) vanish at $M=1$. This means that, in accordance with Eq. (6.64), $E_{\text{em}} \sim \sqrt{1-V^2}$ for $M=1$.

In some applications, an inhomogeneity also arises that can be described by the term $P(u)=\delta(x)$. In particular, it describes injection of bias current into a Josephson junction through a localized region (experimentally this has been realized by Akoh *et al.*, 1985). In the early paper by Eilenberger (1977), the radiation wave field emitted by a kink interacting with this local inhomogeneity has been found by means of the Green's-function technique. The spectral density of the energy emitted by a kink moving across such an inhomogeneity has been found by Malomed (1987d) (for $1-V^2 \ll 1$):

$$\mathcal{E}(k)=\frac{\pi\epsilon^2}{4}\frac{k^2}{1+k^2}\frac{(\sqrt{1+k^2}-k)^{-2}}{[k^2+(1-V^2)^{-1}]^2} \times \operatorname{csch}^2\left[\frac{\pi k}{2}\sqrt{1-V^2}\right]. \tag{6.68}$$

An important difference between this equation and Eqs. (6.62) and (6.62′) is that this time almost all the energy is emitted forwards. The total emitted energy is $E_{\text{em}} \approx 0.79\epsilon^2\sqrt{1-V^2}$.

In Sec. III.B, we also dealt with the inhomogeneities (3.53), (3.63), and (3.65). The energy emitted by a kink scattered by one of these inhomogeneities (Kivshar and Malomed, 1988e, 1989f; Kivshar, Malomed, and Nepomnyashchy, 1988) shows two significant differences from Eq. (6.62): first, the spectral density is symmetric, $\mathcal{E}(k)=\mathcal{E}(-k)$, and, second, at $1-V^2\to 0$ the total emitted energy does not fall as $(1-V^2)^{1/2}$ but, instead, grows $\sim(1-V^2)^{-1/2}$. For instance, in the case of the inhomogeneity from Eq. (3.65) generated by an inductance step, $P=\delta(x)u_x$, the spectral density of the emitted energy has the form

$$\mathcal{E}(k)=\frac{\pi\epsilon^2}{V_0^4}\operatorname{sech}^2\left[\frac{\pi(1-V_0^2)^{1/2}}{2V_0}\sqrt{1+k^2}\right]. \tag{6.69}$$

The corresponding dependence of the total emitted energy $E_{\text{em}}=\int_{-\infty}^{+\infty}\mathcal{E}(k)dk$ on V_0^2 is shown in Fig. 31. In the range $\epsilon \ll V_0^2 \ll 1$, the dependence takes the form

$$E_{\text{em}}=4\pi\sqrt{2}\epsilon^2 V_0^{-7/2}\exp(-\pi/V_0) , \tag{6.70}$$

and in the range $1-V_0^2 \ll 1$

$$E_{\text{em}}\approx 8\epsilon^2/(1-V_0^2)^{1/2} . \tag{6.71}$$

As can be seen from Fig. 31 and Eqs. (6.70) and (6.71), the dependence is monotonically growing [the same pertains to the perturbation (3.63)]. Note that the same inference follows from the numerical results of Sakai, Samuelsen, and Olsen (1987). In Sec. III.C.2 we con-

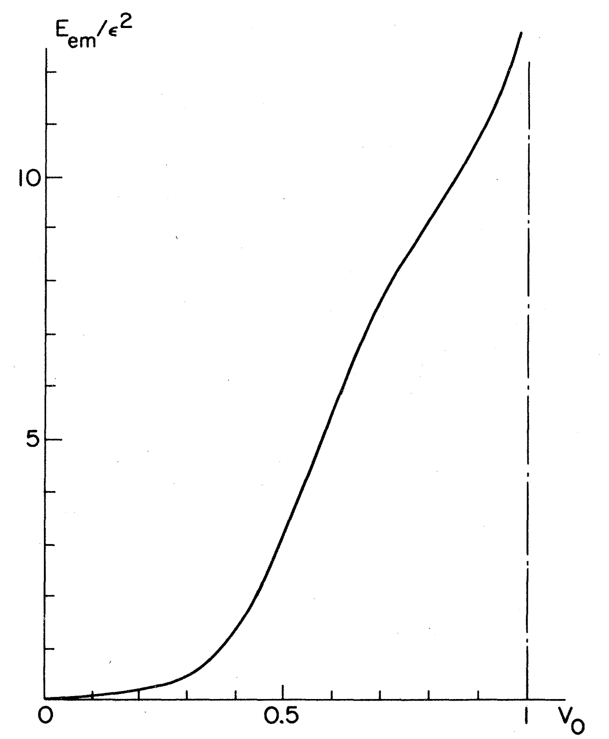

FIG. 31. The normalized total energy emitted by a kink moving past the inhomogeneity described by Eq. (3.65) (and inductance step in a long Josephson junction) vs the kink's velocity V_0.

sidered a generalized local defect in a long Josephson junction described by a linear combination (3.53) of the perturbing terms $\delta(x)\sin u$ and $\beta\delta(u)u_t$. It proves that phases of the plasma waves (radiation) emitted by a moving fluxon under the action of each of these two terms differ by $\pi/2$, so that a full expression for the emitted spectral energy density is the sum of Eqs. (6.62) and (6.69), the latter being multiplied by V_0^2 [Kivshar and Malomed, 1988e; the spectral energy density (6.69), corresponding to the perturbation $\delta(x)u_x$, must be multiplied by V_0^2 in order to correspond to $P=\delta(x)u_t$, because $u_t \approx V_0 u_x$].

It is relevant to mention here that McLaughlin and Scott (1978) proposed employing emission of plasma waves by a fluxon moving through a periodic lattice of microshorts [see the model (3.76)] in the design of a microwave Josephson generator. The results given above suggest that a periodic lattice of dissipative micro-inhomogeneities [described by the model (3.76′) with $\beta_n \equiv \beta$, $x_{n+1}-x_n \equiv a$] would be a better basis for the generator. Using Eq. (6.69) multiplied by $V_0^2(u_t \approx V_0 u_x)$, it is easy to find the energy $W_{\rm rad}$ emitted by a moving kink per unit of time (the power of the one-fluxon generator). In the most interesting case $\gamma a \ll \beta \ll fa \ll 1$, $W_{\rm rad} \approx 2\pi\beta f$. As the power absorbed by the dissipation is $W_{\rm diss}=2\pi f V_0 \approx 2\pi f$ (in the case under consideration $1-V_0^2 \ll 1$), the efficiency of the generator is $W_{\rm rad}/W_{\rm diss} \approx \beta$ (Kivshar and Malomed, 1988e).

Finally, let us consider the local inhomogeneity described by the perturbing term $\epsilon\delta'(x)$. According to Aslamazov and Gurovich (1984), this term describes an Abrikosov vortex lying across a long Josephson junction. The same term describes a local lattice deformation in a CDW system (Brazovsky and Bak, 1978). In the limit $1-V^2 \ll 1$, a kink moving across a local inhomogeneity of this type emits energy in the backward direction, the total emitted energy being $E_{\rm em}=I\epsilon^2(1-V^2)^{-1/2}$, where (Kivshar and Malomed, 1989f)

$$I \equiv 16\int_0^\infty x^4(1+x^2)^{-2}{\rm csch}^2(\pi x)dx \approx 0.005 \ .$$

2. A spatially periodic or random inhomogeneity

Analysis of the energy emission of a kink moving across a localized inhomogeneity gives us a basis for consideration of a periodic lattice of inhomogeneities. The simplest model of a spatially periodic inhomogeneity is based on the perturbed SG equation

$$u_{tt}-u_{xx}+\sin u=\epsilon\sin(\kappa x)\sin u \ . \tag{6.72}$$

Emission of radiation by a kink within the framework of this model was considered by Mkrtchyan and Shmidt (1979), Kivshar (1984), Malomed and Tribelsky (1989), and others. In the lowest approximation, emission takes place under the condition $V^2>(1+\kappa^2)^{-1}$, the radiation wave numbers being

$$k_{1,2}=(1-V^2)^{-1}[V^2\kappa \pm \sqrt{V^2(1+\kappa^2)-1}] \ . \tag{6.73}$$

The emission powers corresponding to the wave numbers (6.73) are

$$W_{1,2}=(\pi^2/8)(1-V^2)^2\epsilon^2(1+\kappa^2)[\kappa \pm 2\sqrt{(1+\kappa^2)V^2-1}]^2[\kappa \mp \sqrt{(1+\kappa^2)V^2-1}]^{-1}$$
$$\times[(1+\kappa^2)V^2-1]^{-1/2}{\rm sech}^2[(\pi/2)(1-V^2)^{-1/2}\kappa \pm \sqrt{(1+\kappa^2)V^2-1}] \ . \tag{6.74a}$$

The dependence of the net emission power $W_{\rm tot} \equiv W_1+W_2$ on the velocity of the kink is shown in Fig. 32(a). As can be seen, there is no emission at $V^2<V_0^2 \equiv (1+\kappa^2)^{-1}$, and $W_{\rm tot}$ diverges $\sim[(1+\kappa^2)V^2-1]^{-1/2}$ at $V^2-V_0^2 \to +0$. However, at small $(V^2-V_0^2)$ the regularizing role of dissipation must be taken into account. The following regularized expression for $W_{\rm tot}$ has been obtained by Malomed and Tribelsky (1989):

$$W_{\rm tot}=(\pi/4\sqrt{2})\epsilon^2\gamma\kappa^6(1+\kappa^2)^{-1/4}{\rm sech}^2\left[\frac{\pi}{2}\sqrt{1+\kappa^2}\right][(\gamma\kappa)^2+4(1+\kappa^2)^2(\delta V)^2]^{-1/2}$$
$$\times[\sqrt{(\gamma\kappa)^2+4(1+\kappa^2)^2(\delta V)^2}-2(1+\kappa^2)\delta V]^{-1/2} \ , \tag{6.74b}$$

where $\delta V \equiv V-V_0$, and it is implied that the dissipative term $-\gamma u_t$ is added to the right-hand side of Eq. (6.72). Both Eqs. (6.74a) and (6.74b) take the same asymptotic form in the overlapping region $(1+\kappa^2)(\gamma\kappa) \ll \delta V \ll V_0$:

$$W_{\rm tot}=(\pi/4\sqrt{2})\kappa^5(1+\kappa^2)^{-3/4}$$
$$\times{\rm sech}^2\left[\frac{\pi}{2}\sqrt{1+\kappa^2}\right]\epsilon^2(\delta V)^{-1/2} \ . \tag{6.75a}$$

At the same time, Eqs. (6.74) and (6.76) exhibit a drastic difference in the region $0<\delta V \ll \gamma\kappa(1+\kappa^2)^{-1}$: While the former diverges $\sim\epsilon^2(\delta V)^{-1/2}$ at $\delta V \to 0$, the latter remains finite and takes the maximum value

$$W_{\max}=(3^{3/4}\pi/2^{7/2})\kappa^{9/2}(1+\kappa^2)^{-1/4}$$
$$\times{\rm sech}^2\left[\frac{\pi}{2}\sqrt{1+\kappa^2}\right](\epsilon^2/\sqrt{\gamma}) \tag{6.76a}$$

at

$$\delta V=\delta V_{\max} \equiv (\gamma/2\sqrt{3})\kappa(1+\kappa^2)^{-1} \ . \tag{6.76b}$$

In the region $\delta V<0$ the dissipationless approximation (Mkrtchyan and Shmidt, 1979) yields $W_{tot}\equiv 0$, while the regularized expression (6.74b) remains finite in this region too. In particular, at $\gamma\kappa(1+\kappa^2)^{-1}\ll-\delta V\ll V_0$ [cf. Eq. (6.75a)]

$$W_{tot}\approx(\pi/2^{9/2})\gamma\kappa^6(1+\kappa^2)^{-9/4}\times\mathrm{sech}^2\left[\frac{\pi}{2}\sqrt{1+\kappa^2}\right]\epsilon^2(\delta V)^{-2}\ . \quad (6.75b)$$

The full regularized dependence $W_{tot}(\delta V)$ in the region $|\delta V|\ll V_0$ is shown in Fig. 32(b).

Finally, in the ultrarelativistic region $1-V^2\ll 1$ the net emission power defined by Eq. (6.74a) takes the asymptotic form

$$W_{tot}\approx W_2\approx(\pi\epsilon/4)^2(1+\kappa^2)(1-V^2)^2\ .$$

At V close to V_0 the radiative braking force W_{tot}/V acting upon a kink has a sharp maximum. This must give rise to a well-pronounced peak at $V=V_0$ on the one-kink velocity-drive characteristic $V(f)$ of the driven damped system described by Eq. (6.72) with the additional terms $-\gamma u_t-f$ on the right-hand side (where f is the drive). That peak and the hysteretic transitions between it and the usual (dissipation-dominated) branch of the velocity-drive characteristic have been investigated in detail by Malomed and Tribelsky (1989). In particular, a stable piece of the characteristic $V(f)$ near $V=V_0$ lies in the region $\delta V<\delta V_{max}$ [see Eq. (6.76b)], while the piece in the region $\delta V>\delta V_{max}$ is unstable. Note that the corresponding dependence $W_{tot}(\delta V)$ [see Eq. (6.75b)] at $\delta V<\delta V_{max}$ does not exist at all in the dissipationless approximation.

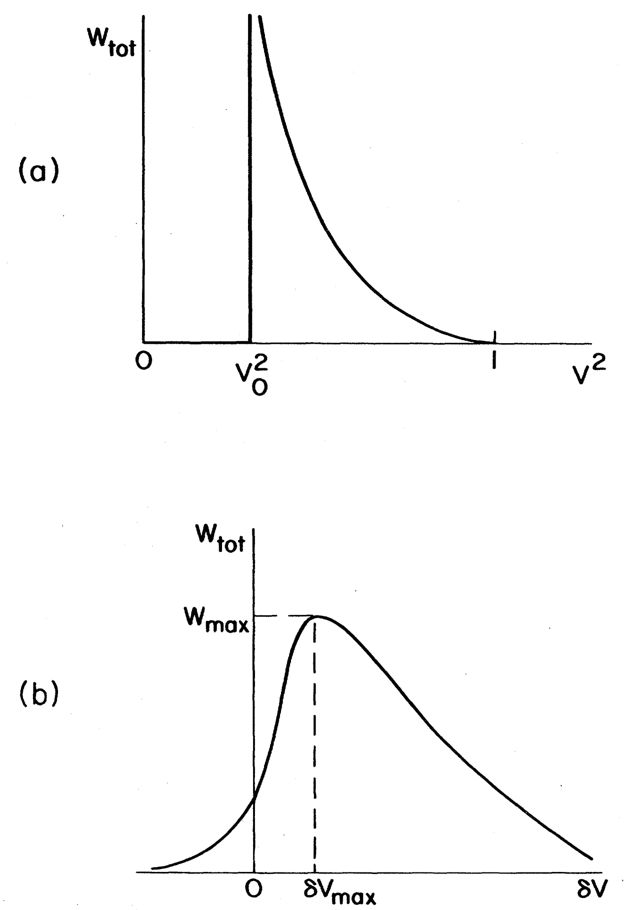

FIG. 32. The kink's net emission power vs its velocity in the model (6.72): (a) the dissipationless approximation (6.74a); (b) the regularized dependence given by Eqs. (6.74b), (6.75a), and (6.75b). The values W_{max} and δV_{max} are given by Eqs. (6.76a) and (6.76b), respectively.

A more realistic model of a lattice of inhomogeneities is described by the SG equation with the perturbation

$$\epsilon\sum_{-\infty}^{+\infty}\delta\left[x-\frac{2\pi}{\kappa}n\right]\sin u\ .$$

In general (for an arbitrary value of the spacing parameter $2\pi/\kappa$), an exact calculation of the energy emission rate is not possible. In the limit $\kappa\gg 1$ this model proves to be equivalent to Eq. (6.72), with ϵ replaced by 2ϵ. However, as can be seen from Eq. (6.74), in the limit $\kappa\gg 1$ the energy emission rate is exponentially small. In the opposite limit, $\kappa\ll 1$, one can find the energy emission rate using Eq. (2.64) (with $\theta=0$) pertaining to an isolated local inhomogeneity. Using these ideas, Kivshar and Malomed (1985) considered radiative braking of a kink moving with a certain initial velocity through a lattice of inhomogeneities. They obtained estimates for a total braking time and for a total distance traveled by the kink.

Let us proceed to the case when a kink moves through a random relief of inhomogeneities. The simplest model is based upon the perturbing term $\epsilon f(x)\sin u$, where $f(x)$ is a random Gaussian function subject to the correlation relations

$$\langle f(x)\rangle=0,\quad \langle f(x)f(x')\rangle=\delta(x-x') \quad (6.77)$$

[cf. Eqs. (3.104) and (3.105)]. This model was analyzed by Mineev, Feigel'man, and Shmidt (1981) in relation to the theory of long inhomogeneous Josephson junctions [generalization to the case of a Gaussian $f(x)$ with a nonzero correlation radius was considered by Kivshar, Konotop, and Sinitsyn (1986)]. The spectral density of the emission power is

$$\mathcal{W}(k)=\frac{\epsilon^2(1-V^2)^2}{16V^5}(kV-\sqrt{1+k^2})^2\times\mathrm{sech}^2\left[\frac{\pi\sqrt{1-V^2}}{2V}\sqrt{1+k^2}\right], \quad (6.78)$$

where V is the kink's velocity (it is assumed that $V^2\gg\epsilon$). The total emission power $W\equiv\int_{-\infty}^{+\infty}\mathcal{W}(k)dk$ can be calculated, as usual, in the two limiting cases:

$$W \approx \frac{\epsilon^2}{2\sqrt{2}} V^{-9/2} e^{-\pi/V} \tag{6.79}$$

for $\epsilon \ll V^2 \ll 1$, and

$$W \approx (2/\pi^3)\epsilon^2 \sqrt{1-V^2} \tag{6.80}$$

for $1-V^2 \ll 1$. As can be seen from Eqs. (6.79) and (6.80), the dependence $W(V^2)$ is increasing at small V^2 and decreasing at small $(1-V^2)$, so that $W(V^2)$ must attain a maximum value at some intermediate V^2. Analysis of the kink's law of motion with regard to this circumstance has led Mineev, Feigel'man, and Shmidt (1981) to the inference that, provided the level of direct dissipative losses be sufficiently low, the dependence of the equilibrium kink's velocity on the driving force (usually described by the constant perturbing term f) is hysteretic (cf. the similar analysis in Sec. VI.D.1 and Figs. 29 and 30).

As another example of a random potential relief, we can take the perturbing term $-\epsilon f'(x)$, where f is the same random function as in Eqs. (6.77) (Malomed, 1987d). When $\epsilon \ll V^2 \ll 1$, the spectral density of the emission power is

$$\mathcal{W}(k) = \frac{\pi\epsilon^2}{V^3} \exp\left[-\frac{\pi}{2V}(1+k^2/2)\right], \tag{6.81}$$

and the corresponding total emission power is $W = 2\pi\epsilon^2 V^{-5/2} \exp(-\pi/2V)$. When $1-V^2 \ll 1$,

$$\mathcal{W}(k) = \pi^{-1}\epsilon^2(1-V^2)(1+k^2)^{-1} \tag{6.82}$$

and $W = \epsilon^2(1-V^2)$. Equation (6.82) differs from (6.78) in its symmetry with respect to the change $k \to -k$ and in the fact that the main part of the emission power is concentrated in the spectral range $k^2 \lesssim 1$, instead of $k^2 \sim (1-V^2)^{-1}$.

3. A ringlike system with inhomogeneities

The emission problem for a kink in an inhomogeneous system acquires specific features if we consider a ringlike system, i.e., a system of finite (but large) length L subject to the periodic boundary conditions

$$u(x+L) \equiv u(x) . \tag{6.83}$$

Given the characteristics of a one-kink emission in an inhomogeneous infinite system, what will be the emission in a finite-length ring system? A solution to the problem has been offered by Malomed and Tribelsky (1989) [see also Malomed and Ustinov (1989b)]. Due to the periodicity condition (6.83), a finite-length system is equivalent to an infinite system containing a chain of kinks with the large period L. The amplitude $b(\lambda)$ of the total emitted radiation is the sum of amplitudes emitted by separate kinks. Summing yields the following results: emission is concentrated at the discrete frequencies

$$\omega_n = \frac{V}{L} 2\pi n, \quad n \ge n_0 = (2\pi L/V) . \tag{6.84}$$

If the one-soliton emission spectrum is described by a spectral density $\mathcal{W}(k)$ [see, for example, Eqs. (6.78), (6.81), and (6.82)], the partial emission power pertaining to the nth spectral line is

$$W_n = \frac{2\pi}{L}\{\mathcal{W}(k=(\omega_n^2-1)^{1/2}) + \mathcal{W}(k=-(\omega_h^2-1)^{1/2})\} . \tag{6.85}$$

An important particular case is that in which the one-kink emission spectrum is discrete itself [as, for instance, in the model (6.72)]. In this case, Eq. (6.85) gives zero if ω_n does not coincide with the discrete frequency Ω of the one-kink spectrum, and infinity if ω_n does coincide. However, this singular behavior will be smoothed if one takes account of the regularizing effect of the dissipation described by the perturbing term $-\gamma u_t$. The corresponding general formula is [cf. Eq. (6.85)]

$$W_n = (2\pi n V)^{-1} \sqrt{\omega_n^2 - 1}\, W_\Omega \gamma [(\gamma/2)^2 + (\omega_n - \Omega)^2]^{-1} , \tag{6.86}$$

where W_Ω is the one-fluxon emission power concentrated at $\omega = \Omega$ in the absence of dissipation, and where it is assumed that the detuning $(\omega_n - \Omega)$ is small. Since both the eigenfrequencies (6.84) and Ω are functions of the kink's velocity V, the quantity $(\omega_n - \Omega)$ may become zero at some discrete values V_n of V. For instance, the frequencies that correspond, according to the dispersion relation $\Omega^2 = 1 + k^2$, to the wave numbers (6.73) are

$$\Omega_{1,2} = \frac{V}{1-V^2}[\kappa \pm \sqrt{V^2(1+\kappa^2)-1}] ,$$

and the equation $\omega_n = \Omega_{1,2}$ determines the spectrum of velocities (Golubov and Ustinov, 1986, 1987):

$$1 - V_n^2 = (L/2\pi n)^2 (4\pi n \kappa / L - 1 - \kappa^2) . \tag{6.86$'$}$$

At those values V_n of the kink's velocity which provide the coincidence of ω_n and Ω, a radiative braking force acting upon the kink will have a sharp maximum. Golubov and Ustinov (1986, 1987) have pointed out that, from the viewpoint of Josephson-junction theory, this resonance will show itself through peculiarities of the current-voltage characteristic of the junction at voltages corresponding to the velocities V_n. This prediction has been corroborated in an experiment of Serpuchenko and Ustinov (1987) [see also Golubov, Serpuchenko, and Ustinov (1988)].

Equation (6.86) provides a detailed description of the peculiarities mentioned. Indeed, according to the above, we may rewrite Eq. (6.86) in the form

$$W_n = (2\pi n V)^{-1} \sqrt{\omega_n^2 - 1} \times W_\Omega \gamma \left[\left[\frac{\gamma}{2}\right]^2 + C_n (V - V_n)^2 \right]^{-1} , \tag{6.87}$$

where C_n is a constant ~ 1. The corresponding energy balance equation is

$$2\pi f V = 8\gamma V^2(1-V^2)^{-1/2} + W_n \tag{6.88}$$

(f is the driving-force constant perturbing term in the SG equation). The dependence $V(f)$ ensuing from Eqs. (6.88) and (6.87) is shown in Fig. 33, where the quantity $f_{\max}$ is $\sim\epsilon^2/\gamma$.

Finally, it is worth noting that a similar analysis applies to a ringlike system with several kinks trapped, provided the overlap of the kinks may be neglected. For this situation, Malomed *et al.* (1988) have predicted and shown in an experiment strongly pronounced peculiarities of the current-voltage characteristic of a long inhomogeneous Josephson junction containing several trapped fluxons. In this experiment, as well as in the above-mentioned experiment of Serpuchenko and Ustinov (1987), the periodic spatial inhomogeneity was realized as a regular lattice of pointlike microresistors [described by the model (3.76)].

The amplification of emission intensity due to in-phase coherent superposition of waves emitted by different fluxons has been called superradiance (Malomed *et al.*, 1988). In the experimental work of Monaco *et al.* (1988), superradiance was realized in a simpler sense, as the in-phase superposition of electromagnetic radiation emitted by solitary fluxons moving synchronously in several long parallel Josephson junctions.

Coherent *antiphase* superposition of emitted waves results in suppression of the emission instead of amplification. The corresponding kinematic relation differs from Eq. (6.86′) by the change $2\pi n \to 2\pi n + \pi$ (Malomed and Tribelsky, 1989). However, antiphase superposition is of less physical importance since, in general, when the emission spectrum contains many frequencies, suppression of one of them does not give a strong effect. An exception is the case in which the suppressed frequency ω is resonant ($|\omega - 1| \ll 1$): As was shown in Sec. VI.C.1, the resonant frequency gives a dominant contribution to the net emission power.

Similar effects have been observed in the numerical experiments of Mistriotis *et al.* (1988) with a Morse or

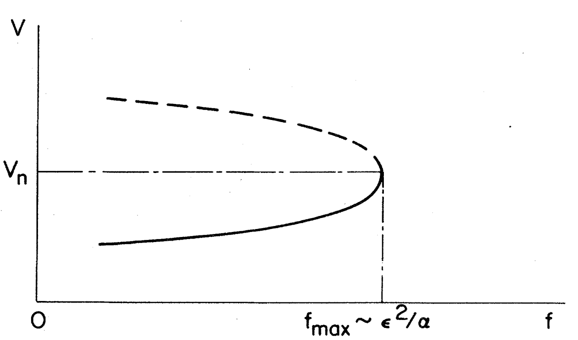

FIG. 33. The near-resonance dependence of a fluxon's velocity V on the driving force f ensuing from Eqs. (6.87) and (6.88). The upper (dashed) branch is unstable, while the lower (solid) one is stable.

Toda lattice closed into a ring and containing a mass impurity. These experiments showed that a moving soliton suffers anomalously fast decay due to resonant excitation of linear waves by periodic collisions of the soliton with the impurity.

It is necessary to note that the above resonant kinematic relations (6.86′) ignores the influence of the inhomogeneity on propagation of the linear waves emitted. As has been demonstrated by Malomed and Ustinov (1989b), this influence becomes important if the emission frequencies ω_n are sufficiently large. In that case, one must take into account not only the defect-induced perturbation of the local maximum supercurrent density accounted for by the term $\epsilon_1\delta(x)\sin u$ [see Eq. (1.19)], but also changes of the local densities of the junction's inductance and capacity, described by the terms $\epsilon_2\delta(x)u_x$ [see Eq. (3.65)] and $\epsilon_3\delta(x)u_{tt}$, respectively. For instance, in the case $\epsilon_3=\epsilon_1$, $\epsilon_2=0$, the coefficient R^2 of reflection of the linear waves from the pointlike defect, defined as the ratio of an energy flux carried by the reflected wave to that transferred by the incident wave, is (Malomed and Ustinov, 1989b)

$$R^2=\epsilon_1^2k^2/(4+\epsilon_1^2k^2)\ ,$$

where k is the radiation wave number ($\omega_n^2=1+k^2$). Thus the defects become impenetrable for the emitted waves ($1-R^2 \ll 1$) if $\epsilon_1^2k^2 \gg 1$. Large wave numbers of emitted radiation correspond to a fluxon velocity sufficiently close to the Swihart (limit) velocity $V_{\rm Sw}=1$. In this limit, the local defects remain "transparent" for the fluxons, while the emitted waves see the junction as divided into short junctions of length a [$a \equiv 2\pi/\kappa$, cf. Eq. (6.72)]. The corresponding resonant kinematic relation analogous to Eq. (6.86′) depends on two integers m,n: $V_{mn} \approx \kappa L(m/n)$, provided $\kappa m \gg 1$ (Malomed and Ustinov, 1989b).

Another circumstance complicating one's interpretation of the experimental data is the fact that, usually, real long Josephson junctions are linear with reflected edges, but not circular. When the junction is itself homogeneous, the edges give rise to another interesting dynamical effect if an external dc magnetic field h is imposed. A corresponding model is based on the SG equation with the right-hand side $-\gamma u_t - f$ [see Eqs. (1.16a) and (1.17)], supplemented by the boundary conditions $u_x(x=0) = u_x(x=l)=h$, where $l \gg 1$ is the length of the junction (Olsen *et al.*, 1986). Depending on the polarity, a fluxon sees the reflecting edge as a potential hill or well of height or depth $4\pi h$. The fluxon cannot be reflected (with a change of polarity) from the edge corresponding to the potential hill if the hill's height exceeds the fluxon's kinetic energy. In this case, the fluxon cannot perform the usual shuttle oscillations described by Fulton and Dynes (1973). However, if h is sufficiently large (and lies beyond the range of applicability of perturbation theory), another mode of motion, called the first Fiske step, is possible: the numerical experiments of Dueholm *et al.* (1981), Erné and Ferrigno (1983), and Soerensen *et al.* (1983) have demonstrated that the fluxon first moves in one direction, then, at an edge, it decays into plasma waves which travel in the opposite direction, and at another edge they reunite into a fluxon.

E. Exponentially weak emission from a kink oscillating in a potential well

Above we encountered an example of exponentially weak emission from a kink oscillating with a small frequency $\omega_0 \ll 1$ [see Eq. (6.46)]. In this section we shall consider this phenomenon in a general form, following the paper by Malomed (1987i). We shall assume that a kink oscillates in an effective potential well of characteristic size $\kappa^{-1} \gtrsim 1$, so that its adiabatic equation of motion coincides with that of a relativistic unit-mass particle moving under the action of the force $F(\xi) = -(\pi\epsilon/4) f(\kappa\xi)$:

$$\frac{d^2\xi}{dt^2} = \left[\frac{\pi\epsilon}{4}\right]\left[1 - \left[\frac{d\xi}{dt}\right]^2\right]^{3/2} f(\kappa\xi) \; . \tag{6.89}$$

This equation can be generated, for example, by the perturbation $\epsilon P = (\pi/2)\epsilon f(\kappa x)\sin(u/2)$ which arises in a variant of the Frenkel-Kontorova model if the substrate potential contains a subharmonic component $\sim \epsilon f(\kappa x)\cos(u/2)$ modulated on a large spatial scale $\sim \kappa^{-1}$ (cf. the model dealt with in Sec. IV.D).

If $f(0)=0$ and $\epsilon f'(0) < 0$, the kink may oscillate with a low frequency $\Omega \sim \sqrt{\epsilon\kappa}$ and an arbitrary amplitude $a \ll 1$ near the bottom of the well $\xi = 0$. According to Sec. VI.B, the intensity of emission at the frequency $n\Omega$ $(n > 1/\Omega)$ is determined by the squared nth Fourier coefficient F_n of the periodic function $F(t,\lambda)$ from Eq. (6.1). Since the frequency Ω is small, we are interested in an asymptotic form of F_n for large n. The smallness of the quantities F_n in this range is determined by the presence of the rapidly oscillating exponent

$$\exp[in\Omega t - ik\xi(t)] \tag{6.90}$$

in an integral that defines F_n. Therefore F_n may be estimated by means of the steepest-descent method. The result proves to be qualitatively different in two different cases. If a periodic solution $\xi(t)$ to Eq. (6.79) (the kink's law of small oscillations), analytically continued to the complex half-plane $\mathrm{Im}\,t > 0$, has a singularity at some point t_0 [in this case the estimate $\mathrm{Im}\,t_0 = \ln(1/\kappa a) + O(1)$ is valid with logarithmic accuracy], the energy emission rate (emission power) is exponentially small

$$W \sim (a\kappa)^{C/\sqrt{\epsilon\kappa}} \tag{6.91}$$

with some positive $C \sim 1$. If singularities of the analytically continued function $\xi(t)$ lie at infinity, the emission power contains, besides the main factor (6.91), an additional factor $[\ln(\epsilon\kappa)^{-1}]^{-S/\sqrt{\epsilon\kappa}}$ with some positive $S \sim 1$. In most cases, one can discern these two possibilities directly, analyzing a corresponding equation of motion of the type (6.90).

An important example (Malomed, 1987d; Kivshar and Malomed, 1988b) is the emission problem for a kink oscillating near the local attractive inhomogeneity (1.19) according to

$$\sinh\xi(t) = \sqrt{(2\epsilon - E)/E}\,\sin(\Omega t), \quad \Omega = \frac{\sqrt{E}}{2} \; , \tag{6.92}$$

where E is the binding energy $(0 < E \le 2\epsilon)$ (in this case $\kappa \sim 1$). Straightforward calculations yield the following results. The basic part of the emission is concentrated at the frequencies $\omega_m = \sqrt{2E}\,([1/\sqrt{2E}\,] + m)$, $m = 1,2,3,\ldots$, the corresponding powers being

$$W_m = (\sqrt{\epsilon}/64\pi)E^{5/4}([1/\sqrt{E}\,]+m) \times (m - \{1/\sqrt{E}\,\})^{-1/2} B^{2([1/\sqrt{E}\,]+m)} \; , \tag{6.93}$$

where $B \equiv (\sqrt{2\epsilon} - \sqrt{E}\,)/(\sqrt{2\epsilon} + \sqrt{E}\,)$, and $\{x\} \equiv x - [x]$. For the typical values $E \sim \epsilon$ (not $E \ll \epsilon$), the total power can be expressed in terms of the standard special function (Bateman and Erdelyi, 1953) $\Phi(z,s,\alpha) \equiv \sum_{n=0}^{\infty} z^n(\alpha+n)^{-s}$:

$$W = \sum_{m=1}^{\infty} W_m \approx (\sqrt{\epsilon}/64\pi)EB^{2([1/\sqrt{E}\,]+1)}\Phi(B^2, \tfrac{1}{2}, 1 - \{1/\sqrt{E}\,\}) \; . \tag{6.94}$$

Because $B < 1$, we may say that the result (6.94) is exponentially small in $\sqrt{E}$, i.e., in $\sqrt{\epsilon}$. It is easy to see that Eq. (6.94) is in accordance with the general formula (6.91).

Clearly Eqs. (6.93) and (6.94) are applicable provided $W \ll E\Omega \sim E^{3/2}$. As is shown in Fig. 34, this condition is not met in close proximity to the points $1/\sqrt{\epsilon} = \sqrt{2}(1-B)(1+B)^{-1}n$, where n are large integer numbers, when the corresponding multiple frequencies of

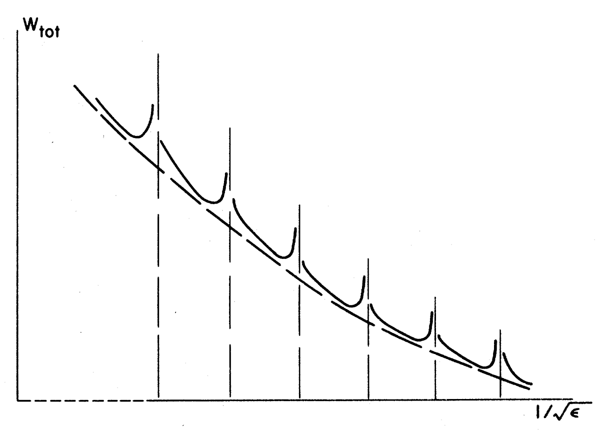

FIG. 34. The dependence $W_{\rm tot}(1/\sqrt{\epsilon})$, when the parameter $B \equiv (\sqrt{2\epsilon} - \sqrt{E}\,)/(\sqrt{2\epsilon} + \sqrt{E}\,)$ is fixed. The dashed envelope line is $W_{\rm tot} = \mathrm{const} \times \epsilon^{3/2} B^{\sqrt{2}(1+B)(1-B)^{-1}/\sqrt{\epsilon}}$ [see Eq. (6.94)].

the kink's motion $2n\Omega$ get too close to the plasma frequency (spectral gap's edge) $\omega_0=1$.

Equations (6.93) and (6.94) are also inapplicable when $E \ll \epsilon$ (the case of a weakly bound kink). In this case the emission of radiation is dominated by a relatively small part of the half-period $\tau/2=2\pi/\sqrt{E}$ of the kink's oscillations, singled out by the condition that the kink be sufficiently close to the emission-generating inhomogeneity: $\xi^2(t)\lesssim 1$. The duration of that part is $t\sim 1/\sqrt{\epsilon}\ll\tau/2$. Therefore, in the first approximation in the small parameter E/ϵ, we may calculate the energy emitted during one half-period of the kink's oscillations as the energy emitted during infinite time by a free kink moving past the inhomogeneity with zero velocity at infinity, i.e., with $E=0$. The corresponding spectral density and total emitted energy have been calculated by Malomed (1985). Dividing them by the half-period $2\pi/\sqrt{E}$, we find the spectral density of the emission power and the total power:

$$\mathcal{W}(q)=\pi^{-1}\sqrt{8\epsilon E}\exp[-2\sqrt{2}(1+q^2/2)/\sqrt{\epsilon}] , \qquad (6.95)$$

$$W=(2^{5/4}/\sqrt{\pi})\sqrt{E}\,\epsilon^{3/4}\exp(-2\sqrt{2/\epsilon}) . \qquad (6.96)$$

Note that these expressions, as well as (6.94), are exponentially small in $\sqrt{\epsilon}$. As can be seen from Eq. (6.95), the emission power is concentrated in the spectral range $q^2\sim\sqrt{\epsilon}$. In the approximation considered, the radiation spectrum (6.95) is continuous. In reality it is discrete, with the frequency gaps between neighboring lines $\Delta\omega\sim\Delta(q^2)\sim\sqrt{E}\ll\sqrt{\epsilon}$, and Eq. (6.95) should be regarded as a version of the genuine spectrum smoothed on a scale $\Delta\omega$, $\sqrt{E}\ll\Delta\omega\ll\sqrt{\epsilon}$.

F. Emission from a low-frequency sine-Gordon breather

Existence of exact breather solutions is a distinguishing feature of exactly integrable systems. While the existence of stationary one-soliton solutions (for instance, kinks) does not purport exact integrability, it is generally believed that only integrable systems possess exact oscillatory states (breathers) which lose no energy through emission (see, for example, Bullough *et al.*, 1980; Eleonskii *et al.*, 1984). It is natural to expect that a perturbed equation has solutions which, being locally close to exact breathers, slowly fade due to energy emission. The problem of evaluation of a perturbation-induced rate of energy emission is of fundamental interest. In this section we shall set forth results obtained so far for low-frequency breathers (2.66) ($\zeta\ll 1$) and also for general-form breathers (2.63) of nonsmall amplitudes ($\tan\mu\sim 1$).

First we consider, following Kivshar and Malomed (1987d, 1988d) radiative damping of a low-frequency breather under the action of perturbation (1.19), $P=\delta(x)\sin u$, assuming that the breather is quiescent and its center coincides with the location of the inhomogeneity ($x=0$) described by the perturbation (1.19).

The emission power is concentrated at the radiation frequencies

$$\Omega=\zeta\{[1/\zeta]+2m+1+\tfrac{1}{2}[1-(-1)^{[1/\zeta]}]\}\equiv\Omega_m , \qquad (6.97)$$

where [ ] represents the integer part, and $m=0,1,\ldots$. The emission power corresponding to the mth frequency (6.80) is

$$W_m=(\epsilon^2\zeta^2/4)[\Omega_m/(\Omega_m^2-1)^{1/2}]e^{-2\Omega_m}|G_m|^2 , \qquad (6.98)$$

where

$$G_m\equiv\Omega_m^2(a_m-b_m)-\Omega_m(a_m-5b_m)-4b_m ,$$
$$a_m\equiv 1+i(\Omega_m^2-1)^{1/2}(b_m/\Omega_m^2) , \qquad (6.99)$$
$$b\equiv[(\Omega_m^2-1)^{1/2}+i]^2/\Omega_m^2 .$$

Equations (6.98) and (6.99) seem rather cumbersome, and they would be far more cumbersome for perturbations other than (1.19). However, we can develop a simpler approach for calculating the total emission rate

$$W_{\rm tot}\equiv\sum_{m=0}^{\infty}W_m ,$$

which is of basic physical interest since, through energy conservation, this total emission rate determines the breather's radiative damping rate $d\zeta/dt$: $dE_b/dt=W$, or

$$d\zeta/dt=W_{\rm tot}(\zeta)/16\zeta , \qquad (6.100)$$

where $E_b\approx 8\zeta^2$ is the binding energy. Indeed, the energy emitted during one half-period $\tau/2=\pi/\zeta$ of the breather's oscillations is dominated by a relatively small portion of the half-period, when the kink and antikink bound inside the breather are strongly overlapped. The duration of that portion is ~ 1, and during it the breather solution (2.66) may be approximated by the limiting form (2.67). So, in the first approximation in ζ, the spectral density of the energy emitted by the breather during a half-period can be calculated as that emitted by the kink-antikink pair (2.67) during an infinite time:

$$\mathcal{E}(k)=(\pi\epsilon^2/4)Q(k)(1+k^2)^{-1}\exp(-2\sqrt{1+k^2}) , \qquad (6.101)$$

where

$$Q(k)\equiv(17k^4+86k^2+72)-(72+50k^2)\sqrt{1+k^2} .$$

It should be noted that the above approach has yielded the discrete (but sufficiently "dense") emission spectrum given by Eqs. (6.98) and (6.99), while the present approximation misses the discreteness of the spectrum and gives its smoothed continuous version (6.101). However, this approximation is sufficient to find $W_{\rm tot}$. Indeed, the total energy corresponding to Eq. (6.101) is

$$E_{\rm tot}=\int_{-\infty}^{+\infty}\mathcal{E}(k)dk=\beta\epsilon^2 , \qquad (6.102)$$

where numerical integration yields $\beta\approx 0.69$. Since the duration of the half-period is π/ζ, the total emission power is

$$W_{\rm tot}=E_{\rm tot}/(\pi/\zeta)=(\beta/\pi)\zeta\epsilon^2 . \qquad (6.103)$$

Then, inserting Eq. (6.103) into Eq. (6.100), we obtain

$$d\zeta/dt=(\beta/16\pi)\epsilon^2 \ . \tag{6.104}$$

For other perturbations calculations are more lengthy. For example, the perturbation (1.25), $P=\sin(2u)$, yields for $E_{\rm tot}$ Eqs. (6.103) and (6.104) with $\beta\approx 9.89$ (Malomed, 1985).

For a breather of a general form, when the parameter $\tan\mu$ is neither large nor small, accurate evaluation of the radiative decay rate poses severe technical difficulties. However, on a level of estimates, it is possible to develop a general description of the perturbation-induced evolution (radiative decay) of the breather, starting from an initial state in the form of a low-frequency breather and finishing at the final asymptotic stage of decay as a small-amplitude breather (Kivshar and Malomed, 1987d). According to Eq. (6.104), the breather remains low frequency during the time $t\lesssim t_1\sim\epsilon^{-2}$ for the perturbation (1.25), and $t\lesssim t_1\sim 10^2\epsilon^{-2}$ for (1.19). For a general-form breather ($\tan\mu\sim 1$), an estimate yields $W\sim\epsilon^2$, so that the duration t_2 of this stage of the breather's evolution is also $\sim\epsilon^{-2}$.

The subsequent radiative decay of the resultant small-amplitude breather, which is a problem of principle interest [Segur and Kruskal (1987); Kivshar and Malomed (1988a)], will be described in detail for various particular perturbations in the next section. To conclude this section, we estimate the time dependence of the characteristic radiation amplitude A. On the time scale $t\lesssim t_1,t_2$, the radiation energy $E_{\rm rad}$ grows with time as $Wt\sim\epsilon^2 t$, and the length l of the spatial region occupied by the radiation grows as $l\sim V_{\rm gr}t\sim kt\sim t$, where $V_{\rm gr}=k/\sqrt{1+k^2}$ is the group velocity and characteristic radiation wave numbers are $k\sim 1$. Thus we arrive at the estimate $A\sim\sqrt{E_{\rm rad}/l}\sim\epsilon$, which is independent of time. At times $t\gg\epsilon^{-2}$, when almost all the energy of the initial breather has already been converted into the radiation energy ($E_{\rm rad}\approx 16$), this energy remains almost constant, while the length l continues to grow $\sim t$, so that at this stage

$$A\sim t^{-1/2} \ .$$

G. Radiative decay of a small-amplitude sine-Gordon breather in external fields

1. Nonresonant case

Proceeding to a study of perturbation-induced emission from the small-amplitude breather (2.65) (Malomed, 1987d), we first consider the perturbation (1.17), $P=f(t)$, with $f(t)=\sin(\Omega t)$ (Malomed, 1987d, 1987e). As in the one-kink problem, we need to renormalize the wave field according to Eq. (6.15). Then, substituting the renormalized perturbation (6.16) [or (6.19)] into the basic formulas (6.1) and (6.2), we obtain in the nonresonant case $\omega-1\gtrsim 1$ (where $\omega=\sqrt{k^2+1}$ is, as before, the radiation frequency)

$$\frac{dB}{dt}=-2i(\pi\tilde{\epsilon}/3)\sqrt{\omega^2-1}\,\mathrm{csch}(\pi\sqrt{\omega^2-1}/2\mu)\sin^3 t \times e^{-it}\sin(\Omega t)e^{i\omega t}+\cdots \ , \tag{6.105}$$

$\tilde{\epsilon}$ being defined in Eq. (6.16). Rather unexpectedly, all the higher terms of the expansion (6.105), though proportional to growing powers of μ^2, yield contributions to dB/dt of the same order as the first term explicitly written in the right-hand side of Eq. (6.88). With regard to this circumstance, we obtain the following results: if Ω belongs to the interval

$$n<\Omega<n+1, \quad n=0,1,2,\ldots , \tag{6.106}$$

the main (most intensive) emission takes place at the frequency

$$\omega=|n-\Omega+\tfrac{1}{2}[1+3(-1)^n]| \ , \tag{6.107}$$

the emission power being

$$W=(\pi C_n\tilde{\epsilon}/3)^2\omega\sqrt{\omega^2-1}\exp(-\pi\sqrt{\omega^2-1}/\mu) \ , \tag{6.108}$$

where C_n are some coefficients depending on n and, generally speaking, on ω. Thus the coefficients C_n are contributed to by all the terms of the expansions (6.105). However, numerical coefficients in front of contributions from higher terms rapidly decrease with the growth of the term's number, and for $n\le 2$ the coefficients C_n are determined, with satisfactory accuracy, by the contribution from the first term of Eq. (6.105): $C_0=3$, $C_1=0$, $C_2=1$. Values of these coefficients can be calculated accurately by means of a more laborious method of Segur and Kruskal (1987) which is based on matched asymptotic expansions.

The case of the constant external field (1.17) ($\Omega=0$) is covered by Eqs. (6.107) and (6.108) with $n=0$ [see (6.106)]; in particular, the corresponding radiation frequency is $\omega=2$, and the emission power

$$W_2=2\sqrt{3}(\pi C_0\epsilon/3)^2\exp(-\sqrt{3}\pi/\mu) \approx(2\pi^2/3^{3/2})\epsilon^2\exp(-\sqrt{3}\pi/\mu) \ . \tag{6.109}$$

The case of the constant external field (1.21) [$P=\sin(u/2)$] requires special consideration. In view of $\mu\ll 1$, we expand the perturbation: $\sin(u/2)=u/2-u^3/24+u^5/3840-\ldots$. The first two terms of the expansion may be comprised in analogous terms of the expansion of the unperturbed SG equation by means of the renormalizing transformation $u=U(1-3\epsilon/16)$, $x=X(1+11\epsilon/32)$, $t=T(1+11\epsilon/32)$. The renormalized equation is

$$U_{TT}-U_{XX}+\sin U=\epsilon[\sin(U/2)-\tfrac{11}{16}\sin U+\tfrac{3}{16}U\cos U] =\epsilon_R U^5+O(\epsilon_R U^7) \ , \tag{6.110}$$

$$\epsilon_R\equiv 3\epsilon/1280 \ . \tag{6.111}$$

The perturbation on the right-hand side of Eq. (6.110) generates an emission at the frequency $\omega=3$, the emission power being

$$W_3=(36\sqrt{2}/5)\pi^2\epsilon^2\exp(-2\sqrt{2}\pi/\mu) . \qquad (6.112)$$

As in the above problem, higher terms of the expansion (6.110) give contributions to W_3 of the same order as (6.112), but with smaller numerical coefficients.

2. Resonant case

In the resonant case $|\delta\Omega|\ll 1$ ($\delta\Omega\equiv\Omega-1$), contributions to the emission power from higher terms of the above-mentioned expansions in powers of μ are immaterial. Due to the presence of the breather's internal frequency $\omega_{\rm br}=\cos\mu\approx 1-\mu^2/2$, the effective perturbation (6.16) may generate resonant emission at two frequencies: Ω and $2\omega_{\rm br}-\Omega$, the corresponding radiation wave numbers being $k^2=2\delta\Omega'$ and $k^2=-2(\delta\Omega'+\mu^2)$, where, as above, $\delta\Omega'\equiv\Omega+a^2/8$ [see Eq. (6.21)], and a is defined by Eq. (6.18). The final expression for the emission power (for the sake of simplicity, we set the dissipative constant $\gamma=0$, but generalization to $\gamma\neq 0$ is straightforward) is (Malomed, 1987d, 1987e)

$$W=(\pi a)^2\{\sqrt{2}\theta(\delta\Omega')(\delta\Omega')^{5/2}(\mu^2+2\delta\Omega')^{-2}\operatorname{csch}^2(\pi\sqrt{\delta\Omega'/2}/\mu) + \tfrac{1}{16}\theta(-\delta\Omega'-\mu^2)\sqrt{2|\delta\Omega'+\mu^2|}[2\delta\Omega'/(2\delta\Omega'+\mu^2)]^2\operatorname{csch}^2(\pi\sqrt{|\Omega'+\mu^2|/2}/\mu)\} , \qquad (6.113)$$

where $\theta(z)=1$ if $z>0$, and $\theta(z)=0$ if $z<0$. Equation (6.113) is valid provided $|\delta\Omega'|\ll 1$, but the relation between the small quantities $\delta\Omega'$ and μ may be arbitrary, and, in contrast to the resonant-emission problem for a kink, we do not need $a^4\ll\delta\Omega'$. Note that, unlike the nonresonant expressions (6.108), (6.109), and (6.112), Eq. (6.113) in general is not exponentially small in μ.

For the time-dependent perturbation (1.21), $P=\sin(\Omega t)\sin(u/2)$, the low-frequency case $\mu^2/2\le\Omega\ll 1$ is the resonant one due to the presence of $\omega_{\rm br}$. The resonant emission takes place at wave numbers $k=\pm\sqrt{2\Omega-\mu^2}$, the emission power being (here we again set $\gamma=0$)

$$W=(\pi\epsilon\Omega/4)^2(2\Omega-\mu^2)^{-1/2}\operatorname{sech}^2(\pi\sqrt{2\Omega-\mu^2}/2\mu) \qquad (6.114)$$

(Malomed, 1987d, 1987e). As in Eq. (6.113), W in Eq. (6.114) is not exponentially small in μ.

Following the approach of Malomed (1987d, 1987e), let us consider emission of radiation by a small-amplitude breather under the action of the time-dependent external field (1.21), $P=f(t)\sin(u/2)$, with random $f(t)$ defined according to Eq. (5.6). The total averaged emission power is dominated by the resonant Fourier components of the random field. As one can conclude from the preceding subsection, for the perturbation (1.21) these are the low-frequency components ($\Omega\ll 1$). In the present problem, it is convenient to calculate the spectral density of the emission power defined with respect to the radiation frequency: $\mathcal{W}(\omega)\equiv\mathcal{W}(k)(dk/d\omega)$, where $\mathcal{W}(k)$ is defined in Eq. (6.5). Calculation of $\mathcal{W}(\omega)$ coincides, actually, with the calculation of the emission power (6.114), with $(\omega-1)$ playing the role of $(\Omega-\mu^2/2)$:

$$\langle\mathcal{W}(\omega)\rangle=(\pi/2^{11/2})\epsilon^2(\omega-1)^{-1/2} \times\left[\frac{\mu^2}{2}+(\omega-1)\right]^2\operatorname{sech}^2(\pi\sqrt{\omega-1}/\sqrt{2}\mu) . \qquad (6.115)$$

The singularity of Eq. (6.115) at $\omega-1=0$ will be smoothed if one takes into account dissipation [cf. the relation between Eqs. (6.13) and (6.12)]. However, the total averaged emission power converges despite the singularity:

$$\langle W\rangle=\int_1^\infty\langle\mathcal{W}(\omega)\rangle d\omega=\tfrac{2}{15}\epsilon^2\mu^5 . \qquad (6.116)$$

Note that the integral (6.116) is dominated by the resonant spectral range $(\omega-1)\sim\mu^2\ll 1$; that is why $\langle W\rangle$ is not exponentially small in μ [compare Eqs. (6.108), (6.110), and (6.112) to Eqs. (6.113) and (6.114)]. If the dissipative term $-\gamma u_t$ is taken into account, Eq. (6.116) remains valid provided $\gamma\ll\mu^2$.

H. Radiative decay of a small-amplitude breather interacting with a localized inhomogeneity

1. Weak perturbation

In this subsection we shall consider the dynamics of a breather interacting with the localized inhomogeneity (1.19), $P=\delta(x)\sin U$, following the approach developed by Malomed (1987d, 1987e). First of all, we must describe the breather's dynamics in the adiabatic approximation. To that end, we employ the Hamiltonian formalism. The full Hamiltonian of the breather (2.63) in the presence of the perturbation (1.19) can be readily found in the first approximation [see also Eq. (5.53)]:

$$H=16\sin\mu\,(1+V^2/2)-8\epsilon\cot^2\mu\,\sin^2\Psi\cosh^2\phi \times(\sin^2\Psi+\cot^2\mu\cosh^2\phi)^{-1} , \qquad (6.117)$$

where $\phi\equiv\xi\sin\mu$ (we assume $V^2\ll 1$). Then the equations for the canonical variables (Zakharov *et al.*, 1980) $4V$, 16μ, 4ϕ, and $-\Psi$ take the form

$$\frac{d\psi}{dt}=\cos\mu+\epsilon S(\tan\mu/\cos^2\mu)\sin^2\Psi\operatorname{sech}^2\phi , \qquad (6.118)$$

$$\frac{d\mu}{dt}=-(\epsilon/2)S\tan^2\mu\sin(2\Psi)\mathrm{sech}^2\phi\ , \qquad (6.119)$$

$$\frac{d\xi}{dt}=V,\quad \frac{dV}{dt}=\epsilon S\tan^2\mu\sin^2\Psi\tanh\phi\ , \qquad (6.120)$$

where

$$S\equiv(\tan^2\mu\sin^2\Psi\,\mathrm{sech}^2\phi-1)(\tan^2\mu\sin^2\Psi\,\mathrm{sech}^2\phi+1)^{-3}\ .$$

The motion of the small-amplitude breather ($\mu^2\ll1$) described by Eqs. (6.118)–(6.120) may be naturally represented as a slow perturbation-induced motion of its center of mass on a background of fast internal vibrations. The simplest way to describe the center-of-mass motion is to average the Hamiltonian (6.117) with respect to the internal vibrations:

$$\langle H\rangle=16(\mu-\mu^3/6)+8\mu V^2-4\epsilon\mu^2\mathrm{sech}^2(\mu x)\ . \qquad (6.121)$$

Further, Eq. (6.121) yields the averaged correction $\langle\Delta\omega_{\mathrm{br}}\rangle$ to the unperturbed frequency $\omega_{\mathrm{br}}^{(0)}=\cos\mu\approx1-\mu^2/2$ of the internal vibrations:

$$\langle\Delta\omega_{\mathrm{br}}\rangle=-(\epsilon\mu/2)\mathrm{sech}^2(\mu\xi)\ . \qquad (6.122)$$

We shall consider the case $\epsilon>0$ only, when, as can be seen from Eq. (6.121), the inhomogeneity attracts the breather. The Hamiltonian (6.117) brings us to the law of motion

$$\mathrm{sech}^2[\mu\xi(t)]=2E/\{(4\epsilon\mu^2+E)-(4\epsilon\mu^2-E)\cos[(\sqrt{\mu E/2}t)]\}\ . \qquad (6.123)$$

Here E is the pinning energy ($0<E\le E_{\max}\equiv4\epsilon\mu^2$); $E=E_{\max}$ corresponds to the ground state $\xi(t)=0$.

In this subsection we assume

$$\epsilon^2\ll8\mu^2 \qquad (6.124)$$

to provide that perturbation-induced corrections to the breather's shape are small ("weak perturbation"); the case when the inequality (6.124) does not hold will be considered in the following subsection.

The superposition of slow external oscillations (6.123) and fast internal vibrations may result in resonantlike emission of long-wave radiation. We shall consider only the case of small oscillations: $P\equiv(E_{\max}-E)/(E_{\max}+E)\ll1$. Then the radiation wave number [with regard to Eq. (6.122)] is

$$k^2=2\sqrt{2\epsilon\mu^3}-\mu^2-\epsilon\mu\ , \qquad (6.125)$$

and the emission power is

$$W^{(1)}=(\epsilon P\mu)^2k^{-1}[(k^2+5\mu^2/2)/(k^2+\mu^2)]^2 \qquad (6.126)$$

[the superscript (1) indicates that the radiation frequency $\omega=\sqrt{1+k^2}$ is close to one; cf. $W^{(3)}$ below]. The emission considered exists provided $k^2>0$, i.e., according to Eq. (6.125), $\mu^2<(17+12\sqrt{2})\epsilon^2$. This inequality may be regarded as compatible with Eq. (6.124). Due to the inharmonicity of the small oscillations [see Eq. (6.123)], the emission spectrum also contains the wave numbers $k_n^2=n\sqrt{2\epsilon\mu^3}-\mu^2-\epsilon\mu$, $n=3,4,5,\ldots$, the corresponding emission powers being proportional to P^n [in the case $n=2$ we return to Eqs. (6.125) and (6.126)].

If one takes into account corrections $\sim\mu^2$, the additional emission takes place at the frequencies

$$\omega_n^{(3)}=3\pm n\sqrt{\mu E/2},\quad n=0,1,2,3,\ldots\ . \qquad (6.127)$$

It is possible to find the total emission power $W^{(3)}$ for the entire multiplet (6.122):

$$W^{(3)}=(3/16\sqrt{2})\epsilon^{3/2}\mu^5\sqrt{E}\times[625+\tfrac{81}{2}(E_{\max}-E)^2/(E_{\max}+E)^2]\ . \qquad (6.128)$$

In contrast to the emission problems for a small-amplitude breather considered in Sec. VI.G, here corrections of higher orders in μ^2 yield relatively small contributions to W; on the other hand, in the present case the nonresonant emission power is not exponentially small in μ [compare Eq. (6.128) with Eqs. (6.108), (6.110), and (6.112)]. It is also possible to find the emission power corresponding to the central line of the multiplet, i.e., to the frequency (6.127) with $n=0$:

$$W_0^{(3)}=(3\sqrt{2}/16\pi^2)\epsilon\mu^4E[16K(\tilde{p})+9E(\tilde{p})]^2\ , \qquad (6.129)$$

where $\tilde{P}\equiv(E_{\max}-E)/E_{\max}$, $K(\tilde{P})$ and $E(\tilde{P})$ are the complete elliptic integrals. For the unsplit singlet (the ground state $E=E_{\max}$), both Eqs. (6.128) and (6.129) yield the same $W_0^{(3)}=625(3\sqrt{2}/16)\epsilon^2\mu^6$. With an increase in $\tilde{P}$, the total power (6.128) subsides monotonically. It is also interesting that at $E\to0$,

$$W_0^{(3)}/W^{(3)}\approx(2048/1331\pi^2)\sqrt{E/E_{\max}}\ln^2(E/E_{\max})\ ,$$

i.e., the central line of the multiplet becomes relatively weaker.

2. Strong perturbation

Let us proceed to the opposite case from Eq. (6.124) ("strong perturbation;" Kivshar and Malomed, 1988b). In this case, the shape of a small-amplitude breather is essentially different from (2.60). Indeed, according to Eq. (1.19), the shape is determined by the unperturbed SG equation with the boundary conditions

$$\begin{aligned}&u(x=+0)=u(x=-0)\ ,\\&u_x(x=-0)-u_x(x=+0)=\epsilon\sin u(x=0)\ .\end{aligned} \qquad (6.130)$$

Following along the lines of the asymptotic method (Kosevich and Kovalev, 1974, 1975; Eleonskii *et al.*, 1984), we look for a solution to the SG equation with the boundary conditions (6.130) in the form expanded in powers of the small amplitude μ,

$$u(x,t)=A(x)\cos(\omega t)+B(x)\cos(3\omega t)+\cdots\ , \qquad (6.131)$$

where

$$A(x)=\mu A_1(x)+\mu^3 A_3(x)+\cdots\ ,$$
$$B(x)=\mu^3 B_3(x)+\cdots\ . \tag{6.132}$$

Substituting Eqs. (6.131) and (6.132) into the unperturbed SG equation yields in the first approximation the relation

$$\omega\equiv\sqrt{1-\mu^2}\approx 1-\mu^2/2\ , \tag{6.133}$$

and the equation for $A_1(x)$:

$$d^2A_1/d(\mu x)^2-A_1+\tfrac{1}{8}A_1^3=0\ . \tag{6.134}$$

The solution to Eq. (6.134) bounded at infinity is well known:

$$A_1(x)=4\,\mathrm{sech}[\mu(x+x_0)]\ , \tag{6.135}$$

where x_0 is an arbitrary constant. Substituting Eq. (6.135) into Eqs. (6.131), and then Eq. (6.131) into the boundary conditions (6.130), we obtain the solution in the form

$$u(x,t)\approx 4\mu\,\mathrm{sech}[\mu(|x|+x_0)]\cos(\omega t)\ , \tag{6.136}$$

where x_0 is now determined by the relation

$$\tanh(\mu x_0)=\epsilon/2\mu\ . \tag{6.137}$$

In the next approximations, substituting Eqs. (6.131) and (6.132) into the SG equation enables us to express the functions $A_3(x), B_3(x), \ldots$ in terms of $A_1(x)$ (see Kosevich and Kovalev, 1974). However, in the approximation $\sim\mu^3$, we cannot satisfy the boundary condition (6.130), since we have no more arbitrary parameters in the solution if we admit only functions vanishing at $|x|\to\infty$. It is known (Kosevich and Kovalev, 1975) that, to resolve this problem, one should supplement the solution (6.131) and (6.132) by traveling waves escaping from the breather. So, with accuracy up to the terms $\sim\mu^3$, the full solution takes the form

$$u(x,t)=4\mu\,\mathrm{sech}z\left[\left[1+\frac{\mu^2}{36}(2-\mathrm{sech}^2 z)\right]\times\cos(\omega t)-\frac{\mu^2}{12}\cos(3\omega t)\mathrm{sech}^2 z\right]+A_0\cos(\sqrt{8}|x|-3\omega t)\ , \tag{6.138}$$

where ω is determined by Eq. (6.133), $z\equiv\mu(|x|+x_0)$, and

$$A_0=-\tfrac{1}{3}\mu\tanh(\mu x_0)\mathrm{sech}^3(\mu x_0)\ . \tag{6.139}$$

Within the present accuracy, Eq. (6.137) for x_0 is replaced by

$$\tanh(\mu x_0)\approx\frac{\epsilon}{2\mu}\{1+\mu^2[\tfrac{217}{18}-\tfrac{73}{6}(\epsilon/2\mu)^2]\}\ . \tag{6.140}$$

The form of the "distorted" breather (6.138)–(6.140) for the two cases $\epsilon\lessgtr 0$ is depicted in Figs. 35(a) and 35(b).

One should keep in mind that Eq. (6.137) has no real solution, i.e., the breather does not exist, unless

$$\mu>|\epsilon|/2\ . \tag{6.141}$$

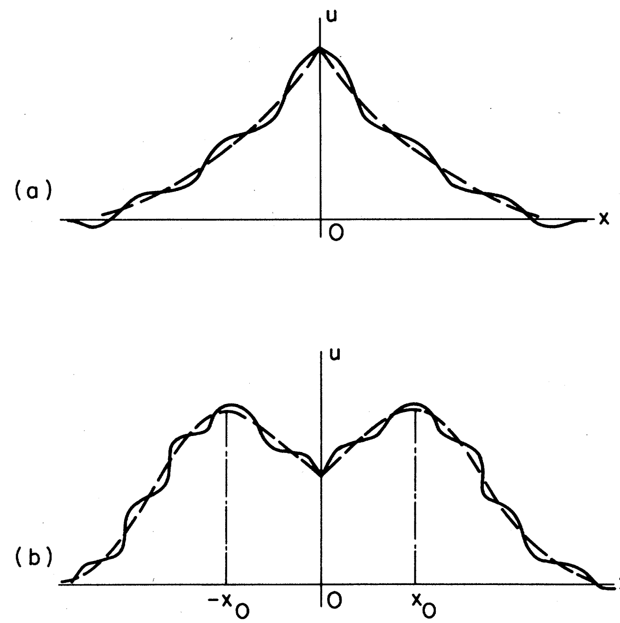

FIG. 35. The shape of the breather (6.138)–(6.140): (a) $\epsilon>0$; (b) $\epsilon<0$. The dashed line corresponds to the first approximation (6.136) and (6.137). The solid line is wavy due to the radiation part of the wave field [the last term on the right-hand side of Eq. (6.138)].

The energy $E_{\rm br}$ of the breather can be easily calculated in the approximation corresponding to Eqs. (6.136) and (6.137):

$$E_{\rm br}=16(\mu-\epsilon/2)\ . \tag{6.142}$$

One can find the rate of energy emission from the breather in an approximation corresponding to Eqs. (6.138)–(6.140):

$$W\equiv-dE_{\rm br}/dt=-\int_{-\infty}^{+\infty}dx\,\langle u_t u_x\rangle\ , \tag{6.143}$$

where the angular brackets stand for the time averaging. The result is

$$W=\frac{3\times 625}{8\sqrt{2}}\epsilon^2[\mu^2-(\epsilon/2)^2]^3\ , \tag{6.144}$$

the radiation frequency being $\omega=3$.

Particularly, in the limiting case

$$0<\nu\equiv\mu-\epsilon/2\ll\epsilon/2 \tag{6.145}$$

the breather (6.136) takes the form

$$u(x,t)\approx 4\nu\exp(-\mu|x|)\cos(\omega_f t)\ ,$$
$$\omega_f=1-\epsilon^2/8\ , \tag{6.146}$$

and, according to Eqs. (6.142) and (6.143),

$$E_{\rm br}=16\nu,\quad W=(3\times 625/8\sqrt{2})\epsilon^5\nu^3\ , \tag{6.147}$$

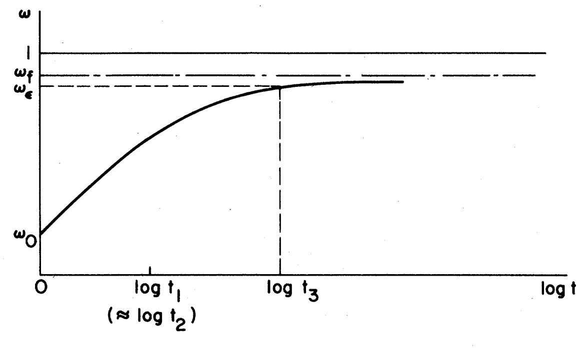

FIG. 36. Evolution of a breather's frequency $\omega \equiv \cos\mu$ over a large time scale under the action of radiative damping. The characteristic values of the frequency are $\omega_f \equiv \omega_{\text{final}} \approx 1-\epsilon^2/8$ [this corresponds to the breather (6.146) with the zero amplitude $\nu \equiv \mu - \epsilon/2 = 0$]; $\omega_f - \omega_\epsilon \sim \epsilon^2$. It is assumed that the initial frequency ω_0 lies in the region $|\epsilon| \ll \omega_0 \ll 1$.

i.e., the parameter ν defined in Eq. (6.145) plays the role of the breather's effective amplitude.[7] The effective amplitude fades, according to Eqs. (6.147), as

$$d\nu/dt = -(3\times 625/128\sqrt{2})\epsilon^5\nu^3 \ . \tag{6.148}$$

As we see from Eq. (6.148), ν fades as $1/\sqrt{\epsilon^2 t}$, provided $t \gg \epsilon^{-5}$.

Now, combining the results of Sec. VI.F with those of this section, we can depict in a general form (Fig. 36) the perturbation-induced evolution of a breather interacting with a localized inhomogeneity (Kivshar and Malomed, 1987d). In Fig. 36, where the breather's frequency is shown as a function of time, $t_3 \sim \epsilon^{-7}$ stands for a characteristic time of evolution during which the breather's amplitude μ falls to a level $\mu \sim \epsilon$, so that the perturbation becomes "strong" in the sense specified above; ω_ϵ is a characteristic internal frequency attained by the breather at $t \sim t_3$.

I. Emission from a nonlinear Schrödinger soliton and sine-Gordon breather scattered by local inhomogeneities

In this section we shall deal with a soliton described by the conservatively perturbed NS equation

$$iu_t + u_{xx} + 2|u|^2 u = -\epsilon f(x) u \tag{6.149}$$

(ϵ is real). Clearly, Eq. (6.149) conserves "the number of plasmons" $N = \int_{-\infty}^{+\infty} |u|^2 dx$. When the soliton, moving with velocity V, is scattered by an inhomogeneity described by the function $f(x)$, it will emit some of the "plasmons" bound inside it. For a localized function $f(x)$, a general expression for the spectral density (2.43b) of the emitted plasmon number has been calculated under the condition $V^2 \gg \epsilon\eta$ by Kivshar, Kosevich, and Chubykalo (1987c, 1987d):

$$n_{\text{rad}}(\lambda) = \epsilon^2 \tilde{n}^{(1)}_{\text{rad}}(\lambda) \times \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' f(x) f(x') e^{i\beta(\lambda)(x-x')} \ , \tag{6.150}$$

where

$$\tilde{n}^{(1)}_{\text{rad}}(\lambda) = (\pi/V^4)\beta^2(\lambda) \times \text{sech}^2\left[\pi\left[\lambda^2 - \frac{V^2}{16} + \eta^2\right]\Big/\eta V\right] \ , \tag{6.151}$$

$$\beta(\lambda) \equiv 4[(\lambda + V/4)^2 + \eta^2]/V \ . \tag{6.152}$$

1. An isolated inhomogeneity

For an isolated inhomogeneity, $f(x) = \delta(x)$, the spectral density (6.150) reduces the form $\epsilon^2 \tilde{n}^{(1)}_{\text{rad}}(\lambda)$, where $\tilde{n}^{(1)}_{\text{rad}}(\lambda)$ is defined in Eq. (6.151). It is relevant to note that, in fact, the same problem can be formulated as the emission of plasma waves from a small-amplitude breather described by the perturbed SG equation

$$u_{tt} - u_{xx} + \sin u = \epsilon\delta(x)\sin u \ . \tag{6.153}$$

Within the framework of this formulation of the problem, the spectral density $\mathcal{E}(k)$ of the emitted energy has been calculated by Malomed (1987d) for two cases: $V^2 \ll \epsilon\mu^2$ and $\mu^2 \ll V^2 < 1$. In the former case,

$$\mathcal{E}(k) = \sqrt{8\epsilon/\mu}\exp[-\sqrt{2/\epsilon\mu^3}(k^2+\mu^2)] \ , \tag{6.154}$$

and the total emitted energy is

$$E \equiv \int_{-\infty}^{+\infty} \mathcal{E}(k)dk = \sqrt{\pi}(32\epsilon^3\mu)^{1/2}\exp(-\sqrt{2\mu/\epsilon}) \ .$$

In the latter case

$$\mathcal{E}(k) = \frac{\pi\epsilon^2}{V^2}(1-V^2)\theta(-k) \times \text{sech}^2\left[\frac{\pi}{2\mu V}[\sqrt{(1+k^2)(1-V^2)} - 1]\right] \ , \tag{6.155}$$

and

$$E = 4\mu\epsilon^2\sqrt{1-V^2}/V^2 \ . \tag{6.156}$$

Since μ is a small parameter, one can see from Eq. (6.155) that the emitted energy is concentrated in a narrow spec-

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[7]As a matter of fact, the breather (6.146) is a well-known inhomogeneity-pinned localized solution of the linearized SG equation with the *linearized* perturbation (1.19). Excitation of a localized mode of this type as a result of a kink-inhomogeneity collision has been studied, in different contexts, by Nakamura (1978), Yoshida and Sakuma (1978, 1982a, 1982b), Watanabe and Toda (1981), Klinker and Latterborn (1983), and others.

tral range of width $\Delta k \sim \mu$ near the point $k=-V/\sqrt{1-V^2}$. It is evident that in the nonrelativistic limit $V^2 \ll 1$ Eq. (6.156) coincides with an analogous expression ensuing from Eqs. (6.151) and (6.152).

Coming back to the formulation based upon Eq. (6.149), it is convenient to describe the scattering by the reflection coefficient R, i.e., the relative number of plasmons bound initially inside the soliton and eventually emitted backwards. According to Kivshar, Kosevich, and Chubykalo (1987c),

$$R \approx R_0 \equiv \epsilon^2/V^2, \quad \eta^2 \ll V^2 \tag{6.157}$$

and

$$R=\pi R_0 \left[\frac{\alpha}{8}\right]^{7/2} e^{-\pi\alpha/2}, \quad \alpha \equiv 4\eta/V, \quad \eta^2 \gg V^2 . \tag{6.158}$$

Note that Eq. (6.157) coincides with the reflection coefficient for the monochromatic wave described by the linearized equation (6.149). The full dependence $R(\eta/V)$ is shown in Fig. 37. Similar numerical and analytical results have been obtained by Li *et al.* (1988).

Using the conservation of the plasmon number and energy, Kivshar, Kosevich, and Chubykalo (1987c) have found the changes in the amplitude and velocity of the soliton due to radiative losses accompanying the scattering.

In addition, scattering of a slow soliton ($V^2 \ll \epsilon\eta$) has been considered, and a threshold velocity that admits radiative capture of the slow soliton by the attractive inhomogeneity ($\epsilon>0$) has been found [Kivshar, Kosevich, and Chubykalo, 1987c; cf. Eq. (6.67)]:

$$V_{\rm thr}^2=\sqrt{2}\pi(\epsilon/\eta)^{3/4}\exp(-2\sqrt{\eta/\epsilon}) .$$

To conclude the discussion of this problem, let us consider, within the framework of the SG equation, radiative effects accompanying the collision of a moving small-amplitude breather with the localized dissipative inhomogeneity described by the perturbing term $-\epsilon\delta(x)u_t$ ($\epsilon>0$) (Malomed, 1987d). The expression for the spectral density of the emitted energy valid in the case $V^2 \gg \mu^2$ differs from Eq. (6.155) by the multiplier $(1-V^2)^{-1/2}$, and by the fact that the multiplier $\theta(-k)$ is absent, i.e., in this case the energy is emitted symmetrically to the left and to the right. Accordingly, the total emitted energy is $2(1-V^2)^{-1}$ times that of Eq. (6.156).

2. A system of inhomogeneities

Let us proceed to the case when the function $f(x)$ in Eq. (6.149) describes two localized inhomogeneities, $f(x)=\delta(x)+\delta(x+a)$ (Kivshar, Kosevich, and Chubykalo, 1987d). If the distance a between the two inhomogeneities is not large in comparison with the soliton's width $\sim\eta^{-1}$, one can observe nonlinear interference in the scattering process. Under the condition of weak nonlinearity $\eta^2 \ll V^2$, the resultant reflection coefficient takes the form

$$R^{(2)}=2R_0\left[1+\frac{2\alpha d}{\sinh(2\alpha d)}\cos(2d)\right] , \tag{6.159}$$

where R_0 is defined in Eq. (6.157), and $d \equiv aV/2$. In the limit $\alpha d \to 0$, Eq. (6.159) goes over into that for the linearized Eq. (6.149). In the opposite case ($\eta^2 \gg V^2$),

$$R^{(2)}=2R(\eta/V)\left[1+\frac{\cos(\frac{1}{2}\alpha^2 d+\gamma)}{(1+c)^{1/4}}\right] , \tag{6.160}$$

where $c \equiv 2a\eta/\pi$, $\gamma \equiv \frac{1}{2}\tan^{-1}c$, and $R(\eta/V)$ is defined in Eq. (6.157). The dependence of $R^{(2)}$ on the quantity $d \equiv aV/2$ is shown in Fig. 38. Similar results have been obtained by Li *et al.* (1988).

Kivshar, Kosevich, and Chubykalo (1987d) have also

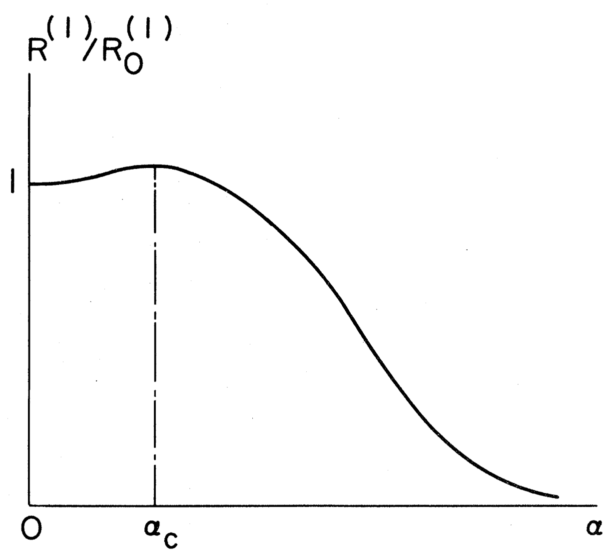

FIG. 37. Normalized reflection coefficient for the NS soliton vs the parameter $\alpha \equiv 4\eta/V$. $R_0^{(1)}$ is the reflection coefficient for a linear wave packet of the same spectral composition as the soliton; $\alpha_c \approx 0.178$.

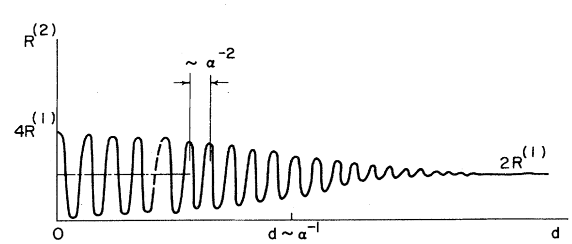

FIG. 38. Reflection coefficient for the NS soliton as a function of the quantity $d=aV/2$ in the case of two identical pointlike impurities (it is assumed that $\alpha \equiv 4\eta/V \gg 1$). $R^{(1)}$ is the reflection coefficient for an isolated pointlike impurity.

considered emission from the soliton scattered by a regular or random set that contains a finite number of the local inhomogeneities.

J. Nonlinear Schrödinger soliton in a model of a nonlinear fiber with dissipation and pumping

The investigation of conditions for propagation of solitons in one-mode optical fibers is of significant current interest (see, for example, Kodama, 1985a, 1985b, 1985c, 1985d). The essential circumstance is the presence of dissipation, which causes damping of a soliton. To compensate for dissipative effects, it was proposed to use periodic pumping in the spatial coordinate. In the simplest case, a waveguide with dissipation and pumping is described by the NS equation with the perturbation (Kodama and Hasegawa, 1982)

$$u_t - iu_{xx} - 2i|u|^2 u = \gamma u \left[-1 + \tau \sum_{n=-\infty}^{+\infty} \delta(t - n\tau) \right] , \tag{6.161}$$

where $u(x,t)$ is the complex envelope of the electromagnetic field, t and x have the sense of the spatial and time variables (note the difference from the usual notation), γ is the dissipation constant, τ is the spatial period of the pumping, and we assume that the dissipation and pumping may be regarded as small perturbations, i.e., $\gamma\tau \ll 1$ and $\gamma/\eta^2 \ll 1$. The ratio of the coefficients in front of the two terms on the right-hand side of Eq. (6.161) is chosen so that it provides, in the first approximation with respect to the two small parameters mentioned, full compensation of the dissipation for all values of the soliton amplitude η and velocity V (Kodama and Hasegawa, 1982). At the same time, it is clear that, under the action of the pumping pulses, a soliton generates radiation in the form of small-amplitude electromagnetic waves, which can be received as "noise" when using the fiber as an optical communication line. In this section, following the approach of Malomed (1986a, 1987a), we give an expression for a stationary level of radiation in the case of one soliton propagating in the fiber, and we demonstrate that the ratio of the number of light quanta scattered in the form of radiation to the number bound inside the soliton is $\gamma\tau/2$, i.e., small due to the conditions adopted. This means that an optical communication line operating with solitonic pulses may be efficient indeed.

Applying the general IST evolution equation (2.47) for the radiation amplitudes to the perturbed system (6.16),[8] one can find an increment of the radiation field amplitude $b(\lambda)$ generated by the soliton under the action of the nth pumping pulse:

$$\Delta_n b(\lambda) = -\pi\gamma\tau\,\mathrm{sech}[\pi(\lambda+V/4)/2\eta] \times \exp[-4i(\eta^2+V^2/16+\lambda V/2)n\tau] . \tag{6.162}$$

With regard to the influence of the dissipation, between the pulses $b(\lambda)$ evolves according to

$$b(\lambda,t) = b(\lambda,0)\exp(-\gamma t + 4i\lambda^2 t) . \tag{6.163}$$

Proceedings from Eqs. (6.162) and (6.163), one can find the amplitude $b(\lambda,t)$ for an arbitrary value of t, $n\tau < t < (n+1)\tau$, as the sum of an infinite geometrical progression:

$$\begin{aligned} b(\lambda,t) &= \sum_{m=0}^{\infty} [\Delta_{n-m} b(\lambda)]\exp[(-\gamma+4i\lambda^2)(t-(n-m)\tau)] \\ &= -\pi\gamma\tau\,\mathrm{sech}[\pi(\lambda+V/4)/2\eta][1-\exp(\{-\gamma+4i[\eta^2+(\lambda+V/4)^2]\}\tau)]^{-1} \\ &\quad \times \exp[(-\gamma+4i\lambda^2)(t-n\tau)-4i(\eta^2+V^2/16+V\lambda/2)n\tau] . \end{aligned} \tag{6.164}$$

Of principal concern are the occupation numbers of the radiation modes (the spectral density of the number of light quanta) $\mathcal{N}(\lambda)$ defined according to Eq. (2.43b). As follows from Eq. (6.164) with regard to the condition $\gamma\tau \ll 1$, in an equilibrium state the occupation numbers take the stationary values

$$\mathcal{N}(\lambda) = (\pi/4)(\gamma\tau)^2\{\mathrm{sech}^2[\pi(\lambda+V/4)/2\eta]\}/\sin^2\{2[\eta^2+(\lambda+V/4)^2]\tau\} , \tag{6.165}$$

provided $|\sin\{2[\eta^2+(\lambda+V/4)^2\tau]\}| \gg \gamma\tau$. If this condition does not hold, i.e.,

$$2[\eta^2+(\lambda+V/4)^2]\tau = \pi q + \phi/2 , \tag{6.166}$$

where $|\phi| \ll \tau\gamma$ and q is an integer, it follows from Eq. (6.164) that

$$\mathcal{N}(\lambda) = \pi(\gamma\tau)^2[(\gamma\tau)^2+\phi^2]^{-1} \times \mathrm{sech}^2[\pi\sqrt{\pi q - 2\tau\eta^2}/(2\eta\sqrt{2\tau})] . \tag{6.167}$$

The regime of the soliton's motion with sufficiently infrequent pumping, i.e., with $\tau\eta^2 \gg 1$, is the most interesting one. In this case Eqs. (6.165)–(6.167) describe a spectral wave packet whose center lies at the point $\lambda = -V/4$ and whose width is $|\lambda+V/4| \sim \eta$. As can be seen from Eq.

[8]As follows from the form of Eq. (6.161), it is convenient to follow the evolution of a wavetrain in the spatial coordinate t instead of the temporal one x (Kodama and Hasegawa, 1982; Kodama, 1985a, 1985b).

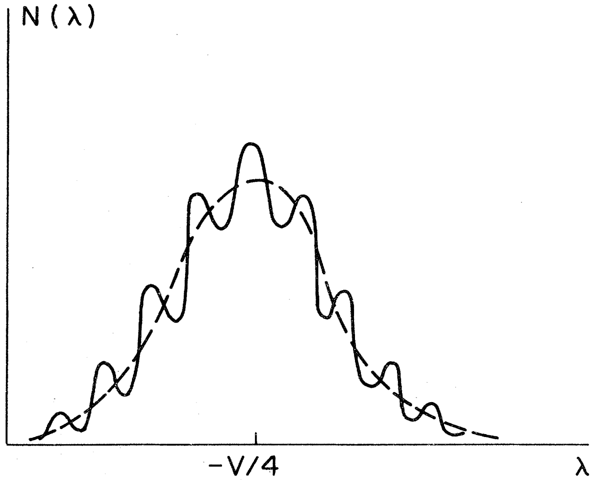

FIG. 39. The spectral composition of radiation (occupation numbers of the normal radiation modes) for the steady regime of motion of a soliton in the model (6.161) of a pumped-damped nonlinear optical fiber: solid curve, the exact equations (6.166) and (6.167); dashed curve, the averaged expression (6.168).

(6.166), inside the packet the spectral density $n(\lambda)$ has a large number $\sim\tau\eta^2$ of sharp maxima separated by the intervals $\Delta\lambda\sim(\eta\tau)^{-1}$; the width of each maximum is $\delta\lambda\sim\gamma/\eta\ll\Delta\lambda$ due to the underlying condition $\gamma\tau\ll1$ (Fig. 39).

If one sums up the occupation numbers instead of summing the amplitudes [see Eq. (6.164)], one obtains the expression

$$\tilde{\mathcal{N}}(\lambda)\equiv\pi^{-1}\sum_{m=0}^{\infty}|\Delta_{n-m}b(\lambda)|^2\exp(-2\gamma\tau m)$$

$$=(\pi/2)\gamma\tau\,\mathrm{sech}^2[(\pi/2)(\lambda+V/4)/\eta]\ , \quad (6.168)$$

which describes the smoothed spectral structure of the wave packet (see the dashed curve in Fig. 39). When calculating integral characteristics of the packet, for instance, the total number of light quanta scattered in the radiation field, $N_{\rm rad}\equiv\int_{-\infty}^{+\infty}\mathcal{N}(\lambda)d\lambda$, the smoothed spectral density (6.168) yields the same results as the exact expressions (6.165)–(6.167). In particular, $N_{\rm rad}=2\gamma\tau\eta$. Comparing this to the number $N_{\rm sol}=4\eta$ of quanta bound inside the soliton (Zakharov *et al.*, 1980), we conclude that in the stationary regime $N_{\rm rad}$ is indeed much smaller than $N_{\rm sol}$: $N_{\rm rad}/N_{\rm sol}=\gamma\tau/2\ll1$.

K. Radiative decay of an envelope soliton under the action of a sound wave

Envelope solitons originate in an interaction of high-frequency (hf) waves with low-frequency (lf) acoustic-type waves. Familiar examples are the Langmuir wave–ion-acoustic wave interaction in plasmas (Zakharov, 1972) and the interaction of intramolecular vibrations which form Davydov solitons with sound in molecular chains (Davydov, 1979). Other examples stem from geophysical hydrodynamics, e.g., interaction of hf and lf gravity-wave perturbations in an atmosphere (Stenflo, 1986). The simplest model of these interactions is the Zakharov system (Zahkarov, 1972) [see also the survey papers of Shukla (1983), Goldman (1984), and Pécseli (1985)]:

$$iu_t+u_{xx}=nu\ , \quad (6.169)$$

$$n_{tt}-n_{xx}=2(|u|^2)_{xx}\ , \quad (6.170)$$

where $u(x,t)$ is the complex envelope of the hf field and $n(x,t)$ is the real (acoustic) field.

The exact solitonic solution of the system (6.169) and (6.170) is

$$u=(1-V^2)^{1/2}2i\eta\,\mathrm{sech}[2\eta(x-Vt)]$$

$$\times\exp[iVx/2+i(4\eta^2-V^2/4)t]\ ,$$

$$n=-2(1-V^2)^{-1}|u|^2\ , \quad (6.171)$$

where η and V are the soliton's amplitude and velocity ($V^2<1$). The system (6.169) and (6.170) also has exact solutions describing free acoustic waves:

$$u\equiv0,\quad n\equiv n_s=n(x\pm t)\ . \quad (6.172)$$

From the viewpoint of both plasma physics and molecular chain theory, the interaction of a soliton with an acoustic wave is of direct physical interest. According to Shul'man (1981), the system (6.169) and (6.170) is nonintegrable. Therefore the interaction is inelastic: the soliton gradually decays into hf radiation. We shall study this process in the situation in which it may be treated in the framework of perturbation theory, i.e., when the acoustic wave's amplitude A is sufficiently small:

$$A\ll\eta^2/(1-V^2)\ . \quad (6.173)$$

Moreover, we shall assume, as is often admitted, that the soliton's velocity V and the emitted wave's group velocity $V_{\rm gr}$ are small as compared to the sound velocity, which is equal to one in the present notation:

$$V,V_{\rm gr}\ll1 \quad (6.174)$$

(the subsonic regime). Following Malomed (1987f, 1988d), we shall investigate the decay in two situations corresponding to physically different problems: interaction of the soliton with a periodic monochromatic acoustic wave and with a random acoustic wave field.

1. Decay of a soliton under the action of a monochromatic acoustic wave

To apply perturbation theory to the system (6.169) and (6.170), let us represent a general solution to Eq. (6.170), which is linear in $n(x,t)$, in the form $n=n_1+n_2$, where a general solution $n_1(x,t)$ to the homogeneous equation is an acoustic wave field $n_{\rm ac}(x,t)$, and a particular solution n_2 to the inhomogeneous equation may be represented as

$$n_2=-2|u|^2[1+O(V^2)] . \qquad (6.175)$$

Using the underlying condition (6.174), we may omit the term V^2 in Eq. (6.175) [the role of an additional perturbation generated by that term in Eq. (6.177) below will be discussed later]. Thus, inserting

$$n=-2|u|^2+n_{ac} \qquad (6.176)$$

into Eq. (6.169), we arrive at the perturbed NS equation

$$iu_t+u_{xx}+2|u|^2u=n_{ac}(x,t)u . \qquad (6.177)$$

Now, with regard to the assumed condition (6.173), it is natural to employ IST perturbation theory.

Let us take the acoustic field (6.172) in the form of the monochromatic wave

$$n_{ac}(x,t)=A\cos(kt-kx+\phi_0) , \qquad (6.178)$$

where ϕ_0 is a phase shift of acoustic oscillations in the center of the soliton ($x=0$) relative to the soliton's internal oscillations, and where, for definiteness, we set $k>0$. Inserting the perturbation from the right-hand side of Eq. (6.177) into the general perturbation-induced evolution equation (2.47) for the radiation amplitudes $B(\lambda,t)$, and assuming the soliton to be quiescent, one obtains

$$\frac{dB}{dt}=(i\pi/8)Ak^2(\lambda^2+\eta^2)^{-1}\operatorname{sech}[\pi(k+2\lambda)/4\eta] \times\exp(i\{[k-4(\lambda^2+\eta^2)]+\phi_0]) . \qquad (6.179)$$

Using Eq. (6.179), we can find the emission intensity in the same way as in Secs. IV.B and IV.C, where emission problems described by a perturbed sine-Gordon equation are dealt with. To that end, let us integrate Eq. (6.179), the right-hand side of which should be multiplied by $\exp(\alpha t)$ with an infinitely small $\alpha>0$. As above, this trick implies adiabatically turning on a perturbation that was absent at $t=-\infty$. Thus we get

$$B^*(\lambda,t)=-(\pi/8)Ak^2(\lambda^2+\eta^2)^{-1}\operatorname{sech}[\pi(k+2\lambda)/4\eta]\{[k-4(\lambda^2+\eta^2)]+i\alpha\}^{-1}\exp(-i\{[k-4(\lambda^2+\eta^2)]t+\phi_0\}) . \qquad (6.180)$$

Then, inserting Eq. (6.179) and (6.180) into the equation

$$\frac{d}{dt}\mathcal{N}(\lambda)=\frac{2}{\pi}\operatorname{Re}\left[B^*\frac{dB}{dt}\right] , \qquad (6.181)$$

where $\mathcal{N}(\lambda)$ is the plasmon number spectral density defined according to Eq. (2.43b), and making use of the relation $\lim_{\alpha\to 0}(\chi+i\alpha)^{-1}\equiv P(1/\chi)-i\pi\delta(\chi)$, where P is the symbol of the principal value, we find

$$\frac{d}{dt}\mathcal{N}(\lambda)=\tfrac{1}{8}(\pi Ak)^2\operatorname{sech}^2[\pi(k+2\lambda)/4\eta] \times\delta[\lambda^2-(k/4-\eta^2)] . \qquad (6.182)$$

The total plasmon number emission rate is

$$\frac{dN}{dt}\equiv\int_{-\infty}^{\infty}d\lambda\frac{d}{dt}\mathcal{N}(\lambda)=\left[\frac{dN}{dt}\right]_+ +\left[\frac{dN}{dt}\right]_- , \qquad (6.183)$$

where

$$\left[\frac{dN}{dt}\right]_\pm=\tfrac{1}{8}(\pi Ak)^2(k-4\eta^2)^{-1/2} \times\operatorname{sech}^2\left[\frac{\pi}{4\eta}(\sqrt{k-4\eta^2}\mp k)\right] . \qquad (6.184)$$

As can be seen from Eqs. (6.182)–(6.184), the emission takes place provided $k>k_0\equiv 4\eta^2$, and it is concentrated at two points of the spectrum,

$$2\lambda_\pm=\pm\sqrt{k-4\eta^2} . \qquad (6.185)$$

The group velocity of the emitted hf envelope waves is

$$V_{gr}=-4\lambda , \qquad (6.186)$$

i.e., the values $\lambda<0$ and $\lambda>0$ correspond to the waves emitted to the right [with the intensity $(dN/dt)_+$] and to the left [with the intensity $(dN/dt)_-$]. As can be seen from Eq. (6.184), the intensity of the waves emitted forward, relative to the sound velocity, is greater than that of the waves emitted backwards. It follows from Eqs. (6.185) and (6.186) that the underlying condition (6.174) takes the form $\eta^2<k/4\ll\frac{1}{16}$.

Using the conservation of the total plasmon number, we can find the soliton's decay rate $d\eta/dt$. Indeed, the number of plasmons bound inside the soliton is $N_{sol}=4\eta$ (Zakharov *et al.*, 1980), so that

$$\frac{d\eta}{dt}=-\frac{1}{4}\frac{dN}{dt} , \qquad (6.187)$$

where dN/dt is defined by Eqs. (6.183) and (6.184). According to the above condition $\eta^2<k/4$, it is natural to distinguish between "heavy" solitons ($\eta\sim\sqrt{k}$) and "light" ones ($\eta\ll\sqrt{k}$). We can estimate a characteristic time τ_1 for the decay of a "heavy" soliton into a "light" one, using Eqs. (6.187), (6.183), and (6.184):

$$\tau_1\sim\left[\eta\Big/\left[\frac{d\eta}{dt}\right]\right]\Bigg|_{\eta\sim\sqrt{k}}\sim(A^2k)^{-1} . \qquad (6.188)$$

As for the "light" soliton, Eq. (6.187) for it takes, with regard to Eqs. (6.183) and (6.184), the form

$$\frac{d\eta}{dt}=-\tfrac{1}{2}(\pi A/8)^2k^{3/2}\exp[-\pi(\sqrt{k}-k)/4\eta]$$

(recall $0<k\ll1$). Integrating this equation gives the eventual expression for the soliton's decay rate at the late stage:

$$\eta \approx \pi\sqrt{k}/4\ln(kA^2t) \ . \tag{6.189}$$

Far from the soliton, the emitted hf envelope wave looks like the monochromatic wave

$$u = a_\pm \exp[-i(4\lambda_\pm^2 t - 2\lambda_\pm x)] \ . \tag{6.190}$$

Equating the plasmon number flux $j_\pm = -4\lambda_\pm a_\pm^2$ carried by the wave (6.190) to the emission rate $(dN/dt)_\pm$ [see Eqs. (6.183) and (6.184)], one can find a general expression for the amplitude of the emitted wave,

$$a_\pm^2 = (4|\lambda_\pm|)^{-1}\left[\frac{dN}{dt}\right]_\pm \ . \tag{6.191}$$

As is well known, monochromatic traveling waves are unstable in the framework of the full NS equation (Benjamin and Feir, 1967). During the time $\tau_2 \sim a_\pm^{-2}$ they must decay into secondary solitons with amplitudes $\sim a_\pm$. One may expect those secondary solitons to decay in turn under the action of the acoustic wave, and so on, but we shall not consider that stage of the process.

To conclude this subsection, let us consider the influence of dissipation on emission of radiation by a soliton under the action of a nondissipative perturbation. In many cases, linear dissipation can be adequately accounted for by the combination of the two terms,

$$P_{\rm diss} = -\gamma_0 u + \gamma_1 u_{xx}, \quad \gamma_0 > 0, \quad \gamma_1 > 0 \tag{6.192}$$

on the right-hand side of a perturbed NS equation [see Eq. (1.8)]. In the absence of dissipation, the basic emission equation (2.47) may be written in the general form [see, e.g., Eq. (6.179)]

$$\frac{dB(\lambda)}{dt} = \epsilon f(\lambda)\exp[i(\Omega - 4\lambda^2)t] \ , \tag{6.193}$$

where Ω is an emission frequency. The dissipative perturbation adds one more term to the right-hand side of Eq. (6.193):

$$\frac{dB(\lambda)}{dt} = -\gamma B(\lambda) + \epsilon f(\lambda)\exp[i(\Omega - 4\lambda^2)t] \ , \tag{6.194}$$

where $\gamma \equiv \gamma_0 + 4\eta^2\gamma_1$. Designating $B(\lambda)e^{\gamma t} \equiv \widetilde{B}(\lambda)$, let us rewrite Eq. (6.194) in the form

$$\frac{d\widetilde{B}^*(\lambda)}{dt} = \epsilon f^*(\lambda)e^{\gamma t}\exp[-i(\Omega - 4\lambda^2)t] \ . \tag{6.195}$$

Multiplying Eq. (6.195) by the evident solution of this equation,

$$\widetilde{B}(\lambda) = [i(\Omega - 4\lambda^2) + \gamma]^{-1}\epsilon f(\lambda)e^{\gamma t}\exp[+i(\Omega - 4\lambda^2)t] \ ,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}|\widetilde{B}(\lambda)|^2 \equiv \mathrm{Re}\left[\frac{d\widetilde{B}^*(\lambda)}{dt}B(\lambda)\right] = \epsilon^2|f(\lambda)|^2 e^{2\gamma t}\gamma/[\gamma^2 + (\Omega - 4\gamma^2)^2] \ . \tag{6.196}$$

Coming back from $\widetilde{B}(\lambda)$ to $B(\lambda)$ and comparing Eq. (6.196) to nondissipative equations of the type (6.182), we infer that the delta functions in the expressions for the spectral density of the emission rate are "smeared" as follows [cf. Eqs. (6.9) and (6.10)]:

$$\delta(\Omega - 4\lambda^2) \to \pi^{-1}\gamma/[\gamma^2 + (\Omega - 4\lambda^2)^2] \ ;$$

this phenomenon may be called Lorenz broadening. Its physical sense is that, due to dissipative absorption, the emitted wave's amplitude decreases when moving away from the soliton, hence the wave is not quite monochromatic (this interpretation has been proposed by V. E. Zakharov).

2. Decay of a soliton under the action of a random acoustic wave field

We define a random acoustic wave field by the random initial conditions $n(x,t=0) = n_0(x)$, $n_t(x,t=0) = m_0(x)$ subject to the Gaussian correlations:

$$\begin{aligned} &\langle n_0(x)\rangle = 0, \quad \langle m_0(x)\rangle = 0 \ , \\ &\langle n_0(x)n_0(x')\rangle = n_0^2\delta(x-x') \ , \\ &\langle m_0(x)m_0(x')\rangle = m_0^2\delta(x-x') \ , \\ &\langle n_0(x)m_0(x')\rangle = 0 \ . \end{aligned} \tag{6.197}$$

The parameters n_0^2 and m_0^2 will be assumed small: $m_0^2 \ll \eta^7$, $n_0^2 \ll \eta^3$. The random initial conditions give rise to acoustic wave packets with both possible values of the group velocities $c_{1,2} = \pm 1$:

$$n_{\rm ac}(x,t) = \sum_{j=1}^{2}\int_{-\infty}^{+\infty} dq\, A_j(q)\exp[i(c_j qt - qx)] \ . \tag{6.198}$$

The correlations (6.197) result in the following correlations for the spectral amplitudes $A_j(q)$:

$$\begin{aligned} &\langle A_j(q)\rangle = 0 \ , \\ &\langle A_1(q)A_2(q')\rangle = (\pi/2)(n_0^2 - m_0^2/q^2)\delta(q-q') \ , \\ &\langle A_1(q)A_1(q')\rangle = \langle A_2(q)A_2(q')\rangle \\ &\qquad = (\pi/2)(n_0^2 + m_0^2/q^2)\delta(q-q') \ . \end{aligned} \tag{6.199}$$

The expression for dB/dt analogous to Eq. (6.199) is

$$\begin{aligned} \frac{dB}{dt} = &(i\pi/4)(\lambda^2+\eta^2)^{-1} \\ &\times \int_{-\infty}^{+\infty} dq\, q^2 \mathrm{sech}[\pi(q+2\lambda)/4\eta] \\ &\qquad \times \sum_{j=1}^{2} A_j(q)\exp\{i[c_j q - 4(\lambda^2+\eta^2)]t\} \ . \end{aligned} \tag{6.200}$$

Integrating Eq. (6.200) in the same way as when proceeding from Eq. (6.179) to (6.180), we get

$$B^*(\lambda)=\frac{\pi}{4}(\lambda^2+\eta^2)^{-1}\times\int_{-\infty}^{+\infty}dq\,q^2\operatorname{sech}[\pi(q+2\lambda)/4\eta]\times\sum_{j=1}^{2}A_j^*(q)\{[c_jq-4(\lambda^2+\eta^2)]+i\alpha\}^{-1}\times\exp\{-i[c_jq-4(\lambda^2+\eta^2)]t\}\ . \quad (6.201)$$

Finally, multiplying Eq. (6.200) by Eq. (6.201) and averaging the product according to Eqs. (6.199), we find the averaged emission rate spectral density:

$$\left\langle\frac{d}{dt}\mathcal{N}(\lambda)\right\rangle\equiv(2/\pi)\mathrm{Re}\left\langle B^*(\lambda)\frac{dB(\lambda)}{dt}\right\rangle=16\pi^3(\lambda^2+\eta^2)^{-2}[n_0^2+m_0^2/16(\lambda^2+\eta^2)^2]\times\sum_{j=1}^{2}\operatorname{sech}^2\{(\pi/2\eta)[\lambda+2c_j(\lambda^2+\eta^2)]\}\ . \quad (6.202)$$

Thus the emission induced by the random acoustic field is described by the smooth spectral density (6.202). It is necessary to remember that, as was mentioned above, the results obtained with the framework of the perturbated NS equation (6.177) are applicable to the underlying system (6.169) and (6.170) provided $\lambda^2\ll\frac{1}{16}$. As can be seen from Eq. (6.202), when $\eta\gtrsim1$ the spectral density is smeared over a broad spectral range $\lambda^2\lesssim\eta^2$, i.e., in this case the transformation of the system (6.169) and (6.170) into Eq. (6.177) is meaningless. If $\eta\ll1$, Eq. (6.202) has exponentially sharp maxima at the spectral points $\lambda_j^{(1)}=-2c_j\eta^2$, $\lambda_j^{(2)}=-c_j/2$. The latter point lies out of the range of applicability of Eq. (6.177), while in the vicinity of the former point Eq. (6.202) takes the form

$$\left\langle\frac{d}{dt}\mathcal{N}(\lambda)\right\rangle\approx\pi^3(m_0^2+16\eta^4n_0^2)\times\sum_{j=1}^{2}\operatorname{sech}^2[\pi(\lambda+2c_j\eta^2)/2\eta]\ . \quad (6.203)$$

Using Eq. (6.209) and the conservation of the total plasmon number, we can find the averaged soliton decay rate [cf. Eq. (6.177)]:

$$\frac{d\eta}{dt}=-\frac{1}{4}\int_{-\infty}^{+\infty}d\lambda\left\langle\frac{d}{dt}\mathcal{N}(\lambda)\right\rangle=-2\pi^2(m_0^2+16\eta^4n_0^2)\eta\ , \quad (6.204)$$

i.e., at the late stage (when $\eta^2\ll m_0/4n_0$) the soliton decays according to the law $\eta(t)\approx\eta_0\exp(-2\pi^2m_0^2t)$, which is much faster than Eq. (6.189).

The important feature distinguishing Eq. (6.204) from (6.187) is the absence of exponential smallness. The reason is that, no matter how small the soliton's amplitude η, there are long-wave components of the random wave field with wave numbers q for which the exponential-smallness condition $\eta^2\ll q$ [see Eq. (6.184)] does not hold.

Of course, one should bear in mind that Eq. (6.204) describes the soliton's decay induced by the long-wave component of the random acoustic wave field [as can be seen from Eqs. (6.201)–(6.203), the dominant range of the acoustic wave numbers is $|q|\sim\eta^2\ll1$]. A possible contribution from a short-wave component is not covered by Eq. (6.177).

Decay of the soliton under the action of a localized acoustic wave packet has been studied by Malomed (1987f, 1988d) too. In this case, the soliton loses only a small share of the plasmons bound inside it. Radiative decay of a Langmuir soliton under the action of other perturbations, viz., external electromagnetic waves and nonlinear Landau damping, was also studied (Malomed, 1988d).

To conclude our discussion of the dynamics of solitons described by the Zakharov system (6.169) and (6.170), it is pertinent to note that perturbation theory based on the proximity of the system to the exactly integrable NS equations under the "subsonic" condition (6.114) can be applied to other problems, e.g., to an inelastic collision of two solitons in the absence of free acoustic waves ($n_{\mathrm{ac}}=0$). In this case a small parameter of the perturbation theory will be V^2 [see Eq. (6.175)], and the perturbation $P=n_2(x,t)u$ will be nonlocal, since $n_2(x,t)$ is determined by the equation $(n_2)_{xx}=-2(|u|^2)_{tt}$ (see also Gibbons, 1978). Following this approach, it is interesting to consider, for example, the emission of acoustic waves by colliding solitons [such an effect was observed in numerical experimental by Degtyarev, Nakhan'kov, and Rudakov (1974)]. Within the framework of perturbation theory, explicit evaluation of the total emitted plasmon number and its spectral density is possible provided the amplitudes of the two colliding solitons are equal, i.e., $\eta_1=\eta_2\equiv\eta$, and their velocities $V_1=-V_2\equiv V$ lie in the interval $\eta^2\ll V^2\ll1$ (cf. a similar problem concerning the emission of radiation from colliding solitons considered in Sec. VII.B.2).

L. "Cherenkov" emission from a soliton in a system of two coupled sine-Gordon equations

A soliton described by a system of two coupled equations may demonstrate a specific type of emission, viz., "Cherenkov" emission. As a physically important example, let us consider a system of two weakly coupled perturbed SG equations, which describes a pair of two weakly interacting parallel Josephson junctions, the distance between the junctions being much larger than the Josephson penetration length (Mineev, Mkrtchyan, and Shmidt, 1981):

$$\varphi_{xx}-\varphi_{tt}-\gamma_1\varphi_t=\sin\varphi+f_1+\alpha\psi_{xx}\ , \quad (6.205)$$

$$\psi_{xx}-a^2\psi_{tt}-\gamma_2ab\,\psi_t=b^2\sin\psi+f_2b+\alpha\varphi_{xx}\ . \quad (6.206)$$

The parameters a and b (~1) are equal, respectively, to the ratios of the Swihart velocities and Josephson penetration lengths of the two junctions, while $\alpha \ll 1$ is a small coupling constant. Let us suppose, for definiteness, $a<1$. Then the equilibrium velocity of a fluxon moving in the second junction,

$$V_2=[a^2+(4\gamma_2/\pi f_2)^2]^{-1/2} \quad (6.207)$$

[cf. Eq. (3.47)], may be larger than the Swihart velocity ($\equiv 1$) of the first junction. In this case, the motion of the fluxon will be accompanied by Cherenkov emission in the first junction, generated by the "tachyonic" motion of the fluxon's image (4.178) in the first junction (Kivshar and Malomed, 1988c). The corresponding energy emission rate is

$$W(V_2)=\frac{2\pi^2\alpha^2 V_2}{(V_2^2-1)^2} \times \operatorname{sech}^2\left[\frac{\pi}{2b}\frac{[1-(aV_2)^2]^{1/2}}{(V_2^2-1)^{1/2}}\right] . \quad (6.208)$$

Equation (6.207) for the fluxon's equilibrium velocity must be corrected on account of the radiative energy losses given by Eq. (6.208). The corrected energy balance equation, which determines the equilibrium velocity, takes the form

$$2\pi V_2 f_2=8\gamma_2 V_2^2+W(V_2) . \quad (6.209)$$

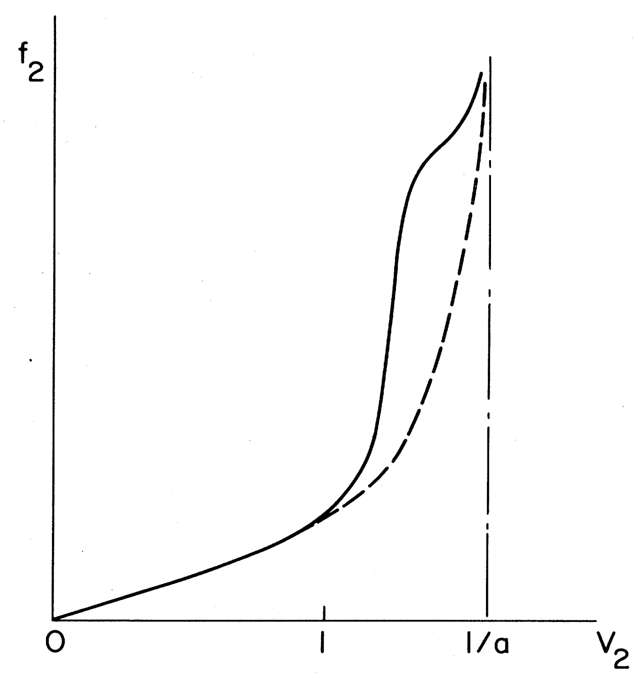

FIG. 40. The I-V (current-voltage) one-fluxon characteristic (6.210) of the model (6.205) and (6.206) of two weakly coupled long Josephson junctions. The dashed line depicts the usual one-fluxon I-V characteristic [Eq. (3.47)] of a long Josephson junction in the absence of a coupled junction.

Inserting Eq. (6.208) into Eq. (6.209), we obtain the equation

$$f_2=\frac{4\gamma_2}{\pi}\frac{aV_2}{[1-(aV_2)^2]^{1/2}} +\frac{\pi\alpha^2}{b^2(V_2^2-1)^2}\operatorname{sech}^2\left[\frac{\pi}{2b}\frac{[1-(aV_2)^2]^{1/2}}{(V_2^2-1)^{1/2}}\right] . \quad (6.210)$$

As is well known, the fluxon's equilibrium velocity is proportional to the voltage across the junction, so that Eq. (6.210) is, in fact, the I-V (current-voltage) characteristic of a long Josephson junction with a trapped fluxon in the presence of a weakly coupled corresponding junction (Fig. 40). As Fig. 40 shows, the differential I-V characteristic df_2/dV_2, as a function of V_2, may acquire two extrema (local maximum and minimum) in the range $1<V_2<a^{-1}$ on account of the "Cherenkov correction," provided α^2 is sufficiently large.

In principle, Cherenkov emission is also possible in a system of two coupled KdV equations, provided the signs of the solitons' velocities in the two subsystems are opposite, and in a system of coupled NS equations, provided the signs in front of the linear dispersion terms are opposite.

M. Radiative instability of a Korteweg–de Vries soliton pinned by a moving dipole

In Sec. III.A.2 we demonstrated that a moving pointlike dipole described by the perturbing term $\epsilon\delta'(x-Vt)$ in the KdV equation [see Eq. (3.9)], with ϵ negative and V positive, may trap a soliton with the amplitude $\kappa=\sqrt{V/2}$. The trapped (pinned) state is stable in the adiabatic approximation. Here we shall show, following Malomed (1988b), that this state is subject to a radiative instability. The instability is possible because the energy

$$E=\int_{-\infty}^{+\infty}(\tfrac{1}{2}u_x^2+u^3+\tfrac{1}{2}Vu^2)dx+\epsilon u\Big|_{x=Vt} , \quad (6.211)$$

conserved by Eq. (3.9), is not positive definite.[9] A general solution to the soliton's adiabatic equations of motion (3.12) and (3.13), which describes oscillations of the pinned soliton, can be written in the following approximate form:

$$\sinh\zeta=\nu^{-1}(-\tfrac{1}{2}\epsilon V^{3/2}-\nu^2)^{1/2}\sin(\nu t) , \quad (6.212)$$

where ν is the frequency taking the values $0<\nu\le\nu_{\max}\equiv\sqrt{-\epsilon/2}V^{3/4}$ [here $\nu=\nu_{\max}$ and $\nu=0$ correspond, respectively, to small oscillations near the equilibrium position (3.14) and to the separatrix (3.14′)]. The value of the energy (6.211) corresponding to the solution (6.212) is

[9]Equation (3.9) can be represented in the explicitly Hamiltonian form $u_t=(\partial/\partial y)(\delta E/\delta u)$, $y\equiv x-Vt$.

$$E(\nu)=V^{-1/2}\nu^2 . \quad (6.213)$$

On the other hand, calculation of the rate W of energy emission from the oscillating soliton yields the result

$$W(\nu)=(8\sqrt{-2\epsilon}V^{11/4})^{-1}(-\epsilon V^{3/2}-2\nu^2)\times(-\epsilon V^{3/2}+6\nu^2) . \quad (6.214)$$

The energy balance equation is

$$\frac{dE(\nu)}{dt}=-W(\nu) . \quad (6.215)$$

Inserting Eqs. (6.213) and (6.214) into Eq. (6.215), one can solve the resultant evolution equation for the parameter ν. The eventual expression for $\nu(t)$ can be represented in the form

$$\tfrac{1}{2}\epsilon t=\ln[(\nu_{\max}+\nu)/(\nu_{\max}-\nu)]+2\sqrt{3}\tan^{-1}(\sqrt{3}\nu/\nu_{\max}) , \quad (6.216)$$

where $\nu_{\max}$ is defined above. The solution (6.216) implies that the soliton is at rest [in the position (3.14)] at $t=-\infty$, remains in an oscillatory state at $t<0$, and becomes free at $t=0$. If one takes an initial state with a small oscillation amplitude $A_0\equiv(\nu_{\max}^2-\nu_0^2)^{1/2}\nu_{\max}^{-1}\ll 1$ (ν_0 is an initial value of ν), the pinned soliton will escape during the finite time $t=2|\epsilon|^{-1}\ln A_0^{-1}$. It also follows from Eq. (6.216) that, in terms of the phase plane shown in Fig. 1, the center corresponding to the equilibrium position (3.14) turns into a slightly unstable focus with the imaginary part of the eigenfrequency $\Omega(\mathrm{Re}\Omega\equiv\nu_{\max})$, $\mathrm{Im}\Omega=\epsilon/4<0$. However, one should bear in mind that, if the Burgers dissipative term γu_{xx} (with $\gamma>0$) is added to the right-hand side of Eq. (3.19), the eigenfrequency will acquire the additional imaginary part $\frac{1}{3}\gamma V>0$, so that the pinned state of the soliton may be genuinely stable if $|\epsilon|\lesssim\gamma V$.

To conclude this section, it is relevant to mention a somewhat surprising result obtained by Malomed (1988b) for the case $V<0$, when the moving dipole emits a quasilinear wave

$$u\approx A\cos(8k^3t+2kx) \quad (6.217)$$

with the wave number $2k=\sqrt{-V}$ and amplitude $A^2=\frac{2}{3}\epsilon^2|V|^{-1}$ [the phase velocity $-4k^2$ of the emitted wave (6.217) coincides with the dipole's velocity V]. If a soliton (2.18) with an amplitude κ collides with the emitting dipole, the monochromatic emission spectrum acquires some finite width Δk. If we designate $t=0$ the moment of collision, i.e., the moment when the pointlike dipole overlaps with the soliton's center, the width Δk of the collision-induced spectrum is exponentially small at $t\to-\infty$ (i.e., long before the collision), in accordance with the fact that the soliton's wave form decays exponentially at large distances:

$$\Delta k\sim\exp[-2\kappa(4\kappa^2-V)t] . \quad (6.218)$$

On the other hand, at $t\to+\infty$ (i.e., long after the collision) the spectrum's width exhibits a power smallness only:

$$\Delta k\sim(|V|t)^{-1} . \quad (6.219)$$

The drastic distinction between Eqs. (6.218) and (6.219) is stipulated by the phase shift $\Delta\phi=4\tan^{-1}(2\kappa/\sqrt{|V|})$ of the emitted wave (6.217) induced by the collision of the emitting dipole with the soliton.

N. Decay of a Korteweg–de Vries soliton into a shelf under the action of a dissipative perturbation

The KdV equation was derived first to describe the evolution of disturbances in a shallow liquid layer. If the layer's depth is nonuniform, one encounters the KdV equation with an effective dissipative perturbation $-\epsilon u$, where ϵ is proportional to a small gradient of the layer's depth [$\epsilon>0$ corresponds to a growing depth; see details in Knickerbocker and Newell (1985)]. Application of the standard formulas of the adiabatic approximation for the KdV soliton to the present perturbation yields the well-known evolution equation for the amplitude κ of the soliton,

$$\frac{d\kappa}{dt}=-\tfrac{2}{3}\epsilon\kappa . \quad (6.220)$$

At the same time, the present perturbed KdV equation gives rise to the following exact evolution equation for the "mass" $M\equiv\int_{-\infty}^{+\infty}u(x)dx$:

$$\frac{dM}{dt}=-\epsilon M . \quad (6.221)$$

Since the soliton's mass $M_{\mathrm{sol}}=-4\kappa$, the two equations (6.220) and (6.221) are in evident contradiction. The problem is resolved if one takes into account the mass of the radiative component of the wave field. That component is generated (emitted) by the soliton under the action of the perturbation. The problem of generation of the shelf has been considered, on the basis of IST perturbation theory, by Kaup and Newell (1978a) and Newell (1980). According to those works, the structure of the shelf can be described directly with the use of the basic perturbation-induced evolution equation (2.24) for the spectral amplitude $b(k)$ of the radiation wave field. Omitting immaterial details, the form of the shelf at the times $\kappa_0^{-2}\ll t\ll|\epsilon|^{-1}$ (where κ_0 is the initial soliton amplitude) is very simple: in the region $0<x<l(t)\equiv\int_0^t 4\kappa^2(\tau)d\tau$ the wave field is approximately uniform, $u\approx-(\epsilon/3)\kappa(t)$, while outside this region $u(x)$ may be set equal to zero. Here $x=0$ is the initial coordinate of the soliton, $l(t)$ is the distance traveled by the soliton up to the moment t, and $\kappa(t)$ evolves in time according to Eq. (6.220). It is obvious that the described structure of the shelf is in accordance with Eq. (6.221). When $\epsilon<0$, the shelf will decay into secondary solitons at larger times $t\gtrsim|\epsilon|^{-1}$. That problem is discussed in more detail in Sec. VIII.C.

The problem of a perturbation-induced shelf generation by a soliton arises also in the framework of the KdV equation (3.8) with variable coefficients. As we have demonstrated in Sec. III.A.1, following the paper by Karpman and Maslov (1982), that equation can be transformed into the standard perturbed form (3.8′) so that the problem can be attacked as explained above. Another approach to the formation of the shelf, based on the method of matched asymptotic expansions (without resorting to IST perturbation theory), has been developed for Eq. (3.8) by Ko and Kuehl (1978). The same approach for a modified KdV equation with variable coefficients has also been developed by Ko and Kuehl (1980).

VII. MANY-SOLITON RADIATIVE EFFECTS

A. Preliminary remarks

In the study of radiative effects accompanying two- and many-soliton interactions, one comes across problems of two types: emission of quasilinear waves and production of new (secondary) solitons. As in the one-soliton case, problems of the first type are solvable by means of IST perturbation theory or other versions of perturbation theory for solitons, while problems of the second type are not amenable to solving by analytical methods. The reason is the same as in the one-soliton situation (see Sec. VIII). Therefore, as in that situation, production of new solitons in many-soliton problems should be studied numerically. For instance, one may expect that a kink-antikink collision in a SG system with a small Hamiltonian perturbation may give rise to the emission of small-amplitude breathers [conversion of a kink-antikink pair into a pair of breathers has been observed in numerical experiments performed by Campbell *et al.* (1986) for the double SG equation].

B. Energy emission from colliding solitons

As we have already mentioned in Sec. IV.B, the only adiabatic effect generated by a conservative perturbation during the collision of two SG kinks amounts to additional phase shifts. Nontrivial effects arise if one takes into account emission of radiation accompanying the two-kink collision. In this section we shall consider this problem for the perturbation (1.25), and an analogous problem (radiative effects in a two-soliton collision) for the NS equation with the perturbation (1.7a). Results on these problems have been obtained by the authors (Kivshar and Malomed, 1986a, 1986c, 1987a). It is relevant to note that, in relation to applications to nonlinear optics, the latter problem was solved numerically by Cowan *et al.* (1986); unfortunately, detailed comparison between those numerical results and our analytical results is not possible since the numerical results were presented by Cowan *et al.* (1986) in the form of pictures only.

1. Sine-Gordon kinks

In this subsection we shall be concerned with the case in which the relative velocity V of the colliding kinks is sufficiently close to the maximum velocity $V_{\max}=1$, i.e., $\nu\equiv(1-V^2)^{-1/2}\gg 1$. It is natural to assume as above (see Sec. VI) that radiation is absent prior to the collision, i.e., $b(\lambda,t=-\infty)=0$. Then the spectral density of the emitted energy $\mathcal{E}(k)$ can be calculated as in the one-soliton case (see Sec. IV.A). Calculations are facilitated by the fact that, due to $\nu\gg 1$, we may (as in Sec. IV.B.1.a) take the field potential in the form (4.46a), i.e., as the sum of the potentials of solitary kinks. Using this "split potential," one obtains [recall that we are dealing with the perturbation (1.25)]

$$\mathcal{E}(k)=\frac{16\pi^3\epsilon^2}{g\nu^{14}}J(\lambda)\,, \tag{7.1}$$

where $\lambda\equiv\frac{1}{2}(k+\sqrt{1+k^2})$ [see Eq. (2.54)], and

$$J(\lambda)\equiv\frac{|Q(\lambda)|^2\sinh^2[(\pi/2\nu)(\lambda-\sigma_1\sigma_2/4\lambda)]}{\lambda^6(\lambda^2+\nu^2)^2(\lambda^2+1/16\nu^2)^2\sinh^2(\pi\lambda/\nu)\sinh^2(\pi/4\lambda\nu)}\,, \tag{7.2}$$

$$\begin{aligned}Q(\lambda)\equiv{}&\lambda^{10}/4-11i\lambda^9\nu/12-\lambda^8\nu^2-i\lambda^7\nu^3/3-5\lambda^6\nu^4/4+7i\lambda^5\nu^5/12+5\lambda^4\nu^4/16-i\lambda^3\nu^3/48\\&+\lambda^2\nu^2/64-11i\lambda\nu/(3\times 2^{10})-2^{-12}\,.\end{aligned} \tag{7.3}$$

Note that $J(\lambda)$ possesses the symmetry property

$$J(\lambda)=J(1/4\lambda)\,. \tag{7.4}$$

As can be seen from Eq. (7.4), with regard to Eq. (2.54), the density $\mathcal{E}(k)$ possesses the symmetry property $\mathcal{E}(k)=\mathcal{E}(-k)$, i.e., equal portions of energy are radiated to the left and to the right.

The dependence $\mathcal{E}(k)$ is plotted in Figs. 41(a) and 41(b) for the cases $\sigma_1\sigma_2=-1$ and $\sigma_1\sigma_2=+1$, corresponding, respectively, to the unlike and like polarities of the colliding kinks. The maxima of the spectral density lie at the two symmetric points (see Fig. 41) $k=\pm k_{\max}\sim\pm\nu$. The maximum value of $\mathcal{E}(k)$ is the same for both cases (a) and (b): $\mathcal{E}_{\max}=\mathcal{E}(\pm k_{\max})\sim\epsilon^2\nu^{-4}$. In the case $\sigma_1\sigma_2=-1$ the value $\mathcal{E}_0\equiv\mathcal{E}(k=0)$ is, according to Eqs. (7.2)–(7.4), $\mathcal{E}_0\sim\epsilon^2\nu^{-6}$, while in the case $\sigma_1\sigma_2=+1$, $\mathcal{E}_0=0$ [Fig. 41(b)]. The function $\mathcal{E}(k)$ falls exponentially at $k^2\gg\nu^2$: $\mathcal{E}\sim\exp(-\pi|k|/\nu)$, while in the range $1\ll k^2\ll\nu^2$ it has

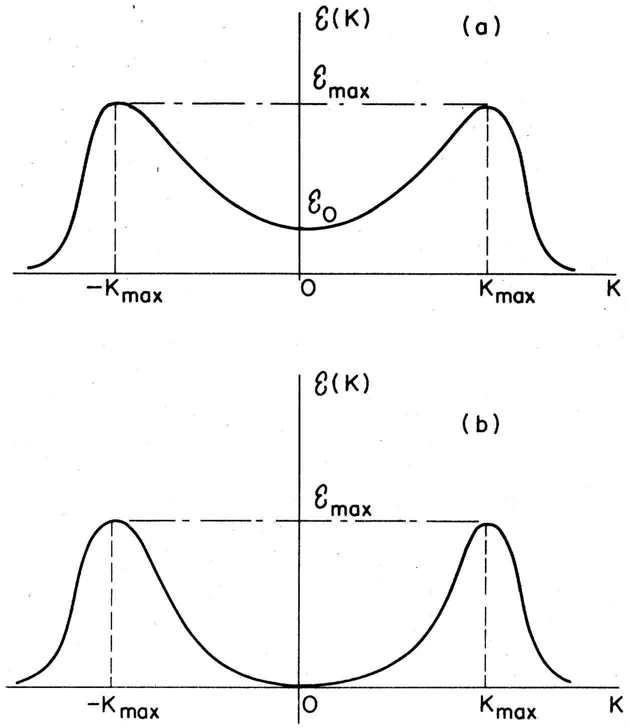

FIG. 41. The spectral density of the energy emitted by colliding SG kinks in the presence of the perturbation (1.25): (a) a kink-antikink pair; (b) a kink-kink pair.

a power asymptotic form $\mathcal{E}\sim\epsilon^2 k^{10}/v^{14}$. The total emitted energy $E_{\rm em}$ can be found from Eqs. (7.1)–(7.4):

$$E_{\rm em}\equiv\int_{-\infty}^{+\infty}\mathcal{E}(k)dk=E_1-\sigma_1\sigma_2 E_2\ , \tag{7.5}$$

where

$$E_1=\epsilon^2 v^{-3}A+O(\epsilon^2 v^{-5})\ , \tag{7.6}$$

$$E_2=\epsilon^2 v^{-5}B+O(\epsilon^2 v^{-7})\ , \tag{7.7}$$

and the constants A and B are defined as some integrals equal approximately to $A=8.78$, $B=1.10$.

An interesting property of Eqs. (7.5)–(7.7) is that, in the first nonvanishing approximation in v^{-1}, the total emitted energy does not depend on the relative polarity $\sigma_1\sigma_2$ of the two kinks. This is somewhat similar to the famous Pomeranchuk theorem asserting the asymptotic equality of total sections for scattering of particles on particles and particles on antiparticles (Schweber, 1961).

Finally, the momentum density $\mathcal{P}(k)$ of the emitted wave field can also be found with the aid of Eq. (2.70b):

$$\mathcal{P}(k)=\frac{16\pi^3\epsilon^2}{9v^{14}}(k/\sqrt{1+k^2})J(\lambda)\ . \tag{7.8}$$

According to Eq. (7.4), the momentum density (7.8) possesses the obvious symmetry property

$$\mathcal{P}(k)=-\mathcal{P}(-k)\ ,$$

which provides that the total emitted momentum is zero.

2. Nonlinear Schrödinger solitons

We shall consider only the case in which the amplitudes η of the two colliding solitons are equal [cf. Eq. (4.83)]. We shall also assume the solitons' velocities $\pm 4\xi$ in the center-of-mass reference frame to be large, $\xi\gg\eta$, i.e., as usual, the two-soliton Jost functions and the wave potential split into "one-particle" expressions. Assuming radiation to be originally absent, one can calculate the total spectral density of the energy emitted during the collision [recall that the radiation wave number is $k=2\lambda$, and that we are dealing with the perturbation (1.7a)]:

$$\mathcal{E}(\lambda)=\frac{4\epsilon^2\eta^6\lambda^2}{\pi\xi^2}\times\left[\left|F\left[\frac{\lambda-\xi}{\eta}\right]\right|^2+\left|F\left[-\frac{\lambda+\xi}{\eta}\right]\right|^2+2\,\mathrm{Re}\left\{e^{i\Delta\phi}\left[G\left[\frac{\lambda-\xi}{\eta}\right]+G\left[-\frac{\lambda+\xi}{\eta}\right]\right]\right\}\right]\ , \tag{7.9}$$

$$F(z)\equiv(\pi/15)Q(z)(z-i)^{-2}\,\mathrm{sech}(\pi z/2)\ , \tag{7.10a}$$

$$G(z)\equiv(2\pi^2/9)(\xi/\eta)^5\exp(-\pi\xi/\eta)Q(z)(z^2+1)^{-2}\times\mathrm{sech}(\pi z/2)\ , \tag{7.10b}$$

$$Q(z)\equiv 49z^4+156z^2+45-4iz(z^2+9)\ . \tag{7.10c}$$

The parameter $\Delta\phi=\phi_1-\phi_2$ in Eq. (7.9) is the phase difference between the two solitons at the moment of collision. As can be seen from Eqs. (7.9) and (7.10), the radiation spectral density maxima lie at the points $\lambda=\pm\lambda_{\max}=\pm\xi+O(\eta)$, the maximum value being

$$\mathcal{E}_{\max}\sim\epsilon^2\eta^6\ . \tag{7.11}$$

The radiation energy is concentrated in the spectral range $\sim\eta$ in the vicinity of the points $\lambda=\pm\lambda_{\max}=\pm\xi+O(\eta)$. Outside this range, the spectral density falls exponentially as $\exp[-(\pi/\eta)|\xi\pm\lambda|]$. Note that Eq. (7.9) is symmetric with respect to changing the sign of λ, which evidently reflects the symmetry between the left and right directions in the present problem. A general form of the emitted energy spectral density described by Eqs. (7.9)–(7.11) is shown in Fig. 42.

The term in curly brackets in Eq. (7.9) which depends on $\Delta\phi$ is exponentially small compared to the first two terms, which are independent of $\Delta\phi$. In Eq. (7.9) we have neglected terms containing faster dependencies on $\Delta\phi$, since they are exponentially small in comparison with those taken into account. As we see, at $\xi\to\infty$ the emitted energy does not depend on the parameter $\Delta\phi$. This inference is even stronger than the above "Pomeranchuk theorem" for the SG kinks: while their relative polarity

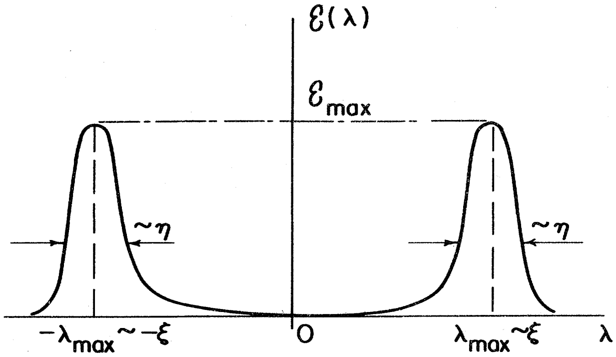

FIG. 42. The spectral density of the energy emitted by colliding NS solitons (with $V \gg \eta$) under the action of the perturbation (1.7a).

σ is the sign parameter assuming only two values ± 1, $\Delta\phi$ is a continuous parameter.

The total emitted energy $E_{\rm em}$ is determined by the formula (2.45a):

$$E_{\rm em} \equiv \int_{-\infty}^{+\infty} \mathcal{E}(\lambda)d\lambda = A\epsilon^2\eta^7 + \epsilon^2\eta^2\xi^5 e^{-\pi\xi/\eta}(B_1 \cos\Delta\phi + B_2 \sin\Delta\phi) \ . \tag{7.12}$$

Here A, B_1, and B_2 are numerical constants determined by some integral representations; cf. Eq. (7.5). Their approximate values are $A \approx 690.5$, $B_1 \approx 1200.5$, $B_2 \approx 173.3$. It is interesting to note that, as can be seen from Eq. (7.12), $E_{\rm em}$ does not vanish at $\xi \to \infty$. In this respect, the NS equation differs from the sine-Gordon equation, in which the total emitted energy falls to zero with the growth of the relative velocity [see Eqs. (7.5)–(7.7)].

The spectral density of the emitted wave momentum, according to Eq. (2.46a), is $\mathcal{P}(\lambda) = -(2\lambda)^{-1}\mathcal{E}(\lambda)$, and the spectral density of the emitted charge ("number of particles") is $\mathcal{N}(\lambda) = (2\lambda)^{-2}\mathcal{E}(\lambda)$, according to Eq. (2.43b). The total emitted momentum is, clearly, zero, and the total emitted charge is $N_{\rm em} \approx E_{\rm em}/4\xi^2$.

The fact that the emitted radiation consists of two wave packets traveling with mean group velocities $V_{\rm gr} = -4\lambda$ close to the solitons' velocities $\pm 4\xi$ is a consequence of the underlying assumption $\xi \gg \eta$. Using this assumption, one can arrive at a following inference (Malomed, 1989c): In the first approximation, the net charges $N_\pm$ of a soliton and a radiative wave packet traveling to the right, and of those traveling to the left, are conserved *separately,* although, of course, only their sum is an exact integral of motion. This fact makes it possible to calculate collision-induced losses of the solitons' amplitudes, $\Delta\eta_\pm = -\frac{1}{4}(N_{\rm em})_\pm$, even in the case when the collision is not symmetric (for instance, if the solitons' amplitudes are different). Going this way, Malomed (1989c) investigated radiative losses in a collision of two solitons driven by the perturbations (3.24) or (3.26) [see Eqs. (3.25) and (3.25′)]. It has been demonstrated that, while a single soliton persists in the presence of those dissipative perturbations (Malomed, 1988d), collision-induced losses result in complete decay of a *rarefied gas* of solitons into radiation.

C. Fusion of a kink-antikink pair into a breather

Collision of a kink with an antikink described by a conservatively perturbed SG equation may result in fusion of the kink-antikink pair into a breather. This process was discovered in the numerical experiments of Peyrard and Campbell (1983). Independently, the problem was considered by Malomed (1985) within the framework of the perturbed SG equation with the term (1.25). The analytical calculations involved are extremely ponderous; therefore we shall give here only final results. The basic characteristic of the process is the maximum (threshold) relative velocity $W_{\rm thr}$ of the colliding kinks which permits the fusion. It can be presented in the form

$$W_{\rm thr}^2 = (\alpha\epsilon)^2 + O(\epsilon^4) \ , \tag{7.13}$$

where the constant $\alpha \approx 2.224$. As to the spectral composition of the emitted radiation, the energy is concentrated in the range of the radiation wave numbers $|k| \lesssim 1$. The collision of two kinks, in contrast with a kink-antikink collision, cannot result in fusion. It has been demonstrated by Malomed (1985) that in the case of a small relative velocity W the total energy emitted by two colliding unipolar kinks is exponentially small in w.

The collision-induced emission and the related fusion problem were extensively studied numerically by Ablowitz, Kruskal, and Ladik (1979), Peyrard and Remoissenet (1982), Campbell *et al.* (1983), Peyrard and Campbell (1983), Campbell and Peyrard (1986), Campbell *et al.* (1986), and Belova and Kudryavtsev (1988) within the framework of nonintegrable models. An important example is the so-called φ^4 model,

$$\varphi_{tt} - \varphi_{xx} + \varphi - \varphi^3 = 0 \tag{7.14}$$

(this equation admits an exact kinklike one-soliton solution). Campbell *et al.* (1983) [see also Belova and Kudryavtsev (1988)] have determined several threshold velocities $W_0, W_1, \ldots, W_n$ such that the kink-antikink collision results in annihilation (fusion) if the relative velocity W lies inside any of the intervals $0 < W^2 < W_0^2, W_1^2 < W^2 < W_2^2, \ldots$, while in the "windows" $W_0^2 < W^2 < W_1^2, W_2^2 < W^2 < W_3^2, \ldots$ the collisions are nondestructive. Campbell and Peyrard (1986) have developed a semianalytical approach that enables them to explain the existence of the "windows" (that approach is not related to IST perturbation theory). Similar numerical results have been obtained by Peyrard and Remoissenet (1982), Peyrard and Campbell (1983), and Campbell *et al.* (1986) for a strongly perturbed SG equation. In the papers mentioned, a semianalytical approach to an explanation of those results has been developed too.

Since the SG equation with the perturbation (1.25) and Eq. (7.14) are time reversible, it is clear that, in principle, the inverse process is possible: a weakly bound (approximate) breather may decay into a kink-antikink pair under the action of incident radiation. For the SG equation with the perturbation (1.25), this process has been studied in detail by Malomed (1985).

D. Radiative effects in fluxon-antifluxon collisions in long Josephson junctions

A standard model of a dc-driven damped long Josephson junction (see Barone and Paterno, 1982; Likharev, 1985) is

$$u_{tt}-u_{xx}+\sin u=-f-\gamma u_t \ . \tag{7.15}$$

Recall that the equilibrium velocity V_0 of a fluxon (kink) described by Eq. (7.15) is given by Eq. (3.47). Emission of energy in the fluxon-antifluxon collision has been studied by Kivshar and Malomed (1986d, 1986e). In the "ultrarelativistic" limit $f \gg \gamma$, when, according to Eq. (3.47), $1-V_0^2 \ll 1$, the spectral density of the emitted energy takes the "self-similar" form

$$\mathcal{E}(k)=\gamma^4 f^{-2}F(k\gamma/f) \ . \tag{7.16}$$

The dependence (7.16) coincides qualitatively with that shown in Fig. 41(a). In particular, $k_{\max}\sim f/\gamma$, $\mathcal{E}_{\max}\sim\gamma^4/f^2$, $\mathcal{E}_0\sim\gamma^6/f^4$. The total emitted energy is

$$E_{\mathrm{em}}=(4/\pi)^3 D\gamma^3/f+O(\gamma^5/f^3) \ , \tag{7.17}$$

where $D\approx 0.568$. It is interesting to note that both perturbing terms on the right-hand side of Eq. (7.15) give contributions of the same order to Eq. (7.17).

In a homogeneous long Josephson junction, the collision-induced radiative losses are always much smaller than the direct dissipative losses, so that radiation-dominated fluxon-antifluxon annihilation is not possible. Kivshar and Malomed (1986f) have studied in detail radiative effects accompanying a fluxon-antifluxon collision in an *inhomogeneous* long Josephson junction, demonstrating, in particular, that radiation-dominated annihilation is possible if the inhomogeneity is sufficiently strong.

E. Emission from a wobbler

As is well known, the double SG equation, i.e., that with the perturbation (1.21), $f(t)\equiv 1$, is not exactly integrable. Nevertheless, it has an exact one-soliton solution (a 4π kink), which in the case $0<\epsilon\ll 1$ has the form (Bullough *et al.*, 1980)

$$u\approx 4\tan^{-1}(\sqrt{\epsilon/2}\sinh z) \ , \quad z=(x-Vt)/(1-V^2)^{1/2} \ . \tag{7.18}$$

This may be interpreted as a bound state of two sine-Gordon 2π kinks (Newell, 1978b). As was demonstrated by Newell (1978a, 1978b), the 4π kink can also exist in an excited state, when the internal 2π kinks oscillate relative to each other:

$$u\approx 4\tan^{-1}\left[\frac{W\sinh x}{\cosh\chi}\right] \ , \tag{7.19}$$

where

$$\chi=a\sqrt{2/\epsilon}\sin(\sqrt{\epsilon}t) \ , \tag{7.20a}$$

$$W=\sqrt{\epsilon/2}+a\cos(\sqrt{\epsilon}t) \ , \tag{7.20b}$$

and where a is a small amplitude of the internal oscillations [Eqs. (7.20) are valid provided $a^2\ll\epsilon$]. The excited 4π kink [Eqs. (7.19) and (7.20)] is sometimes called a wobbler (Bullough *et al.*, 1980). Equation (7.19) is written in the wobbler's rest frame of reference ($V=0$). Since the double SG equation is not exactly integrable, it is clear that Eqs. (7.19) and (7.20), unlike (7.18), may provide an approximate solution only. In fact, the wobbler's internal oscillations slowly fade due to emission of radiation.

The combination of the Jost functions which should be inserted into the general formulas (6.1) and (6.2) takes the following form for the wobbler (7.19):

$$\{[\Psi^{(2)*}(x,t;\lambda)]^2-[\Psi^{(1)*}(x,t;\lambda)]^2\} = \frac{\exp[-ik(\lambda)x]}{\left[\sinh^2 x+\frac{2}{\epsilon}\cosh^2\chi\right]} \times\left[\sinh^2 x+\frac{2}{\epsilon}\cosh^2\chi\left[\frac{\lambda+i/2}{\lambda-i/2}\right]^4\right] .$$

Using this expression, we find that, in the first approximation, the emission spectrum consists of discrete lines corresponding to the emission frequencies

$$\omega_m=\sqrt{\epsilon}\{n_0+2m+1+\tfrac{1}{2}[1+(-1)^{n_0}]\} \ ,$$

where $n_0=[1/\sqrt{\epsilon}]$, $m=0,1,2,\ldots$, and [ ] stands for the integer part. For the mth spectral line the emission power is (assuming $\epsilon\ll a\ll\sqrt{\epsilon}$)

$$W_m=\frac{32\epsilon^3(\omega_m^2-1)^2}{\omega_m^4\cosh^2\left[\frac{\pi}{2}(\omega_m^2-1)^{1/2}\right]}\left[\frac{a}{\pi\sqrt{\epsilon}}\right]^{2\omega_m/\sqrt{\epsilon}} \tag{7.21}$$

(Kivshar and Malomed, 1987b). Since we assume a/ϵ to be a small parameter, Eq. (7.21) is exponentially small ($\sim\exp[-(2/\sqrt{\epsilon})\ln(a/\sqrt{\epsilon})]$) in $\sqrt{\epsilon}$. For the same reason, the total emission power $\sum_m W_m$ is dominated by W_0. Using energy conservation, we can write an equation to determine the fade rate of the wobbler's small oscillations. Indeed, the energy E_{osc} of the wobbler's internal oscillations can be easily calculated to be a^2, i.e.,

$$\frac{d}{dt}(a^2)\approx -W_0 \sim -\epsilon^3\left[\frac{\epsilon n_0^2}{2}-1\right]^2\left[\frac{a}{\pi\sqrt{\epsilon}}\right]^{2n_0} . \tag{7.22}$$

As can be seen from Eq. (7.22), the small internal oscillations of the wobbler fade very slowly, which accords with the numerical observations of Bullough *et al.* (1980).

Now let us consider, following Malomed (1987h), the radiative damping of large-amplitude internal oscillations, $a \gg \sqrt{\epsilon}$. The two constituent 2π kinks inside the wobbler, separated by a large distance ξ, may be considered as relativistic particles with masses $m=8$ interacting via the attractive potential

$$U(\xi)=2\epsilon\xi . \tag{7.23}$$

The oscillation frequency ω is related to the amplitude a, i.e., the maximum distance between the oscillating 2π kinks, as

$$\omega=\pi\sqrt{\epsilon}/\sqrt{a(8+\epsilon a)} . \tag{7.24}$$

The radiative energy losses are dominated by the relatively short stage of the oscillation period $2\pi/\omega$ when the two kinks are strongly overlapped. At that stage, we may regard the 2π kinks as free, moving with velocities $\pm V$ determined by the energy balance, $16/\sqrt{1-V^2}=2\epsilon a$; see Eq. (7.23). Calculations analogous to those that yielded the results set forth in Sec. VII.B give the following expression for the spectral density of the energy emitted during the collision between the two 2π kinks:

$$\mathcal{E}(k)\approx\frac{16\pi\epsilon^2(1-V^2)k^2}{[1+k^2(1-V^2)]^2}e^{-\pi|k|\sqrt{1-V^2}} , \tag{7.25}$$

provided $1-V^2 \ll 1$, i.e., $\epsilon a \gg 4$ [in this case $\omega\approx\pi/a$; see Eq. (7.33)]. Integration of Eq. (7.25) yields the total emitted energy

$$E_{\rm em}=2\int_0^\infty \mathcal{E}(k)dk=16\pi C\epsilon^2/\sqrt{1-V^2} , \tag{7.26}$$

where $C\equiv \mathrm{si}(\pi)-\pi\mathrm{ci}(\pi)\approx 0.06$. Using Eqs. (7.24)–(7.26) and energy conservation, we can find the law of radiative damping for the wobbler's large-amplitude oscillations:

$$da/dt=-\pi C\epsilon^2 . \tag{7.27}$$

The same expressions (7.23)–(7.27) pertain to radiatively damped large-amplitude internal oscillations of a breather described by the double SG equation, i.e., a bound state of the 2π kink and 2π antikink [the dynamics of this breather have been studied in the adiabatic approximation by Kosevich and Kivshar (1982)].

Finally, Eqs. (7.25) and (7.26) multiplied by four give the radiative energy losses during a collision of two unexcited wobblers (4π kinks) moving with the velocities $\pm V$.

F. Radiative effects in systems of coupled equations

Interaction of solitons in coupled systems of nearly integrable equations gives rise to new types of radiative processes. First, the collision of two solitons belonging to the same subsystem can result in generation of radiation in a second subsystem where solitons are absent. Second, interaction of solitons belonging to different subsystems can result in emission of radiation in both subsystems. Moreover, under certain conditions uniform motion of one soliton may be accompanied by Cherenkov-type emission in another subsystem (see Sec. VI.I.).

1. Coupled Korteweg–de Vries equations

Leapfrogging motion of solitons in the pair of coupled KdV equations (1.3) and (1.4), investigated in Sec. IV.C.1, is accompanied by the emission of radiation. This effect was revealed numerically by Gear and Grimshaw (1984) and Gear (1985). Within the framework of IST perturbation theory the effect has been investigated by Malomed (1989d) and Kivshar and Malomed (1989d), whose analysis is based on perturbation-induced evolution equations for the amplitudes $B_j(k,t)$ of the radiation part of the wave field in the two subsystems $(j=1,2)$. The amplitudes $B_j(k,t)$ are related to the standard Jost coefficients $b_j(k,t)$ defined in Sec. II.B as follows: $B_j(k,t)=b_j(k,t)\exp(8ik^3t)$. The evolution equation for the case of equal soliton amplitudes is

$$\frac{dB_1}{dt}=-\frac{64}{21}\frac{i\kappa^4}{k}[\epsilon_3+\tfrac{1}{10}(\epsilon_1+\epsilon_2)]\zeta(t)e^{-8i\kappa^2kt} +O(\zeta^2,(\kappa_1-\kappa_2)^2) , \tag{7.28}$$

where $\zeta\equiv\kappa(z_1-z_2)$ is the dimensionless distance between the centers of the two solitons (see Sec. IV.C.1), and where it is assumed that we are considering small oscillations, i.e., $\zeta^2 \ll 1$. The evolution equation for $j=2$ can be written analogously. Our aim is to find the emission power, i.e., the energy emission rate. According to Eq. (2.22b), for $|B_j|^2 \ll 1$ the spectral density of the radiation energy can be cast in the form

$$\mathcal{E}_j(k)=(16/\pi)k^4|B_j(k)|^2 . \tag{7.29}$$

Using Eqs. (7.29) and (7.28), we can calculate the spectral density of the emission power,

$$\mathcal{W}_1(k)=\frac{d}{dt}\mathcal{E}_1(k) =(32/\pi)k^4\,\mathrm{Re}\left[B_1^*(k)\frac{dB_1(k)}{dt}\right] . \tag{7.30}$$

To do this, we shall follow the approach of Sec. VI.A. Inserting Eq. (7.28) into Eq. (7.30), we assume that the solitons perform small oscillations,

$$\zeta=a\sin(\omega_0 t) ,$$

where $a^2 \ll 1$, and where ω_0 is determined by Eq. (4.137). Eventually, we find

$$\mathcal{W}_1(k)=(\tfrac{8}{21}ak\omega_0)^2[\epsilon_3+\tfrac{1}{10}(\epsilon_1+\epsilon_2)]^2\delta\left[k-\frac{\omega_0}{8\kappa^2}\right]. \tag{7.31}$$

According to Eq. (7.31), the total emission power is

$$\begin{aligned}W_1&=\int_0^\infty \mathcal{W}_1(k)dk\\&=\frac{2}{\pi}(\tfrac{16}{105})^3(\epsilon_1+\epsilon_2+10\epsilon_3)^2\\&\quad\times(\epsilon_1+\epsilon_2+\tfrac{5}{2}\epsilon_3)\kappa^8a^2(1+\beta\alpha)\,.\end{aligned} \tag{7.32}$$

For the second subsystem ($j=2$) the formulas are similar.

Far from the soliton, the emitted wave looks like

$$u_1=A_1\cos(8\xi^3t+2\xi x)\,,$$

where the wave number $2\xi=\omega/4\kappa^2$; see Eq. (7.31) (in the rest reference frame of the bound state's center of mass, a wave frequency corresponding to this ξ coincides with ω). To find the wave's amplitude A_1, we may employ the energy balance. Indeed, a soliton moving with velocity $V_{\rm sol}=4\kappa^2$ leaves behind an emitted wave field with the averaged energy density

$$\langle h\rangle=\langle\tfrac{1}{2}u_x^2+u^3\rangle\approx\xi^2A_1^2\,,$$

where the angular brackets indicate averaging in fast oscillations like $\cos(8\xi^3t+2\xi x)$. The energy balance equation $W_1=V_{\rm sol}\langle h\rangle$ yields

$$A_1^2=W_1/(4\kappa^2\xi_0^2)\,. \tag{7.33}$$

If two solitons with equal amplitudes κ collide (instead of forming a bound state), the relevant quantity is the total emitted energy $E_{\rm em}$ (instead of the energy emission rate). When the relative velocity W of the colliding solitons is much smaller than $(V_{\rm sol})_1=4\kappa^2$, calculations yield

$$E_{\rm em}=\frac{2^{12}}{\pi^7}W\kappa^3[C_1(\alpha+\epsilon_3)^2+C_2\alpha(\alpha+\epsilon_3)+C_3\alpha^2]\,, \tag{7.34}$$

where $\alpha\equiv\frac{1}{24}(\epsilon_1+\epsilon_2)$, $C_1\approx1.20$, $C_2\approx2.95$, $C_3\approx2.78$. Malomed (1987g) and Kivshar and Malomed (1989d) have also calculated the spectral density of the emitted energy.

2. Coupled nonlinear Schrödinger equations

Let us consider two NS equations coupled by the derivative linear terms (1.14) with a small real coupling coefficient ϵ. As has been demonstrated in Sec. IV.C.2, in the adiabatic approximation collision of two solitons belonging to different subsystems does not result in changes of the solitons' amplitudes and velocities. Therefore it is of interest to find collision-induced radiative losses. Straightforward calculations yield the following expressions for the spectral densities $\mathcal{N}_{1,2}(\lambda;V,\eta)$ of numbers of plasmons emitted in both subsystems [recall that we are considering a collision of two solitons with equal amplitudes η and velocities $\pm V$ (Kivshar and Malomed, 1986c)]:

$$\begin{aligned}\mathcal{N}_1(\lambda;V,\eta)&=\frac{\pi^2\epsilon^2\eta^2}{V^2[(\lambda-V/4)^2+\eta^2]}\\&\quad\times(A^2+B^2+2AB\cos\Delta\phi)\,,\end{aligned} \tag{7.35}$$

$$\mathcal{N}_2(\lambda;V,\eta)=\mathcal{N}_1(-\lambda;V,\eta)\equiv\mathcal{N}_1(\lambda;-V,\eta)\,, \tag{7.36}$$

where A and B are the following real functions:

$$\begin{aligned}A&=\frac{ah(-b)}{\sinh(\pi a/2)\cosh(\pi b/2)}\,,\\B&=\frac{f(c)h(+d)}{\sinh(\pi c/2)\cosh(\pi d/2)}\,,\end{aligned} \tag{7.37}$$

$h(x)$ and $f(x)$ are the polynomes

$$h(x)=x^2+(V/2\eta)x+1,\quad f(x)=\eta^2x-2\eta(\lambda-V/2)\,,$$

and

$$\begin{aligned}a&=(\lambda^2+\lambda V/2-7V^2/16+\eta^2)/\eta V\,,\\b&=+(\lambda^2-\lambda V/3+5V^2/16+\eta^2)/\eta V\,,\\c&=+(\lambda^2+\lambda V/2+V^2/16+\eta^2)/\eta V\,,\\d&=+(\lambda^2-\lambda V/2-3V^2/16+\eta^2)/\eta V\,.\end{aligned}$$

The total emitted plasmon numbers

$$(N_{1,2})_{\rm em}=\int_{-\infty}^{+\infty}d\lambda\,\mathcal{N}_{1,2}(\lambda)$$

can be found explicitly from Eqs. (7.35)–(7.37) in the limit $\eta\ll V$:

$$(N_1)_{\rm em}=(N_2)_{\rm em}=24.64\epsilon^2\eta\equiv N_{\rm em}\,. \tag{7.38}$$

Quite analogously one can calculate the spectral densities of the energies emitted in both subsystems and (in the limit $\eta\ll V$) the total emitted energies,

$$(E_{\rm em})_1=(E_{\rm em})_2\equiv E_{\rm em}=(V^2/4)N_{\rm em}\,. \tag{7.39}$$

Note that, as in the case of a collision of two fast solitons described by a single perturbed NS equation, the expressions (7.38) and (7.39) do not depend on the phase difference $\Delta\phi$ of the colliding solitons (a correction dependent on $\Delta\phi$ is exponentially small in V/η).

Using conservation of the total energy E_1+E_2 and total plasmon number N_1+N_2, and taking account of the circumstance that, due to symmetry, the changes of parameters of the two solitons are related by $\Delta\eta_1=\Delta\eta_2\equiv\Delta\eta$, $\Delta V_1=-\Delta V_2\equiv\Delta V$ (this is true in the limit $\eta\ll V$, when the dependence on $\Delta\phi$ disappears), we find

$$\Delta\eta=N_{\rm em},\quad \Delta V=\tfrac{5}{12}(N_{\rm em}/\eta V)\,.$$

In principle, one can also find the rate of radiative damping for an oscillating bisoliton, but that quantity proves to be exponentially small in the oscillation amplitude.

3. Coupled sine-Gordon equations

In this subsection we shall consider radiative effects accompanying collisions between kinks described by the systems of coupled equations (4.175) and (4.176) and (4.196) and (4.197). Results for the system (4.175) and (4.176) have been obtained by Kivshar and Malomed (1988c), and for (4.196) and (4.197) by Braun, Kivshar and Kosevich (1988). First let us consider collisions of fluxons in the system of two weakly interacting parallel long Josephson junctions described by the system (4.175) and (4.176). If the fluxons belong to the same junction we mean a fluxon-antifluxon collision, while in the case when they belong to different junctions either polarity of the fluxons relative to each other is possible.

In the former case a distinctive feature of the problem is that colliding fluxons generate radiation in both junctions. In their own (the first) junction, the emission is generated on account of the perturbing terms $-f$ and $-\gamma_1\varphi_{1t}$ in the corresponding perturbed SG equation (Kivshar and Malomed, 1986e). This emission has been discussed in Sec. VII.D. Here we shall consider the perturbation-induced emission of radiation in the corresponding (second) junction where there are no fluxons. The corresponding emitted energy can be calculated explicitly in two limiting cases. In the first, as in Sec. VII.D, the colliding fluxon and antifluxon are "relativistic," $v_1^{-2}=1-V_1^2\ll 1$, i.e., $f_1\gg\gamma_1$. Calculations yield the following spectral density of the emitted energy:

$$\mathcal{E}_2(k)=\frac{\pi^3\alpha^2\sinh^2[(\pi/2v_1)\sqrt{1+k^2}]}{4v_1^4\sinh^2[(\pi/2v_1)(\sqrt{1+k^2}+k)]\sinh^2[(\pi/2v_1)(\sqrt{1+k^2}-k)]}\,, \tag{7.40}$$

where V_1 is the equilibrium velocity (3.47) and k is the radiative wave number. The total emitted energy can be readily obtained from Eq. (7.40):

$$(E_{\rm em})_2\equiv\int_{-\infty}^{+\infty}\mathcal{E}_2(k)dk\approx\tfrac{4}{3}\alpha^2v_1\,. \tag{7.41}$$

In the second tractable limiting case, $f_1\ll\gamma_1$ (i.e., $V_1^2\ll 1$). Then calculations for the spectral density of the emitted energy yield

$$\mathcal{E}_2(k)=\frac{\alpha^2}{\pi}\left|\int_{-\infty}^{+\infty}dx\,[1-\sqrt{1+k^2}(\sinh x)(\tanh x)]\times\exp(-ikx-\cosh x\sqrt{1+k^2})\right|^2\,, \tag{7.42}$$

and the total emitted energy is

$$(E_{\rm em})_2\approx 38.4\alpha^2\,. \tag{7.43}$$

As we see from Eqs. (7.40) and (7.42), when $v_1^2\gg 1$ (fast fluxons) the emitted energy is concentrated in the short-wave spectral range $k^2\lesssim v_1^2$, while in the case $V_1^2\ll 1$ (slow fluxons) it is concentrated in the long-wave range $k^2\lesssim 1$.

It is pertinent to note that, provided $(E_{\rm em})_2$ defined in Eq. (7.43) is larger than the total kinetic energy $8V_1^2$ of the colliding fluxon and antifluxon, the radiative losses considered will result in fluxon-antifluxon annihilation into a breather.

When the colliding fluxons belong to different junctions, the emitted energy can be found explicitly for all values of the fluxons' velocities unless the relative velocity $W=V_1-V_2$ is very small, $W^2\lesssim\alpha^2$. If $W^2\lesssim\alpha^2$, the collision may result in radiative fusion of the fluxons into a bifluxon (dissipative fusion was considered in Sec. IV.A.2).

Straightforward calculations yield the following expressions for the spectral densities of the energy emitted in both junctions:

$$\mathcal{E}_1(k)=\frac{\pi^3\alpha^2(1-V_1^2)^2}{(V_2-V_1)^4\sinh^2(\pi\kappa_1/2)\cosh^2(\pi\kappa_2/2)}\,, \tag{7.44}$$

$$\mathcal{E}_2(k)=\mathcal{E}_1(k;V_1\rightleftarrows V_2)\,, \tag{7.45}$$

where

$$\kappa_1=(1-V_1^2)^{1/2}(\sqrt{1+k^2}-kV_2)/|V_2-V_1|\,,$$

$$\kappa_2=(1-V_2^2)^{1/2}(\sqrt{1+k^2}-kV_1)/|V_2-V_1|\,,$$

and where $V_{1,2}$ are determined according to Eq. (2.47). The total emitted energies can be calculated in the same two limiting cases described above: $1-V_j^2\ll 1$ (we assume $V_1V_2<0$) and $\alpha^2\ll(V_2-V_1)^2\ll 1$. In the former case

$$(E_{\rm em})_1=8\alpha^2v_2^3/(3v_1^2)\,,\qquad (E_{\rm em})_2=8\alpha^2v_1^3/(3v_2^2)\,, \tag{7.46}$$

where $v_j\equiv(1-V_j^2)^{-1/2}$, $j=1,2$. In the latter case,

$$(E_{\rm em})_1\approx(E_{\rm em})_2=16\pi^3\alpha^2(V_2-V_1)^{-7/2}\times\exp(-2\pi/|V_2-V_1|)\,. \tag{7.47}$$

Equations (7.44)–(7.47) demonstrate the same qualitative features as (7.40)–(7.42).

Similar results can be obtained for the system (4.196) and (4.197), except for the fact that, for small relative velocity, the total energy emitted by colliding kinks belonging to the different subsystems is $\sim\exp(-4\pi/|V_1-V_2|)$ instead of $\exp(-2\pi/|V_1-V_2|)$ in Eq. (7.47).

Finally, let us consider the system of two coupled double SG equations (4.198) and (4.199) with $Q=1$. In Sec. IV.C.4 we considered internal oscillations of a bikink described by this system. For small-amplitude oscillations the rate W of radiative energy losses is exponentially small, $W\sim a^{{\rm const}/\sqrt{\epsilon}}$. The case of large-amplitude oscillations ($\epsilon a\gg 4$) is more interesting. As has been demonstrated by Malomed (1987h), radiative damping of these oscillations is described by the same expressions (7.25)–(7.27) as pertain to the radiative damping of large-amplitude oscillations of a wobbler or a breather described by the single double SG equation. Moreover, Eq. (7.26) multiplied by four gives the total energy loss accompanying a collision between an unexcited wobbler moving with velocity V $(1-V^2 \ll 1)$ and another unexcited wobbler or bikink moving with velocity $-V$. A collision between two unexcited bikinks is not accompanied by the emission of radiation because, as explained in Sec. IV.C.4, this collision can be described by the unperturbed SG equation.

4. The coupled sine-Gordon–d'Alembert system

The system of coupled equations

$$\phi_{tt}-\phi_{xx}+\sin\phi=-\alpha U_x \sin\phi-\beta V_x \cos\phi \ , \tag{7.48}$$

$$U_{tt}-S_L^2 U_{xx}=-\alpha(\cos\phi)_x \ , \tag{7.49}$$

$$V_{tt}-S_T^2 U_{xx}=\beta(\sin\phi)_x \tag{7.50}$$

describes nonlinear magnetization waves in an elastic ferromagnet (Maugin and Miled, 1986) and similar waves in an elastic ferroelectric system (Pouget and Maugin, 1984, 1985a). Here $U(x,t)$ and $V(x,t)$ stand for the longitudinal and transverse acoustic wave fields, respectively, S_L and S_T are the corresponding sound velocities, and α and β are regarded as small coupling constants. Radiative effects (Cherenkov emission) accompanying the motion of a single kink (domain wall) in the ϕ subsystem can be analyzed as in Sec. VI.L. Here we shall concentrate on effects produced by a collision between two kinks [in the adiabatic approximation, the collision has been considered by Pouget and Maugin (1985a)]. Clearly, the simplest and most important collision-induced radiation effect is the emission of transverse and longitudinal sound waves in the V and U subsystems. Since Eqs. (7.49) and (7.50) are linear d'Alembert equations, the emitted wave fields can be calculated in a straightforward way with the aid of the Green's function for the d'Alembert equation (Pouget and Maugin, 1984). Here we present the expressions for the emitted energy found by Kivshar and Malomed (1989f).

First let us consider a kink-antikink collision with zero relative velocity at infinity. Using the corresponding exact unperturbed solution (2.67), one can calculate the spectral density $\mathscr{E}(k)$ of the emitted energy, where k is the sound wave number. In the case $S_L^2, S_T^2 \ll 1$ it takes the form

$$\mathscr{E}(k)\approx 2\pi^2\left[\frac{\alpha^2}{S_L^2}+\frac{\beta^2}{S_T^2}\right]k^3 \operatorname{csch}(\pi k) \ . \tag{7.51}$$

The total emitted energy corresponding to Eq. (7.51) is

$$E_{\rm em}\equiv\int_{-\infty}^{+\infty}\mathscr{E}(k)dk=(\pi^2/2)\left[\frac{\alpha^2}{S_L^2}+\frac{\beta^2}{S_T^2}\right] . \tag{7.52}$$

It is noteworthy that in real elastic ferromagnets the sound velocities are quite small compared to the maximum magnon velocity, i.e., the assumption $S_L^2, S_T^2 \ll 1$ is indeed relevant. As can be seen from Eq. (7.52), this circumstance may render the radiative energy losses significant. Using Eq. (7.52), one can find a threshold condition for kink-antikink annihilation into a breather as a result of sound emission.

Let us proceed to the opposite case, when kinks colliding with velocities $\pm V$ are "ultrarelativistic" $(1-V^2 \ll 1)$. In contrast to the similar problem considered in Sec. VII.B.1 for the single SG equation perturbed by the term (1.25), this time the result depends substantially on the relative polarity σ of the colliding kinks. In the longitudinal (U) subsystem, the spectral density of the emitted energy is

$$\mathscr{E}_-(k)\approx(\pi^3/2)\alpha^2(1-V^2)^4k^6\cosh^2(\pi k\sqrt{1-V^2}) \times\operatorname{csch}^4(\pi k\sqrt{1-V^2}) \tag{7.53}$$

for $\sigma=-1$, and

$$\mathscr{E}_+(k)\approx(\pi^3/2)\alpha^2(1-V^2)^4k^6\operatorname{csch}^4(\pi k\sqrt{1-V^2}) \tag{7.54}$$

for $\sigma=+1$. The corresponding total emitted energies are

$$(E_{\rm em})_-\approx(\alpha^2/6)(1+\pi^2/21)\sqrt{1-V^2} \ , \tag{7.55}$$

$$(E_{\rm em})_+\approx(\alpha^2/6)(1-2\pi^2/21)\sqrt{1-V^2} \ . \tag{7.56}$$

It follows from these expressions that

$$(E_{\rm em})_+/(E_{\rm em})_-\approx\tfrac{1}{25} \ . \tag{7.57}$$

Thus, according to Eq. (7.66), in the present problem a kink-antikink collision gives rise to much stronger emission than does a kink-kink collision.

In the transverse (V_-) subsystem, this property shows itself in an even more pronounced form. For $\sigma=-1$,

$$\mathscr{E}_-(k)=(\pi^3/2)\beta^2(1-V^2)^4k^6\operatorname{csch}^2(\pi k\sqrt{1-V^2}) \tag{7.58}$$

[cf. Eq. (7.53)], and

$$(E_{\rm em})_-=(\pi^2/42)\beta^2\sqrt{1-V^2} \tag{7.59}$$

[cf. Eq. (7.55)], while for $\sigma=+1$, $\mathscr{E}_+(k)$ is zero in the approximation considered. A nonzero result arises if one takes into account the first correction in the small parameter S_T^2:

$$\mathscr{E}_+(k)\approx 2\pi^5\beta^2S_T^2(1-V^2)^5k^8\operatorname{csch}^4(\pi k\sqrt{1-V^2}) \ , \tag{7.60}$$

$$(E_{\rm em})_+\approx(8/\pi^2)\beta^2S_T^2\sqrt{1-V^2} \ . \tag{7.61}$$

According to Eqs. (7.59) and (7.61), this time $(E_{\rm em})_+/(E_{\rm em})_-\sim S_T^2$ [cf. Eq. (7.57)]. It should be noted that, from the physical viewpoint, both cases $\sigma=\pm1$ are possible. Indeed, in a real (damped) system kinks must be driven by a force produced by the additional perturbation $\sim\sin(\phi/2)$, which describes an external field applied to the system (Pouget and Maugin, 1985b). In view of this fact, it is easy to see that a collision between two kinks specified by the boundary conditions $\phi(x=-\infty)=\phi(x=+\infty)$ corresponds to $\sigma=-1$, while a collision specified by $|\phi(x=+\infty)-\phi(x=-\infty)|=4\pi$ corresponds to $\sigma=+1$.

VIII. CREATION OF SOLITONS BY A PERTURBATION

A. General discussion of the problem

As was mentioned in Sec. VI, in general the problem of the creation of solitons under the action of a perturbation is not amenable to solution by existing analytical methods. In terms of IST perturbation theory, the reason can be explained as follows. The birth of a new soliton purports the appearance of a new zero of the forward scattering amplitude $a(\lambda)$ at some (generally speaking, complex) value λ_0 of the spectral parameter λ (see Sec. II). It proves that, in the vicinity of the point λ_0, the effective parameter of a perturbative expansion is not simply ϵ, but rather ϵ/λ in the case of a perturbed KdV equation (Karpman and Maslov, 1977; in this case new solitons appear with an infinitely small amplitude, i.e., at $\lambda_0=0$) or $\epsilon^2/(\lambda-\lambda_0)$ for perturbed SG and NS equations (Kivshar, 1984). In the case of a SG equation, the newborn soliton are breathers with an infinitely small amplitude, which correspond to a pair of points $\lambda_0=\pm\frac{1}{2}\sqrt{(1+V)/(1-V)}$, where V is the velocity of the breather; in the case of a NS equation, λ_0 may be arbitrary real, which corresponds to an infinitely-small-amplitude soliton with velocity $V=-4\lambda_0$. Thus at $\lambda\to\lambda_0$ an effective perturbative parameter becomes large, and perturbation theory becomes irrelevant.

Due to these difficulties, the generation of new solitons is often studied by means of numerical methods (Patoine and Warm, 1982; Akylas, 1984; Malanotte-Rizzoli, 1984; Bussac *et al.*, 1985; Cole, 1985; Ertekin, Webster, and Wehausen, 1986; Kaup and Hansen, 1986; Mei, 1986; Wu, 1987). Nevertheless, there have been attempts to develop a qualitative analytical description of these physically important phenomena (Karpman and Maslov, 1978; Wright, 1980; Kivshar, 1984; Kaup, 1986). Here we shall set forth the ideas of Kaup and Hansen (1986), which are rather close to the technique of IST perturbation theory.

Kaup and Hansen (1986) considered the NS equation on the semiaxis $x\geq 0$ with a prescribed boundary value $u(x=0,t)=q(t)$. As usual, the NS equation is represented as a condition of compatibility of the two linear equations (2.2) and (2.3) with the operators $\hat{L}$ and $\hat{A}$ defined by Eqs. (2.27) and (2.28). However, this time the auxiliary scattering problem (2.2) must be considered on the semiaxis $(0,+\infty)$, instead of the whole axis $(-\infty,+\infty)$. According to Kaup and Hansen (1986), this distinction from the standard situation means that the trivial evolution equations (2.35) for the scattering amplitudes $a(\lambda)$ and $b(\lambda)$ are replaced by the equations

$$i\frac{dB}{dt}=(2\lambda^2-|q|^2)B+(2i\lambda q-p)A\ , \tag{8.1}$$

$$i\frac{dA}{dt}=-(2\lambda^2-|q|^2)A-(2i\lambda q^*+p^*)B\ , \tag{8.2}$$

where $B(\lambda,t)\equiv b(\lambda,t)\exp(2i\lambda^2 t)$, $A(\lambda,t)\equiv a(\lambda,t)\times\exp(2i\lambda^2 t)$, and $p(t)\equiv u_x(x=0,t)$. Though the quantity $q(t)$ is specified by the boundary condition, $p(t)$ is not known *a priori*, and it can be found only together with a full solution of the boundary problem under consideration, so the inverse scattering transform for this problem cannot be cast in a closed form. The idea of Kaup and Hansen (1986) is to set $p(t)=0$, which is sometimes suggested by the results of numerical simulations. Then the usual technique can be applied, provided the linear system (8.1) and (8.2) with $p(t)=0$ and given $q(t)$ can be solved analytically. A comparison of results obtained by means of this approach with numerical data has convinced Kaup and Hansen (1986) that good qualitative and satisfactory quantitative agreement can be achieved. Earlier, Kaup and Neuberger (1984) (see also Kaup and Hansen, 1987) developed a similar approach for a Toda lattice with a boundary condition.

In the next two sections we shall consider some special soliton generation problems, which can be analyzed in terms of a more consistent perturbative technique.

B. Generation of solitons by a pulse of an external force

1. Generation of fluxons by a bias current pulse in a semi-infinite Josephson junction

In this subsection we shall deal with the model of a semi-infinite Josephson junction put forward by Sakai and Samuelsen (1987),

$$u_{tt}-u_{xx}+\sin u=-\gamma u_t+h(t)\delta(x)\ , \tag{8.3}$$

where the additional condition $u(-x)\equiv u(x)$ is implied (only the range $x\geq 0$ is physical). The term $h(t)\delta(x)$ on the right-hand side of Eq. (8.3) describes a bias current injected through the junction's edge $(x=0)$, $h(t)=h_0+f(t)$, where h_0 is a permanent component and the function $f(t)$ describes a short pulse. Sakai and Samuelsen (1987) considered a pulse of a triangular form, and they have demonstrated numerically that, for sufficiently large h_0, the pulse gives rise to a fluxon escaping from the edge of the junction. They also developed an analytical approach which, however, did not seem reliable. In an experiment, the generation of a fluxon by a bias current pulse was observed by Sakai *et al.* (1984, 1985a, 1985b).

Here, following Kivshar and Malomed (1989a), we shall describe a consistent analytical approach to the birth of a fluxon for small h_0 and a boxlike pulse

$$f(t)=\kappa_1[\theta(t)-\theta(t-T)]\ , \tag{8.4}$$

where

$$T \ll 1, \quad \kappa_1 T \sim 1 . \tag{8.5}$$

When one assumes the conditions (8.5), during the time $0 < t < T$ of the action of the bias current pulse one may neglect the term $\sin u$ in Eq. (8.3). The corresponding linear inhomogeneous d'Alembert equation has the following solution at the moment $t = T$ (see Fig. 43):

$$u(x,T) = \frac{\kappa_1}{2}\left[-x\,\mathrm{sgn}x + \tfrac{1}{2}(x+T)\left[1+\frac{\gamma}{4}(x-T)\right]\mathrm{sgn}(x+T) + \tfrac{1}{2}(x-T)\left[1-\frac{\gamma}{4}(x+T)\right]\mathrm{sgn}(x-T)\right], \tag{8.6}$$

$$u_t(x,T) = \frac{\kappa_1}{4}\left[1-\frac{\gamma T}{2}\right][\mathrm{sgn}(x+T) - \mathrm{sgn}(x-T)] . \tag{8.7}$$

Using the wave form (8.6) and (8.7) as an initial condition for the unperturbed SG equation (the pulse is switched off at $t > T$), one can solve the direct scattering problem (2.2) and (2.27) approximately for these initial conditions under the assumptions (8.5). The Jost coefficient $a(\lambda)$ [defined in Eq. (2.12)] corresponding to these initial conditions may be calculated explicitly for $k(\lambda) \equiv \lambda - 1/4\lambda \ll \kappa_1 \sim T^{-1}$:

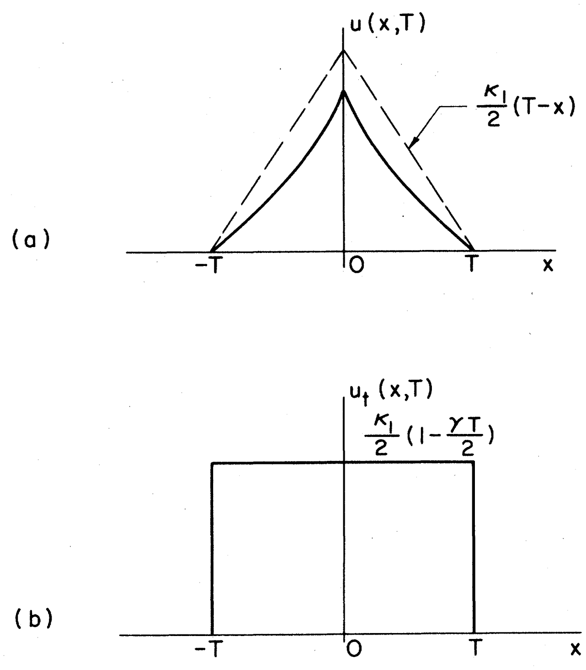

FIG. 43. The wave-field configuration created by the pointlike pulse (8.4) and (8.5).

$$a(k) = e^{ikT}\left[\left[1-\frac{ikT}{2}\right]\cos\frac{BT}{2} + \frac{\gamma BT^2}{4}\sin\frac{BT}{2} - \frac{ik}{B}\sin\frac{BT}{2}\right], \quad B \equiv \frac{\kappa_1}{2}\left[1-\frac{\gamma T}{2}\right] . \tag{8.8}$$

Due to the adopted additional condition $u(-x) \equiv u(x)$, a fluxon in a real semi-infinite junction ($x > 0$) corresponds to a kink-antikink pair on the whole axis $-\infty < x < +\infty$. The threshold for the birth of the first kink-antikink pair is reached when the quantity $a(k=i)$ vanishes. As follows from Eq. (8.8), the pair exists in the parametric range

$$|\kappa_1| T \ge 2\pi(1+\gamma T) + \frac{4T}{\pi} . \tag{8.9}$$

Quite analogously, one can find thresholds for the birth of a larger number of fluxons (Kivshar and Malomed, 1989a).

In addition to fluxons, the bias current pulse generates radiation (plasma waves) described by the Jost coefficient $b(\lambda)$ defined in Eq. (2.12). In particular, at the threshold for the birth of one fluxon [i.e., when Eq. (8.9) takes the form of an equality],

$$b(\lambda) \equiv b(k) = -i\left[1-\frac{ikT}{2}\right]\cos\left[\frac{T}{\pi} + \frac{\pi\gamma T}{4}\right] \tag{8.10}$$

for $k \ll T^{-1}$. Using the general expression (2.69) for the total energy of the radiation wave field and Eq. (8.10), one can show that the share $E_{\mathrm{fl}}/E_{\mathrm{tot}}$ of the total energy input from the pulse that is expanded on the birth of the fluxon at the threshold is very small: $E_{\mathrm{fl}} \approx 8$;

$$E_{\mathrm{tot}} \approx (4/\pi)\int_0^\infty |b(k)|^2 dk \sim \frac{1}{T}\int_0^1 \left| b\left[\frac{u}{T}\right]\right|^2 dk \sim \mathrm{const}/T ,$$

and $E_{\mathrm{fl}}/E_{\mathrm{tot}} \sim T$.

So far, we have neglected the dissipative term in Eq. (8.3) (at times $t > T$) and the presence of the permanent bias current h_0. Further analysis demonstrates that in the presence of dissipation the generated fluxon indeed escapes provided its initial velocity V_0 exceeds the critical value

$$(V_0)_{\mathrm{cr}} = \gamma \ln|h_0|^{-1} . \tag{8.11}$$

According to Eq. (8.8), the initial velocity of the created fluxon may be represented as

$$V_0^2 = (k_0^2 - 1)/k_0^2 , \tag{8.12}$$

where k_0 ($k_0 > 1$) is a root of the equation $a(ik_0) = 0$:

$$k_0 = -B\left[1+\frac{\gamma BT^2}{4}\tan(BT/2)\right] \Big/ \left[\frac{BT}{2} + \tan\frac{BT}{2}\right] .$$

If V_0^2 is less than the value (8.11) (in particular, if $\gamma \ln|h_0|^{-1} > 1$), the fluxon returns eventually to the edge of the junction and annihilates there due to dissipative losses.

Quite similarly, one can find thresholds for the creation of one or more breathers by the bias current pulse (Kivshar and Malomed, 1989a). Physically, a breather describes an oscillating fluxon pinned by the edge of the junction. However, this problem is less important, since the breather will eventually be damped by the dissipation.

2. Generation of magnetic solitons by a broad field pulse

A quasi-one-dimensional easy-plane ferromagnet in the presence of a strong constant magnetic field H (lying in the easy plane) and an additional variable field $h(x,t)$ is described by the following perturbed SG equation for the orientation angle $u(x,t)$ of the magnetization vector lying in the easy plane (Kosevich *et al.*, 1983; Lyubchansky *et al.*, 1987):

$$u_{tt}-u_{xx}+[1+h(x,t)\cos\chi]\sin u=-\gamma u_t+h(x,t)\sin\chi \tag{8.3'}$$

[cf. Eq. (8.3)], where χ is an angle between the vectors H and h in the easy plane. An interesting physical problem is related to a field pulse of the form [cf. Eq. (8.4)]:

$$h(x,t)=0 \quad \text{for } t<0 \text{ and } t>T\ , \qquad h(x,t)=\begin{cases} h_0, & |x|\le L/2\ ,\\ 0, & |x|>L/2 \end{cases} \tag{8.4'}$$

for $0<t<T$. The initial conditions are the same as above, $u(x,0)=u_t(x,0)=0$, and we assume again $T\ll 1$ [see Eq. (8.5)]. The important difference between this problem and the preceding one is that this time the pulse is not pointlike. We shall concentrate on the case of a broad pulse, $L\gg T$ [the results given below were obtained by Kivshar and Malomed (1989b)].

The solution of the problem follows closely the pattern described above: using $T\ll 1$, we solve an effective inhomogeneous d'Alembert equation on the time interval $0<t<T$. The form of the functions $u(x,T)$ and $u_t(x,T)$ found from that equation is shown in Fig. 44, where we have made use of the assumption $L\gg T$. The subsequent solution of the direct scattering problem for the unperturbed SG equation yields the result [cf. Eq. (8.8)]

$$a(k)=e^{ikL/2}\left[\cos(\kappa L)-\frac{ik}{2\kappa}\left[1+\frac{a^2T^4}{24}\right]\sin(\kappa L)\right]\ , \tag{8.8'}$$

where $a\equiv h_0\sin\chi$, $\kappa^2=k^2/4+a^2T^2/16$, and we assume $\tan\chi\gg T/L$. Analysis of the equation $a(k)=0$ with $a(k)$ defined by Eq. (8.8') leads to the conclusion that a breather (i.e., magnetic soliton) is generated under the condition

$$aLT>2\pi\ . \tag{8.13a}$$

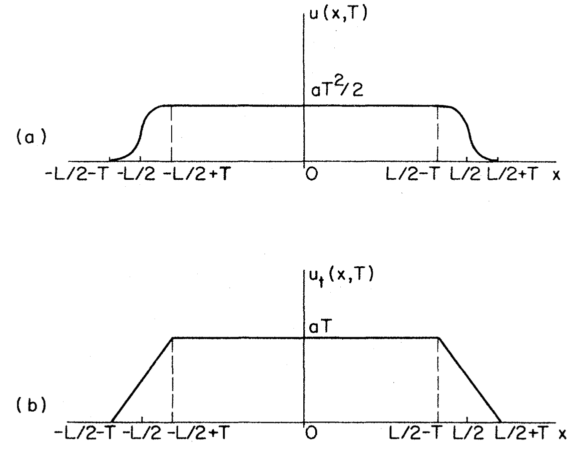

FIG. 44. The wave-field configuration created by the broad pulse (8.4') with $T\ll 1$, $L\gg T$.

With an increase in the parameter aLT, a kink-antikink pair (i.e., two domain walls of opposite polarities) may be generated instead of a breather. The velocities $\pm V$ of the kink and antikink are determined by Eq. (8.12): $V^2=(k^2-1)/k^2$, where this time k ($k>1$) is a real root of the transcendental equation

$$Z\cot Z=-[(aLT/4)^2-Z^2]^{1/2}[1+\tfrac{1}{24}(aT^2)^2]\ , \qquad Z\equiv\tfrac{1}{2}L[(aT/2)^2-k^2]^{1/2}\ . \tag{8.13b}$$

Further analysis of Eq. (8.13b) demonstrates that the kink-antikink pair is generated under the condition $aLT\ge F(L)$, where $F(L)$ is a certain monotonically growing function that is determined by Eq. (8.13b) on replacing $Z\to\frac{1}{4}\sqrt{F^2(L)-4L^2}$, $aLT\to F(L)$ [the term $(aT^2)^2$ gives a small correction]. For $L\to\infty$, $F(L)\approx 2L+4\pi^2/L$, $F(0)=2\pi$, and $F(L)>2L$ for all values of L [cf. Eq. (8.13a)].

The influence of dissipative losses on the evolution of the generated solitons can be investigated as in the preceding subsection. In particular, the threshold condition (8.13a) takes the form

$$aLT>2\pi(1+\gamma T)\ .$$

C. The spectral singularity in the perturbation theory for the Korteweg–de Vries equation and the problem of multiple soliton production

The presence of the multiplier k^{-1} in the perturbation-induced evolution equation (2.23) for the continuous-spectrum scattering data gives rise to the well-known divergence in the perturbation theory for the KdV equation (integrals over the spectral parameter k diverge at $k\to 0$). We shall consider this problem for the perturbed equation

$$u_t - 6uu_x + u_{xxx} = -\epsilon[\alpha(u + 2xu_x) - \beta(2u + xu_x)] , \tag{8.14}$$

where ϵ is a small parameter, while α and β are arbitrary. In the particular case $\beta = 2\alpha$ the perturbation on the right-hand side of Eq. (8.14) goes over into that described by Eq. (1.2) (with $\alpha_2 = 0$). This perturbation has the sense of dissipation or pumping, respectively, in the cases $\epsilon\alpha < 0$ and $\epsilon\alpha > 0$. It has been demonstrated by Newell (1980) that, in the latter case, at times $t \ll |\epsilon|^{-1}$ the divergence results in formation of a long shelf on the background of the soliton. It was speculated that at larger times the shelf would decay into "secondary" solitons, but this stage was not subjected to accurate investigation. In this section we shall present the recent results of Benilov and Malomed (1988), which give some additional insight into the problem.

It is important to note that some kinds of perturbations do not give rise to divergence (Karpman, 1978). In particular, when $\alpha = 0$ divergences are absent at least in the first order of the perturbation theory. Another remarkable property of this particular case is its exact integrability (Calogero and Degasperis, 1982): Eq. (8.14) with $\alpha = 0$ has the exact (L, A) pair

$$-\Psi_{xx} + [u - k^2 \exp(-2\beta t)]\Psi = 0 , \tag{8.15}$$

$$\Psi_t - 2\left[u - \frac{\epsilon\beta}{2}x + 2k^2 \exp(-2\beta t)\right]\Psi_x + \left[u_x - \frac{\epsilon\beta}{2}\right]\Psi = 0 . \tag{8.16}$$

There is another integrable case, $\beta = 0$ (Calogero and Degasperis, 1982), the corresponding (L, A) pair being

$$-\Psi_{xx} + \left[u - \frac{\epsilon\alpha}{2}x - k^2 \exp(2\epsilon\alpha t)\right]\Psi = 0 , \tag{8.17}$$

$$\Psi_t - 2[u + 2k^2 \exp(2\epsilon\alpha t)]\Psi_x + u_x\Psi = 0 . \tag{8.18}$$

Nevertheless, formal application of the standard perturbation theory to the latter case entails the appearance of the divergence.

The starting point of the present consideration is the fact that when $\beta = 0$ the form of the scattering data defined by the unperturbed L equation [see (2.9a)] is drastically distinct from that defined by the "perturbed" equation (8.17). Due to the infinite increase of the effective potential $u_{\text{eff}} = u + \epsilon\alpha x/2$ in Eq. (8.17) at $x \to \pm\infty$, a discrete spectrum is absent, and the reflection coefficient satisfies the equality

$$|r(k)| \equiv 1 \tag{8.19}$$

[all information about $u(x)$ is mapped into the phase $\arg r(k)$ of the reflection coefficient]. On the other hand, treating the "integrable" perturbation $\epsilon\alpha(u + xu_x)$ in the spirit of the standard perturbation theory, one would employ the unperturbed L equation and, consequently, a quite inadequate approximation of the genuine scattering data. This seems to be the cause of the appearance of divergences in this case. Note that at $\alpha = 0$ (the divergence-free case) the effective potential in the "perturbed" (L, A) pair (8.15) does not grow at $x \to \pm\infty$ and, accordingly, no divergences appear in the perturbation theory.

Of course, in the integrable cases $\alpha = 0$ and $\beta = 0$ the perturbation theory is not needed at all. Let us see how to construct a divergence-free perturbation theory for a generic (nonintegrable) perturbation of the type (8.14). The above observation suggests that in this case the L equation should be properly modified on a level with the A equation. A natural way to realize this idea is to construct an (L, A) pair for Eq. (8.14) in the form of a power series in ϵ [something like this was also discussed by Kodama (1986) and Menyuk (1986a, 1986b)]:

$$-\Psi_{xx} + \left[u + \sum_{n=1}^{\infty} \epsilon^n u_n - k^2\right]\Psi = 0 , \tag{8.20}$$

$$\Psi_t - 2\left[u + \sum_{n=1}^{\infty} \epsilon^n w_n + 2k^2\right]\Psi_x + \left[u + \sum_{n=1}^{\infty} \epsilon^n w_n\right]_x \Psi = 0 . \tag{8.21}$$

Straightforward calculations yield the following expressions for the two lowest terms of the series:

$$u_1 = 2k^2(\beta - \alpha)t - \frac{\alpha}{2}x , \tag{8.22}$$

$$w_1 = 4k^2(\alpha - \beta)t - \frac{\beta}{2}x , \tag{8.23}$$

$$u_2 = -2k^2(\beta - \alpha)^2 t^2 + \tfrac{3}{2}\alpha\beta t x + v , \tag{8.24}$$

$$w_2 = 4k^2(\alpha - \beta)^2 t^2 + \tfrac{3}{2}\alpha\beta t x + v , \tag{8.25}$$

where the function $v(x,t)$ satisfies the linear equation

$$v_t + 6(uv)_x + v_{xxx} = 9\alpha\beta t(xu)_x . \tag{8.26}$$

It is easy to see that the effective potential in Eq. (8.20) contains the term $\sim x$ already in the first order of the perturbation theory [see Eq. (8.22)]. The only exception is the case $\alpha = 0$ [the same pertains to the second order; see Eq. (8.24)]. It is remarkable that, as mentioned above, this is actually the only case when divergences do not appear in the standard perturbation theory.

A correct version of perturbation theory could be based upon a truncation of the series in Eqs. (8.20) and (8.21). However, due to the secular dependence of u_n and w_n on t [see Eqs. (8.22)–(8.26)], such truncated expansions cease to be applicable at times $t \gtrsim |\epsilon|^{-1}$. On the other hand, comparison of Eqs. (8.20)–(8.26) in the exactly integrable cases with the corresponding exact (L, A) pairs (8.15) and (8.16) and (8.17) and (8.18) demonstrates that one may hope to sum up all the secular terms into something like $\exp[\epsilon f(\alpha, t)t]$. However, this program remains to be completed.

At the same time, there is a simpler approach to the problem. It is easy to see that the transformation

$$t \to \tilde{t} = (3\beta\epsilon)^{-1}[1-\exp(-3\beta\epsilon t)] ,$$
$$x \to \tilde{x} = x \exp(-\beta\epsilon t) , \quad (8.27)$$
$$u \to \tilde{u} = u \exp(-2\beta\epsilon t)$$

eliminates the second term on the right-hand side of Eq. (8.14):

$$\tilde{u}_{\tilde{t}} - 6\tilde{u}\,\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = \gamma(\tilde{t})(\tilde{u} + 2\tilde{u}_{\tilde{x}}\tilde{x}) , \quad (8.28)$$

$$\gamma(\tilde{t}) \equiv -\epsilon\alpha/(1-3\epsilon\beta\tilde{t}) . \quad (8.29)$$

In contrast to Eq. (8.14) with $\beta=0$, Eq. (8.28) is not exactly integrable due to the presence of the time-dependent coefficient $\gamma(\tilde{t})$. Nevertheless, if the characteristic time scale of a solution that we are interested in is much smaller than the characteristic time during which the coefficient $\gamma(\tilde{t})$ alters significantly, we may treat Eq. (8.28) in the adiabatic approximation, i.e., first assume $\gamma=$const and write a corresponding exact solution, and then insert into it the function (8.29) instead of const.

To illustrate this scheme, let us apply it to the following exact solution of Eq. (8.28) with $\gamma=$const (Calogero and Degasperis, 1982):

$$\tilde{u} = \left[\frac{\gamma}{2}\right]^{2/3} U\left[\left[\frac{\gamma}{2}\right]^{1/3}\tilde{x} - z(\tilde{t}), \rho(\tilde{t})\right] , \quad (8.30)$$

$$U(y,\rho) \equiv 2\rho[2\,\mathrm{Ai}'(y)\mathrm{Ai}(y) + \rho(\mathrm{Ai}(y))^4 G(y)]G(y) , \quad (8.31)$$

$$G(y) \equiv [1+\rho(\mathrm{Ai}'(y))^2 - \rho y(\mathrm{Ai}(y))^2]^{-1} , \quad (8.32)$$

where $\mathrm{Ai}(y)$ is the Airy function, and $z(\tilde{t}) = z_0\exp(-2\gamma\tilde{t})$, $\rho(\tilde{t}) = \rho_0\exp(-2\gamma\tilde{t})$. Replacing in the latter expressions the constant γ by the function (8.29), one obtains

$$z(\tilde{t}) = z_0(1-3\epsilon\beta\tilde{t})^{-2\alpha/3\beta} ,$$
$$\rho(\tilde{t}) = \rho_0(1-3\epsilon\beta\tilde{t})^{-2\alpha/3\beta} . \quad (8.33)$$

Equations (8.30)–(8.33) give an approximate solution to Eq. (8.27) provided $|\epsilon|\tilde{t} \gg 1$. According to Eq. (8.27), this condition makes sense only for the case $\epsilon\beta<0$, i.e., for a pumping-type perturbation, when it means simply $|\epsilon|t \gg 1$. Thus, in contrast with the standard perturbative technique (Newell, 1980), which is relevant for the early stage of evolution ($|\epsilon|t \lesssim 1$), this approach works at the late stage and is meaningful only for the pumping case. In particular, the approximate solution (8.30)–(8.33) may be interpreted as describing the generation of an infinite number of new solitons [see a graph of the function $U(y,\rho)$ in Calogero and Degasperis, 1982] at the late stage of evolution of an initial profile $u_0(x)$. Note, however, that, due to the property $\int_{-\infty}^{+\infty} U(y,\rho)dy = 0$, the initial profile must be restricted by $\int_{-\infty}^{+\infty} u_0(x)dx = 0$. Earlier, the singularity of the perturbation theory was interpreted by Wright (1980) just as a manifestation of the production of new solitons.

D. Transformation of a Korteweg–de Vries soliton passing a zero-dispersion point

In Sec. III.A.1 we mentioned the KdV equation (3.8) with variable coefficients. The important particular case is that in which either coefficient α or β vanishes at some $t=t_0$ (in what follows we set $t_0=0$). An example is an equation describing the propagation of disturbances in a two-layer shallow liquid with the ratio of depths of the two layers subject to a long-range modulation (Kakutani and Yamasaki, 1978; Helfrich, Melville, and Miles, 1984). The same equation describes ion acoustic waves in a plasma with negative ions, the ion density being slowly modulated in space (Watanabe, 1984). In both these models, there are critical values of the relevant parameters (for instance, the ratio of the two depths in the two-layer liquid equal to one) at which the coefficient α in front of the nonlinear-dispersion term uu_x in Eq. (3.8) vanishes. It has been demonstrated in the papers mentioned that, generally speaking, in this case the higher nonlinear term u^2u_x must be taken into account. So we arrive at a modified KdV equation with variable coefficients:

$$u_t - 6\alpha(t)uu_x - 6\gamma u^2 u_x + u_{xxx} = 0 . \quad (8.34)$$

Here we regard the coefficient in front of u_{xxx} as constant. To describe phenomena related to the vanishing of the coefficient α, we set $\alpha \equiv -t$. We assume that at $t<0$ there is one soliton, and we are interested in the result of its transformation induced by a change of the sign of α. The problem has been studied numerically by Helfrich, Melville, and Miles (1984) and analytically by Malomed and Shrira (1989), whose consideration is based upon the two exact integrals of motion of Eq. (8.34), viz., the mass

$$M = \int_{-\infty}^{+\infty} u(x)dx \quad (8.35)$$

and the momentum

$$P = \tfrac{1}{2}\int_{-\infty}^{+\infty} u^2(x)dx . \quad (8.36)$$

For $\alpha=$const, the one-soliton solution to Eq. (8.34) is [cf. Eq. (2.18)]

$$u_{\mathrm{sol}} = -4\kappa^2(\alpha^2-4\gamma\kappa^2)^{-1/2} \times \{\cosh[2\kappa(x-4\kappa^2 t)] + \alpha(\alpha^2-4\gamma\kappa^2)^{-1/2}\}^{-1} , \quad (8.37)$$

where κ is the soliton's amplitude.

A general sketch of the evolution of the initial soliton pulse is as follows: due to vanishing of α, the amplitude κ varies in time, and, simultaneously, the soliton gives rise to a long small-amplitude shelf (cf. Sec. VI.N). The evolution law for the amplitude, valid as long as the soliton's amplitude remains much greater than that of the shelf, follows from Eq. (8.36) (for sufficiently small γ, the soliton's momentum is $P_{\mathrm{sol}} \approx \frac{8}{3}\kappa^3\alpha^{-2}$):

$$\kappa(t) = \kappa_0[\alpha(t)/\alpha_0]^{2/3} = \kappa_0(t/t_0)^{2/3} \quad (8.38)$$

[recall that we have set $\alpha(t)=-t$; $t_0<0$ is an initial moment, and $\kappa_0\equiv\kappa(t_0)$]. The distance $l(t)$ traveled by the soliton is simultaneously the length of the shelf generated by it (cf. Sec. VI.N). As follows from Eq. (8.38), the total length of the shelf is

$$l(0)=4\int_{t_0}^{0}\kappa^2(t)dt\approx\tfrac{12}{7}\kappa_0^2|t_0| \ . \tag{8.39}$$

Next, using mass conservation (for sufficiently small γ, the soliton's mass is $M_{\rm sol}\approx-2\alpha^{-1}\kappa$), one can find the value $u(t)$ of the wave field inside the shelf [in the first approximation, the shelf is uniform, i.e., $u(t)$ does not depend on x]:

$$u(t)\approx\tfrac{7}{6}\kappa_0^{-1}|t_0|^{-5/3}|t|^{-1/3} \ . \tag{8.40}$$

The momentum concentrated in the shelf is of the same order as the soliton's momentum at $|t|=t_{\rm cr}\sim\kappa_0^{-9/2}|t_0|^{-1/2}$, so that at $|t|<t_{\rm cr}$ Eqs. (8.38) and (8.40) are not valid. Nevertheless, using the natural assumption

$$|t_0|\kappa_0^3\gg1 \tag{8.41}$$

(this means that the frequency $\omega_{\rm sol}\sim\kappa_0^3$ of the initial soliton is much greater than the inverse evolution time $|t_0|^{-1}$), one can verify that Eq. (8.39) for the final length of the shelf remains valid, and the shelf's amplitude takes the final value

$$u(0)=(\sqrt{14}/3)(\kappa_0/|t_0|^3)^{1/2} \ . \tag{8.42}$$

The results (8.39) and (8.42) are also valid if $\gamma=0$ in Eq. (8.34).

It is important to note that the sign of $u(t)$ [see Eqs. (8.40) and (8.42)] is opposite to that of $u_{\rm sol}$ [see Eq. (8.37)]. Therefore, after the change of sign of α, the shelf may be regarded as an initial wave-field configuration that will give rise to new (secondary) solitons. The polarity (sign) of the secondary solitonic wave field is reversed from that of the initial soliton (8.37). Let us assume that at $t>0$ the coefficient $\alpha(t)$ attains, sufficiently quickly, a certain constant value $\alpha_f<0$. Then the number N of secondary solitons can be found easily if one treats the shelf as a rectangularlike initial pulse. Using Eqs. (8.39) and (8.42), one obtains

$$N^2\approx(\sqrt{14}/\pi^2)\kappa_0^{9/2}\sqrt{|t_0|}\,|\alpha_f| \ . \tag{8.43}$$

The amplitude of the nth soliton ($1<n<N$) is

$$\kappa_n\approx\pi n/l(0)=(7\pi/12)(\kappa_0^2|t_0|)^{-1}n \ . \tag{8.44}$$

Setting $n=N$ in Eq. (8.44), one finds the maximum amplitude,

$$\kappa_N\approx(7\times14^{1/4}/12)\kappa_0^{1/4}|t_0|^{-3/4}\sqrt{|\alpha_f|} \ . \tag{8.45}$$

Even if the coefficient $\alpha(t)$ does not attain a constant value sufficiently quickly, the qualitative inference remains true: the change of sign of the nonlinear-dispersion coefficient in the KdV equation results in the transformation of an initial soliton into a set of secondary solitons with reversed polarity. This inference does not depend on the presence of the additional term γu^2u_x in Eq. (8.34), provided γ is sufficiently small.

Malomed and Shrira (1989) have also considered the case when the linear-dispersion coefficient $\beta(t)$ in Eq. (3.8) changes the sign. It is convenient to set $\beta(t)\equiv-t$. Then the evolution of the soliton's amplitude takes the opposite form from Eq. (8.38),

$$\kappa(t)=\kappa_0(t_0/t)^{2/3} \ . \tag{8.46}$$

Using Eq. (8.46), it is possible to demonstrate that, contrary to the preceding case, the shelf generated by the soliton has the same polarity as the soliton. This implies that, after the change of sign of β (at $t>0$), the shelf will not give rise to secondary solitons. Instead, it will degrade into quasilinear (dispersive) waves. Thus a change of sign of the linear-dispersion coefficient in the KdV equation results in the decay of an initial soliton.

Malomed and Shrira (1989) have considered analogous problems for a NS equation and a Burgers equation with variable coefficients. In those cases, only decay of a NS soliton or Burgers shock wave, respectively, will take place if a dispersion coefficient changes its sign.

IX. NONSOLITON AND "SEMICLASSICAL" WAVE TRAINS

In this section we shall deal with perturbation-dominated dynamics of two special types of nonlinear wave packets: nonsoliton (dispersive) ones and the so-called semiclassical wave trains (Zakharov *et al.*, 1980), i.e., those containing a large number ($N\gg1$) of solitons. We shall be primarily concerned with the two different NS equations,

$$iu_t+u_{xx}\pm2|u|^2u=i\epsilon P(u) \ . \tag{9.1}$$

Equation (9.1) with the upper and lower signs in front of the nonlinear term on the left-hand side is often referred to as the nonlinear Schrödinger equation "with attraction" [NSE(+)] and "with repulsion" [NSE(−)]. Of greatest interest is the dissipative perturbation (1.8), which, for the time being, we shall write in a more convenient form,

$$\epsilon P=-\epsilon_1u+\tfrac{1}{8}\epsilon_2u_{xx}-2\epsilon_3|u|^2u \ . \tag{9.2}$$

The perturbation (9.2) occurs in a number of physical problems mentioned in Sec. I.B.

Dissipative effects are of special concern near the linear instability threshold, where ϵ_1 changes sign. [At $\epsilon_1<0$ the trivial solution of Eq. (9.1) $u\equiv0$ is unstable against small long-wave perturbations; in such a situation the term $-i\epsilon_1u$ on the right-hand side of Eq. (9.1) describes linear pumping.] Therefore we shall pay basic attention to the case $\epsilon_1=0$.

Just as the NS equation is a universal one for describing one-dimensional weakly nonlinear envelope waves (see, for example, Taniuti, 1974), the KdV equation (as mentioned in Sec. I.B) appears as a universal equation in the theory of weakly nonlinear waves in media with weak dispersion (see, for example, Zakharov *et al.*, 1980). The dissipatively perturbed KdV equation,

$$u_t - 6uu_x + u_{xxx} = -\epsilon_1 u + \tfrac{1}{8}\epsilon_2 u_{xx} - 8\epsilon_3 u^3 , \qquad (9.3)$$

naturally arises when one takes dissipation into account. The parameters ϵ_1, ϵ_2, and ϵ_3 in Eq. (9.3) have the same sense as in Eq. (9.1), and the linear instability threshold corresponds again to $\epsilon_1 = 0$. Equations of the type (9.3) occur in various physical problems (see, for example, Kawahara and Toh, 1985a, 1985b); of special interest is the particular case of Eq. (9.3) with $\epsilon_1 = \epsilon_3 = 0$ (the KdV-Burgers equation).

A. Nonlinear Schrödinger and sine-Gordon wave packets

In Secs. IX.A and IX.B we shall consider the evolution of dissipatively damped nonsoliton wave packets. In this context, of basic interest is the study of the asymptotic ($t \to \infty$) stage of the evolution. The results presented in these sections were obtained by Malomed (1986b, 1987j).

1. A general case

In the spirit of IST perturbation theory, let us proceed from Eq. (9.1) to the evolution equations for the quantities $n(\lambda)$ and $\psi(\lambda)$, which, in the absence of perturbations, have the sense of canonical actions and angles (Bullough *et al.*, 1982; Malomed, 1982, 1986b). As is well known (see, for example, Zakharov *et al.*, 1980), these quantities can be expressed in terms of the continuous-spectrum scattering data:

$$n(\lambda) = \frac{1}{\pi}\ln[1 - |b(\lambda)|^2] , \qquad (9.4)$$

$$\psi(\lambda) = \arg b(\lambda) . \qquad (9.5)$$

The unperturbed evolution equations for the quantities (9.4) and (9.5) are trivial (Zakharov *et al.*, 1980);

$$\frac{dn}{dt} = 0, \quad \frac{d\psi}{dt} = 4\lambda^2 . \qquad (9.6)$$

The values of $n(\lambda)$ are determined by initial conditions. For instance, the simplest localized initial condition,

$$u_0(x) = \begin{cases} a_0, & 0 \le x \le l_0 , \\ 0, & x < 0 \text{ or } x > l_0 , \end{cases} \qquad (9.7)$$

yields for the NSE(+) (Zakharov *et al.*, 1980)

$$n(\lambda) = \frac{1}{\pi}\ln(1 + \{\sin^2[l_0(\lambda^2 + a_0^2)^{1/2}]\}/\{\lambda^2/a_0^2 + \cos^2[l_0(\lambda^2 + a_0^2)^{1/2}]\}) . \qquad (9.8)$$

In the case of a localized initial condition of a more general form than Eq. (9.7), the same qualitative inference is true as follows from Eq. (9.8): the size Λ_0 of the wave train in λ space is

$$\Lambda_0 \sim l_0^{-1} , \qquad (9.9)$$

and in the spectral range $|\lambda| \lesssim \Lambda_0$ the actions take the values

$$n(\lambda) \lesssim l_0^2 a_0^2 , \qquad (9.10)$$

where l_0 and a_0 are the characteristic size and amplitude of a wave train in x space at the initial moment. It is easy to verify that, provided $a_0 l_0 \lesssim 1$, the estimates (9.9) and (9.10) pertain to the NSE(−) as well.

In this subsection we shall deal with wave packets containing no solitons. For the initial configuration (9.7) the condition for absence of solitons takes the form

$$a_0 l_0 < \pi/2 . \qquad (9.11)$$

For initial conditions of a more general form this relation takes a similar form $a_0 l_0 <$ const, where const is ~ 1. For the NSE(−) we shall also presume $a_0 l_0 \lesssim 1$.

As the wave packet does not contain solitons, it spreads under the action of dispersion. At times

$$t \gg \Lambda_0^{-2} , \qquad (9.12)$$

the asymptotic stage of the evolution commences (Zakharov *et al.*, 1980): at distances $x^2 \gg t$ the wave train may be represented in the form

$$a^2(x,t) \equiv |u(x,t)|^2 \approx \frac{1}{4t} n(\lambda = -x/4t) , \qquad (9.13a)$$

$$\phi(x,t) \equiv \arg u(x,t) \approx \frac{x^2}{4t} + \tfrac{1}{2} n(\lambda = -x/4t)\ln(t\Lambda_0^2) . \qquad (9.13b)$$

In λ space, the range $x^2 \gg t$ corresponds to $\lambda^2 \gg t^{-1}$. In this range the quantities $n(\lambda)$ have a direct physical sense: They are occupation numbers of normal modes (Zakharov *et al.*, 1980).

The small dissipative terms in Eq. (9.1) result in the slow decay of the wave field, i.e., slow decrease of the action variables $n(\lambda)$ (Bullough *et al.*, 1982; Malomed, 1982). Equations describing the perturbation-induced evolution of the quantities (9.4) and (9.5) at the asymptotic stage (9.12) have been derived (assuming $\epsilon_2 = 0$) by Malomed (1982). These equations have a rather cumbersome form. However, it has been demonstrated by Bullough *et al.* (1982), with the aid of the adiabatic invariant theorem, that in the case $\epsilon_2 = \epsilon_3 = 0$ the evolution equation for $n(\lambda,t)$ averaged over the rapid unperturbed phase oscillations takes a very simple form,

$$\frac{dn}{dt}=-2\epsilon_1 n \ . \tag{9.14}$$

Substituting Eq. (9.13) into the right-hand side of Eq. (9.1), we notice that, on proceeding from $u(x,t)$ to $n(\lambda,t)$, the perturbing terms $\sim\epsilon_2,\epsilon_3$ reduced to the one $\sim\epsilon_1$ with the effective coefficient $(\epsilon_1)_{\rm eff}=(\epsilon_2/2)\lambda^2+(\epsilon_3/2t)n$, i.e., the averaged evolution equation for $n(\lambda,t)$ corresponding to the perturbed NS equation (9.1) with $\epsilon_1=0$ takes the form (Malomed, 1986b)

$$\frac{dn}{dt}=-\epsilon_2\lambda^2 n-(\epsilon_3/t)n^2 \ . \tag{9.15}$$

The exact solution of Eq. (9.15) is

$$n(\lambda,t)=n_0(\lambda)\{\exp[-\epsilon_2\lambda^2(t-t_0)]\}/\{1+\epsilon_3 n_0(\lambda)\exp(\epsilon_2\lambda^2 t_0)[E(\epsilon_2\lambda^2 t_0)-E(\epsilon_2\lambda^2 t)]\} \ , \tag{9.15$'$}$$

where $t\geq t_0$, $n_0(\lambda)\equiv n(\lambda,t=t_0)$, and $E(z)\equiv-\int_z^\infty e^{-t}dt/t$. Equation (9.15) and the solution (9.15′) make sense if the moment t_0 satisfies Eq. (9.12). At the same time, it is natural to take t_0 as a moment when the perturbation effects are still immaterial, i.e.,

$$1\ll t_0\Lambda_0^2\ll\epsilon_2^{-1}, \quad \epsilon_3 n_0\ln(t_0\Lambda_0^2)\ll 1 \ . \tag{9.16}$$

Then we may set

$$n_0(\lambda)\approx n(\lambda,t=0) \ , \tag{9.17}$$

where it is implied that at the moment $t=0$ we have a wave packet of a general form, with a nonsmall amplitude $\sim a_0$. Using Eq. (9.16) and the evident asymptotic formula $E(z)\approx\ln(z^{-1})$, valid at small z, one can write Eq. (9.15′) in a simple form that does not contain the auxiliary quantity t_0:

$$n(\lambda,t)=n_0(\lambda)[\exp(-\epsilon_2\lambda^2 t)] \times\{1+\epsilon_3 n_0(\lambda)\ln[(t\Lambda_0^2)/(1+\epsilon_2\lambda^2 t)]\}^{-1} \ . \tag{9.18}$$

This expression was obtained in the logarithmic approximation, i.e., it was assumed

$$\ln(t\Lambda_0^2)\gg\ln(t_0\Lambda_0^2) \ ; \quad \ln\epsilon_2^{-1}\gg\ln(t_0\Lambda_0^2) \ , \tag{9.19}$$

which is compatible with the condition (9.16).

Let us proceed to an investigation of the approximate solution (9.18). The solution in x space corresponding to (9.18) can be immediately obtained with the aid of Eq. (9.13a):

$$a^2(x,t)=(4t)^{-1}n_0(-x/4t)[\exp(-\epsilon_2 x^2/16t)] \times\{1+\epsilon_3 n_0(-x/4t)\ln[t^2\Lambda_0^2/(t+\epsilon_2 x^2/16)]\}^{-1} . \tag{9.20}$$

The expression for the phase $\phi(x,t)$ corresponding to Eq. (9.20) can be easily written in the range $\epsilon_2 x^2\ll t$:

$$\phi\approx x^2/4t+(2\epsilon_3)^{-1}\ln[1+\epsilon_3 n_0(-x/4t)\ln(t\Lambda_0^2)] \ . \tag{9.21}$$

It might seem that the above consideration makes no use of the exact integrability of the unperturbed NS equation, and that analogous calculations could be accomplished for a small-amplitude wave train in any weakly nonlinear system with dispersion. However, it is only in the case of a nearly integrable system that a solution of the type (9.18)–(9.21), which describes a small-amplitude wave packet, may be explicitly written in terms of the initial data $n_0(\lambda)$ pertaining to an initial state with a nonsmall amplitude [see Eq. (9.17)].

The wave train (9.18) and (9.20) is structured as follows: at times $t\lesssim(\epsilon_2\Lambda_0^2)^{-1}$,

$$n(\lambda,t)\approx n_0(\lambda)/[1+\epsilon_3 n_0(\lambda)\ln(t\Lambda_0^2)] \ , \tag{9.22}$$

$$a^2(x,t)\approx(4t)^{-1}n_0(-x/4t) \times[1+\epsilon_3 n_0(-x/4t)\ln(t\Lambda_0^2)]^{-1} \ , \tag{9.23}$$

so that the wave train's size l in x space grows with time as in the absence of perturbations:

$$l\sim\Lambda_0 t \tag{9.24}$$

(provided $t\gg l_0/\Lambda_0$), while the wave train's size Λ in λ space remains constant: $\Lambda=\Lambda_0$. At larger times

$$t\gg(\epsilon_2\Lambda_0^2)^{-1} \ , \tag{9.25}$$

the size of the wave train's "head" [described by Eq. (9.22) in λ space] decreases:

$$\Lambda\sim(\epsilon_2 t)^{-1/2} \ , \tag{9.26}$$

and in x space, where the "head" is described by Eq. (9.23), its size increases with time more slowly than (9.24):

$$l\sim 4(t/\epsilon_2)^{1/2} \ , \tag{9.27}$$

while the ranges $(\epsilon_2 t)^{-1}\ll\lambda^2\lesssim\Lambda_0^2$ and $16t/\epsilon_2\ll x^2\lesssim\Lambda_0^2 t^2$ in the x and λ spaces, respectively, are occupied by an exponentially small "tail." It is important that the size of the "head" (9.27) is always much bigger than the size of the internal range $x^2\lesssim t$, where the asymptotic formulas (9.13) and (9.14) are inapplicable.

As can be seen from Eqs. (9.22) and (9.23), at the final stage of evolution, determined by

$$\ln(t\Lambda_0^2)\gg(\epsilon_3 n_0)^{-1} \ , \tag{9.28}$$

the quantities $a^2(x,t)$ and $n(\lambda,t)$ inside the 'head" become independent both of the initial data $n_0(\lambda)$ and of x or λ, respectively:

$$a^2(t)\approx[4\epsilon_3 t\ln(t\Lambda_0^2)]^{-1} \ , \tag{9.29}$$

$$n(t)\approx[\epsilon_3\ln(t\Lambda_0^2)]^{-1} \ . \tag{9.30}$$

2. A threshold for soliton creation

The analysis developed in the preceding subsection is irrelevant if initial conditions are taken at a threshold corresponding to the birth of a soliton with an infinitely small amplitude. For instance, in the case of the initial conditions (9.7), this threshold corresponds to the values of the parameters a_0 and l_0 subject not to the inequality (9.11), but to the relation

$$a_0 l_0 = \pi/2 \ . \tag{9.31}$$

As can be seen from Eq. (9.8), in this case

$$n_0(\lambda) \approx \frac{1}{\pi}\ln(a_0^2/\lambda^2) \tag{9.32}$$

in the range $\lambda^2 \ll a_0^2$. The solution (9.18)–(9.27), obtained under the condition (9.11), is inapplicable in the case (9.31) because of the logarithmic singularity in Eq. (9.32). This singularity is inherent in any initial state corresponding to a soliton creation threshold. Here we shall concentrate on the evolution of a wave packet with the initial data (9.32). We shall be concerned with the spectral range over which threshold effects must be substantial. [According to Eqs. (9.31) and (9.32), a_0 plays the role of the initial size of the wave packet in λ space.] This range is equivalent to the x-space range $x^2 \ll a_0^2 t^2$.

Direct substitution into the unperturbed NSE(+) demonstrates that this equation possesses the following solutions: in the range $x^2 \gg t\ln(ta_0^2)$ (hereafter to be called the external region),

$$a^2 \approx (4\pi t)^{-1}\ln(t^2 a_0^2/x^2) \ , \tag{9.33}$$

$$\phi \approx x^2/4t + \frac{1}{4\pi}[\ln(t^2 a_0^2/x^2)]\ln(x^2 a_0^2) \tag{9.34}$$

[cf. Eqs. (9.13) and (9.14)]; in the range $x^2 \ll t$ (hereafter to be called the internal region),

$$a^2 \approx (4\pi t)^{-1}\ln(ta_0^2) \ , \tag{9.35}$$

$$\phi \approx (4\pi)^{-1}\ln^2(ta_0^2) + x^2/4t - x^2/[4t\ln(ta_0^2)] \ . \tag{9.36}$$

In the "underthreshold" problem (9.11), the internal region is that where the asymptotic stage of the unperturbed evolution has not as yet commenced.

The perturbed solution corresponding to the unperturbed one (9.33)–(9.36) has a form that can again be checked by direct substitution into the perturbed Eq. (9.1) (as before, we consider the case $\epsilon_1=0$): in the external region,

$$a^2 \approx (4\pi t)^{-1}[\ln(t^2a_0^2/x^2)]\exp(-\epsilon_2 x^2/16t) \times\{1+(\epsilon_3/2\pi)[\ln(t^2 a_0^2/x^2)] \times \ln(t^2 a_0^2 x^2)/(t+\epsilon_2 x^2/16)^2\}^{-1} \tag{9.37}$$

[cf. Eq. (9.20)], and in the internal region

$$a^2 \approx (4\pi t)^{-1}[\ln(ta_0^2)]/[1+(\epsilon_3/2\pi)\ln^2(ta_0^2)] \ . \tag{9.38}$$

In λ space the external region is $\lambda^2 \gg t^{-1}\ln(ta_0^2)$. Using Eq. (9.13), we obtain from Eq. (9.38) the expression for the occupation numbers in this region:

$$n(\lambda,t) = \pi^{-1}[\exp(-\epsilon_2\lambda^2 t)][\ln(a_0^2/\lambda^2)] \times\{1+(\epsilon_3/2\pi)[\ln(a_0^2/\lambda^2)] \times\ln[\lambda^2 t^2 a_0^2/(1+\epsilon_2\lambda^2 t)^2\}^{-1} \tag{9.39}$$

[cf. Eq. (9.18)]. To obtain the expression for $n(\lambda,t)$ in the internal region, one should take into account the fact that the logarithmic singularity at $\lambda=0$ [see Eq. (9.32)] must persist, since it may be interpreted as an infinitely-small-amplitude soliton [recall that the condition (9.31) defines the soliton creation threshold], and it is well known that the dissipation in Eq. (9.1) damps the soliton, but cannot eliminate it (Karpman and Maslov, 1977; Kaup and Newell, 1978a, 1978b). Thus we obtain in the internal region

$$n(\lambda,t) = \pi^{-1}[\ln(a_0^2/\lambda^2)]/[1+(\epsilon_3/2\pi)\ln^2(ta_0^2)] \ . \tag{9.40}$$

As can be seen from Eq. (9.40), the size of the wave-train "head" in λ space and x space is the same as in the "underthreshold" situation (9.11), i.e., given by Eqs. (9.26) and (9.23). However, in the present case the structure of the "head" differs from Eqs. (9.22) and (9.23). In particular, the final stage [see Eqs. (9.28)–(9.30)] commences in the internal region (which always lies inside the "head") at times

$$\ln(ta_0^2) \gg \epsilon_3^{-1/2} \tag{9.41}$$

[cf. Eq. (9.28)], and at this stage

$$a^2(t) \approx [2\epsilon_3 t\ln(ta_0^2)]^{-1} \ , \tag{9.42}$$

$$n(\lambda,t) \approx (2/\epsilon_3)[\ln(a_0^2/\lambda^2)]/\ln^2(ta_0^2) \ . \tag{9.43}$$

As we see, Eq. (9.42) does not coincide with Eq. (9.29), and Eq. (9.43), unlike Eq. (9.30), depends on λ. In the external regions of the x and λ spaces, the final stage is characterized by the conditions

$$\ln(t^2 a_0^2/x^2) \gg (2\pi/\epsilon_3)/\ln(x^2 a_0^2) \ , \tag{9.44}$$

$$\ln(t^2\lambda^2 a_0^2) \gg (2\pi/\epsilon_3)/\ln(a_0^2/\lambda^2) \ , \tag{9.45}$$

i.e., the beginning of the final stage in the external region depends on x or λ, in contrast with Eqs. (9.28) and (9.41).

At the final stage, Eqs. (9.37) and (9.39) go over into

$$a^2(x,t) \simeq [2\epsilon_3 t\ln(x^2 a_0^2)]^{-1} \ . \tag{9.46}$$

Unlike Eqs. (9.29) and (9.42), Eq. (9.46) depends upon x.

Thus we have described the evolution of the local amplitude $a(x)$ (in x space) and occupation numbers $n(\lambda)$ (in λ space). Evolution of the phase $\phi(x)$ can also be effectively described (Malomed, 1986b, 1987j).

3. Sine-Gordon wave packets

Now let us proceed to the dissipatively perturbed SG equation

$$u_{tt}-u_{xx}+\sin u=-\epsilon(1+\alpha\cos u)u_t+\epsilon_2 u_{txx}\ , \qquad (9.47)$$

which occurs in the theory of long Josephson junctions and in the theory of convection. [The first term on the right-hand side describes, according to Levring *et al.* (1984), dissipative losses due to tunneling of normal electrons across the barrier with regard to interference with the superconductive tunneling current; in the theory of convection, according to Coullet and Huerre (1986), it describes the propagation of transverse modulations in a system of forced convective rolls in a low-Prandtl-number liquid.] In the absence of perturbations ($\epsilon=\epsilon_2=0$), the nonsoliton wave field is described at large times by the asymptotic formulas (Zakharov and Manakov, 1976; Newell, 1978b) analogous to Eqs. (9.13) and (9.14),

$$u=a(x,t)\sin\phi(x,t)\ , \qquad (9.48)$$

$$a^2(x,t)=(16/t)(\lambda+1/4\lambda)\nu(\lambda)\ , \qquad (9.49)$$

$$\phi(x,t)=\sqrt{t^2-x^2}+\cdots\ , \qquad (9.50)$$

where $\nu(\lambda)$ is expressed in terms of the continuous-spectrum scattering data (2.33), $\nu(\lambda)\equiv(1/2\pi)\ln|a(\lambda)|^{-2}$, and

$$\lambda^2\equiv\tfrac{1}{4}(t-x)/(t+x) \qquad (9.51)$$

[$\nu(\lambda)=\nu(-\lambda)$ due to the reality of $u(x,t)$; see Eq. (2.55)]. At the asymptotic stage, the nonlinear terms in Eq. (9.47) may be expanded to cast this equation in the form

$$u_{tt}-u_{xx}+u-\tfrac{1}{6}u^3=-\epsilon_1 u_t+\epsilon_2 u_{txx}-\epsilon_3 u^2 u_t\ , \qquad (9.52a)$$

where $\epsilon_1\equiv\epsilon(1+\alpha)$ and $\epsilon_3\equiv-(\epsilon/2)\alpha$ (we assume $\alpha>-1$). Inserting Eqs. (9.48)–(9.51) into Eq. (9.52a) and performing calculations analogous to those described in Sec. IX.A.1, one obtains the evolution equation for $\nu(\lambda)$:

$$\frac{d\nu}{dt}=-2[\epsilon_1+(\lambda+1/4\lambda)^2\epsilon_2+(4/t)(\lambda+1/4\lambda)\nu]\nu\ . \qquad (9.52b)$$

Equation (9.52b) is similar to Eq. (9.15), and all subsequent considerations of the perturbed SG equation (9.47) can be developed by analogy with the perturbed NS equation (9.1). Particularly, the threshold for creation of a soliton in the NS equation corresponds to the threshold for the birth of a breather in the SG equation.

B. Korteweg–de Vries wave packets

If we deal with an initial condition for the perturbed KdV equation (9.3) in the form of a localized nonsoliton wave packet of amplitude $\sim a_0$ and size $\sim l_0$ [cf. Eq. (9.8)], the size of the wave packet in the space of the spectral parameter k (related to the linear spectral problem employed for solving the unperturbed KdV equation; see Sec. II.B) is, provided $a_0 l_0^2\lesssim 1$,

$$K_0\sim\min\{\sqrt{a_0},a_0 l_0\} \qquad (9.53)$$

[cf. Eq. (9.9)]. The asymptotic stage begins at $t\gg K_0^3$ [cf. Eq. (9.12)]. At this stage, the unperturbed wave field in the range

$$-x^3\gg t \qquad (9.54)$$

may be represented by a slowly varying amplitude $a(x,t)$ and a rapidly varying phase $\phi(x,t)$ (Zakharov *et al.*, 1980),

$$u(x,t)=a(x,t)\cos\phi(x,t)\ , \qquad (9.55)$$

$$\phi(x,t)=(2/3^{3/2})(-x^3/t)^{1/2}+\cdots\ , \qquad (9.56)$$

$$a^2(x,t)=(12t)^{-1}n(k=\sqrt{-x/12t}\,) \qquad (9.57)$$

[cf. Eqs. (9.14) and (9.13)]. Here $n(k)$ are the action variables that are expressed in terms of the continuous-spectrum scattering data (2.12) [cf. Eq. (9.4)]: $n(k)=(2k/\pi)\ln[1+|B(k)|^2]$. The values of $n(k)$ in the spectral range $k\lesssim K_0$ can be estimated as

$$n(k)\lesssim a_0 l_0\ . \qquad (9.58)$$

In the absence of perturbations, the quantities $n(k)$ remain constant [cf. Eq. (9.6)]. The dissipative perturbations in Eq. (9.3) result in an evolution equation similar to Eq. (9.15):

$$\frac{dn}{dt}=-2\epsilon_1 n-8\epsilon_2 k^2 n-(\epsilon_3/8t)n^2 \qquad (9.59)$$

[the derivation of Eq. (9.59) involves averaging with respect to the fast unperturbed oscillations; see Eqs. (9.55) and (9.56)]. Equation (9.59) is relevant in the k-space region

$$k\gg t^{-1/3}\ , \qquad (9.60)$$

which is equivalent to the x-space region (9.54).

As in the case of the perturbed NS equation (9.1), dissipation produces nontrivial effects near the linear instability threshold, i.e., when $\epsilon_1=0$. Then the solution to Eq. (9.59) coincides, with logarithmic accuracy [which in the present case implies $\ln(tK_0^3)\gg\ln(t_0K_0^3)$, $\ln(K_0/\epsilon_2)\gg\ln(t_0K_0^3)$; cf. Eq. (9.19)], with Eq. (9.18):

$$n(k,t)=n_0(k)[\exp(-\epsilon_2k^2t)]\times\{1+\epsilon_3 n_0(k)\ln[tK_0^3/(1+\epsilon_2k^2t)]\}^{-1}\ . \qquad (9.61)$$

Despite this coincidence, in the present case the evolution of a wave packet essentially differs from that of Eq. (9.1). As can be seen from Eq. (9.61), at times

$$t\gg(\epsilon_2K_0^2)^{-1} \qquad (9.62)$$

[cf. Eq. (9.25)], the size K of the wave train in k space is [cf. Eq. (9.26)]

$$K\sim(\epsilon_2 t)^{-1/2}\ , \qquad (9.63)$$

i.e., comparing Eq. (2.63) to Eq. (9.60), one sees that at times $t\gg\epsilon_2^{-3}$ the "head" as a whole lies outside the region where Eqs. (9.55)–(9.57) and all the above considerations hold. The evolution of the wave train in x space also reveals an important difference from the case of the NS equation: Substituting Eq. (9.61) into Eq. (9.57), one can see that at times $t \lesssim (\epsilon_2 K_0^2)^{-1}$ the size of the "head" grows similarly to Eq. (9.24), $l \sim K_0^2 t$, while at times (9.62) l becomes constant,

$$l \sim \epsilon_2^{-1} ,$$

in contrast with the increasing size represented by Eq. (9.27).

C. "Semiclassical" wave packets

In this section we shall study, following Malomed (1987j), the evolution of a "semiclassical" wave-train solution to Eq. (9.1), which can be represented in the form

$$u(x,t)=a(x,t)\exp\left[i\int p(x,t)dx\right] , \tag{9.64}$$

where the real local amplitude $a(x,t)$ and wave number $p(x,t)$ are assumed to be slowly varying functions of x:

$$|a_{xx}| \ll \max\{|a|^3, |a|p^2\} . \tag{9.65}$$

Here, it is convenient to rewrite Eqs. (9.1) and (9.2) in the form [cf. Eqs. (1.8)]

$$iu_t+u_{xx}\pm 2|u|^2u=-i\epsilon_1 u+\epsilon_2 u_{xx}-i\epsilon_3|u|^2u . \tag{9.66}$$

Inserting Eq. (9.64) into Eq. (9.66) results, with regard to Eq. (9.65), in a system of quasilinear first-order equations,

$$a_t=-ap_x-2pa_x-\epsilon_1 a-\epsilon_3 a^3-\epsilon_2 ap^2 , \tag{9.67}$$

$$p_t=-2pp_x\pm 4aa_x . \tag{9.68}$$

In the case of the NSE($-$) [the lower sign in Eq. (9.68)], this system is hyperbolic, and the method for solving it is well known (Whitham, 1974). Two families of corresponding characteristics on the (x,t) plane are determined by the equations

$$\frac{dx}{dt}=2(p-a), \quad \frac{dx}{dt}=2(p+a) . \tag{9.69}$$

Evolution of a and p along the characteristics is governed by the ordinary differential equations

$$\frac{d}{dt}(2a-p)=-2\epsilon_1 a-2\epsilon_3 a^3-2\epsilon_2 ap^2 , \tag{9.70}$$

$$\frac{d}{dt}(2a+p)=-2\epsilon_1 a-2\epsilon_3 a^3-2\epsilon_1 ap^2 . \tag{9.71}$$

Equations (9.69)–(9.71) solve, in principle, the evolution problem for the NSE($-$) quasiclassical wave packet.

For NSE($+$) the system of Eqs. (9.67) and (9.68) is elliptic. However, in this case one can employ IST perturbation theory. According to Zakharov *et al.* (1980), under the condition

$$|u_{xx}| \ll |u|^3 \tag{9.72}$$

[which is stronger than Eq. (9.65)], a quasiclassical wave packet may be approximated by a nonlinear superposition of a large number of solitons with different amplitudes η_n (n is the soliton number) and zero velocities, where the solitons' characteristic size η_n^{-1} is much less than the size l of the whole wave train:

$$l\eta_n \gg 1 \tag{9.73}$$

[$l\eta_n$ may be regarded as a large "semiclassicality parameter"]. The amplitudes η_n vary in time according to the perturbation-induced equation (2.48).

Inserting into Eq. (2.48) the perturbation from Eq. (9.66), we note that, by virtue of Eq. (9.72), the term $\epsilon_2 u_{xx}$ may be neglected in comparison with $\epsilon_3|u|^2u$. According to Zakharov *et al.* (1980), under the condition (9.72) the Jost functions that enter Eq. (2.48) can be found explicitly in the semiclassical approximation:[10]

$$\psi^{(1)}(x,\eta_n)=\text{const}\times q^{-1}\cos\left[\int q\,dx\right] , \tag{9.74}$$

$$\psi^{(2)}(x,\eta_n)=\text{const}\times[iu(x)q]^{-1} \times\left[q\sin\left[\int q\,dx\right]+\eta_n\cos\left[\int q\,dx\right]\right] , \tag{9.75}$$

where

$$q(x,\eta_n)\equiv[|u(x)|^2-\eta_n^2]^{1/2} . \tag{9.76}$$

Inserting Eqs. (9.74)–(9.76) into Eq. (2.48), we can cast the integral

$$M_n\equiv\int_{-\infty}^{+\infty}dx\{[\psi^{(1)}(x,\eta_n)]^2(\epsilon_1 u+2\epsilon_3|u|^2u) -[\psi^{(2)}(x,\eta_n)]^2(\epsilon_1 u^*+2\epsilon_3|u|^2u^*)\} \tag{9.77}$$

in Eq. (2.48) into the form

$$M_n=-(\text{const})^2\int_{x_1(\eta_n)}^{x_2(\eta_n)}dx\,u(x)[|u(x)|^2-\eta_n^2]^{-1} \times[\epsilon_1+2\epsilon_3|u(x)|^2] , \tag{9.78}$$

where the points $x_{1,2}(\eta_n)$ are determined by the equation $|u(x)|^2=\eta_n^2$. In the logarithmic approximation, i.e., regarding $\ln(l\eta_n)$ as a large parameter [see Eq. (9.73)] and neglecting corrections of the relative order $\sim[\ln(l\eta_n)]^{-1}$, the integral (9.78) can be readily simplified to

$$M_n=-(\text{const})^2(\epsilon_1+2\epsilon_3\eta_n^2) \times\int_{x_1(\eta_n)}^{x_2(\eta_n)}dx\,u(x)[|u(x)|^2-\eta_n^2]^{-1} . \tag{9.79}$$

Now let us note that the adiabatic invariant theorem applied in the Sec. IX.A to the occupation numbers of a

[10]The semiclassical approximation for the Zakharov-Shabat linear system $\hat{L}\Psi=\lambda\Psi$ with the operator $\hat{L}$ given by Eq. (2.27) has been developed in detail by Lewis (1985).

small-amplitude nonsoliton wave packet may, as well, be applied to solitons, for which the action variables are the quantities $4\eta_n$ (Bullough *et al.*, 1982). Thus, at $\epsilon_3=0$, the evolution equation for the amplitudes η_n averaged over the rapid phase oscillations reduces to [cf. Eq. (9.14)]

$$\frac{d}{dt}\eta_n=-2\epsilon_1\eta_n \ . \tag{9.80}$$

According to Eq. (9.79), for $\epsilon_3\neq 0$ the dissipation constant ϵ_1 in Eq. (9.80) should be replaced by the effective constant $(\epsilon_1)_{\rm eff}\equiv\epsilon_1+2\epsilon_3\eta_n^2$, and the final equation for η_n takes the form (in the logarithmic approximation)

$$\frac{d}{dt}\eta_n=-2(\epsilon_1+2\epsilon_3\eta_n^2)\eta_n \ . \tag{9.81}$$

It is opportune to note here that, according to Eq. (3.21a), the amplitude of an isolated soliton varies in time under the action of dissipation according to the same equation (9.80) if $\epsilon_2=\epsilon_3=0$. However, if $\epsilon_3\neq 0$, the evolution equation for the one-soliton amplitude ensuing from Eq. (3.21a),

$$\frac{d}{dt}\eta=-2\eta(\epsilon_1+\tfrac{16}{3}\epsilon_2\eta^2) \ , \tag{9.82}$$

does not coincide with Eq. (9.81). The difference between Eq. (9.81) and Eq. (9.82) is stipulated by the overlapping of a given soliton with other solitons inside the "semiclassical" wave packet.

Let us note that if we considered, instead of Eq. (9.1), a NS equation with a conservative perturbation [e.g., with the term $\epsilon|u|^4u$ describing the higher nonlinear dispersion], the integral analogous to M_n [see Eqs. (9.77) and (9.78)] would be equal to zero, while the contribution from the same conservative perturbation to the quasilinear equations (9.67) and (9.68) for the NSE$(-)$ would be different from zero.

The study of the evolution of a "semiclassical" wave packet is also meaningful for the perturbed KdV equation (9.3). It is natural to take as a zero-order approximation an exact multiphase solution. General equations describing the evolution of parameters of that solution under the action of dissipation have been derived by Krichever (1988). Note that evolution of a one-phase (Whitham's) wave packet in the $K\,dV$ equation under the action of dissipation has been studied in detail in numerical simulations of Avilov *et al.* (1987).

D. Resonant absorption of energy by a semi-infinite Josephson junction

Interaction of an external radio-frequency field with a semi-infinite damped Josephson junction occupying the half-axis $x\geq 0$ is described by the perturbed SG equation

$$u_{tt}-u_{xx}+\sin u=-\gamma u_t, \quad x>0 \ , \tag{9.83}$$

with the boundary condition

$$u_x(x=0)=-(\epsilon/2)\cos(\omega t) \ , \tag{9.84}$$

where the amplitude $(\epsilon/2)$ of the drive is, in general, a sum of two terms accounting for injection of an ac bias current (with frequency ω) into the junction through the edge $x=0$, and/or an external ac magnetic field (with the same frequency ω) applied to the junction.

Equations (9.83) and (9.84) are equivalent to the perturbed SG equation

$$u_{tt}-u_{xx}+\sin u=-\gamma u_t+\epsilon\delta(x)\cos(\omega t) \ , \quad |x|<\infty \ , \tag{9.85}$$

subject to the additional constraint

$$u(-x)\equiv u(x) \ . \tag{9.86}$$

It has been demonstrated by Olsen and Samuelsen (1986b) that under the condition $\omega<1$ the model (9.85) and (9.86) admits hysteresis: there may be two stable states, one corresponding to small-amplitude quasilinear plasma oscillations with the frequency ω, another corresponding to oscillations of a fluxon (magnetic quantum flux) pinned by the edge of the junction. Due to the condition (9.86), an oscillating pinned fluxon is approximately described by a breather solution of the unperturbed SG equation.

The aim of the present section is to find, following Malomed (1987i), the rate of energy absorption by a semi-infinite Josephson junction in the near-resonant case,

$$\omega=1+\Omega, \quad |\Omega|\ll 1 \ , \tag{9.87}$$

when the external drive frequency ω is close to the junction's plasmas frequency $\omega_0=1$, and the results of Samuelsen and Olsen (1986b) are inapplicable. We shall also consider the case in which the external drive contains both dc and near-resonant ac components.

The external drive supports a small-amplitude plasma wave, which can be represented in the form

$$u_{\rm pl}(x,t)=a_0e^{-\gamma|x|/2k}\sin(\omega t-k|x|+\delta_{\rm pl})+O(a_0^3) \ , \tag{9.88}$$

where $\delta_{\rm pl}=\tan^{-1}(\gamma/2k^2)$,

$$a_0^2=8(k^2-2\Omega-\gamma^2/4k^2) \ , \tag{9.89}$$

and the wave number k is determined by the equation

$$32(k^2)^4-64\Omega(k^2)^3-\epsilon^2(k^2)^2-16\Omega\gamma^2(k^2)-2\gamma^4=0 \tag{9.90}$$

[recall that $\Omega\equiv\omega-1$, $|\Omega|\ll 1$]. In what follows, we shall confine our attention to the case of weak dissipation, $\gamma\ll\max\{|\epsilon|,|\Omega|\}$. Then the roots of Eq. (9.90) may be approximately written as

$$k_1^2=\Omega+\sqrt{\Omega^2+\epsilon^2/32} \ , \tag{9.91}$$

$$k_{2,3}^2=(-8\Omega\pm\sqrt{64\Omega^2-2\epsilon^2})\gamma^2/\epsilon^2 \ . \tag{9.92}$$

(There is also a fourth root, the twin to k_1^2, which is always negative.) The roots $k_{2,3}^2$ are positive provided

$$\Omega < \Omega_{\rm thr} \equiv -|\epsilon|/4\sqrt{2} \ . \tag{9.93}$$

According to Eq. (9.89), the corresponding values of the squared amplitude are

$$(a_0^2)_1 = 8(-\Omega + \sqrt{\Omega^2 + \epsilon^2/32}) \ , \tag{9.94}$$

$$(a_0^2)_{2,3} = 2[-8\Omega + \epsilon^2/(8\Omega \mp \sqrt{64\Omega^2 - 2\epsilon^2})] \ . \tag{9.95}$$

The dependence of the squared wave number and amplitude on the resonance frequency detuning Ω is shown in Fig. 45. The intermediate branch $k_2^2(\Omega), (a_0^2)_2(\Omega)$ corresponding to an unstable solution, is depicted by a dashed line in Fig. 45.

At $\Omega \to \infty$ (in fact, at $\Omega \gg |\epsilon|$), we have the single root

$$k_1^2 \approx 2\Omega, \quad (a_0^2)_1 \approx \epsilon^2/8 \ . \tag{9.96}$$

In this limit, the solution (9.88) goes over into the nonresonant driven plasma wave

$$u_{\rm pl}(x,t) \approx (\epsilon/2\sqrt{\omega^2-1})\exp(-\kappa|x|) \times \sin(\omega t - \sqrt{\omega^2-1}|x|) \ , \tag{9.97}$$

where $\kappa \equiv \gamma\omega/2\sqrt{\omega^2-1}$. In the opposite limiting case $\Omega<0$, $|\Omega| \gg |\epsilon|$, Eqs. (9.91), (9.92), (9.94), and (9.95) take the asymptotic form

$$k_1^2 \approx \epsilon^2/64|\Omega|, \quad (a_0^2)_1 \simeq 16|\Omega| + \epsilon^2/8|\Omega| \ , \tag{9.98}$$

$$k_2^2 \approx 16|\Omega|\gamma^2/\epsilon^2, \quad (a_0^2)_2 \approx 16|\Omega| - \epsilon^2/8|\Omega| \ , \tag{9.99}$$

$$k_3^2 \approx \gamma^2/8|\Omega|, \quad (a_0^2)_3 \approx \epsilon^2/8|\Omega| \ . \tag{9.100}$$

In this limit, the solution (9.88) and (9.98) goes over into the nonresonant breatherlike equilibrium solution, while Eq. (9.88), with k^2 and a_0^2 from Eqs. (9.100), recovers the nonresonant small-amplitude plasma oscillation solution (Olsen and Samuelsen, 1986b)

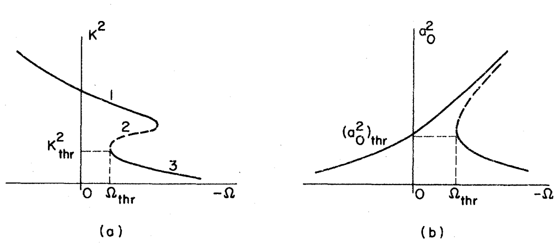

FIG. 45. The dependence of (a) the plane wave's squared wave number k^2 and (b) the plane wave's squared amplitude a_0^2 on the resonance frequency detuning for the model (9.83) and (9.84). The numbers 1, 2, and 3 refer to the three branches determined by Eqs. (9.91), (9.92), (9.94), and (9.95). The threshold value of the detuning $\Omega_{\rm thr}$ separating the nonhysteretic region from the hysteretic one is given by Eq. (9.93), and, according to Eqs. (9.91) and (9.95), $(a_0^2)_{\rm thr} = \sqrt{2}|\epsilon|$, $(k^2)_{\rm thr} = \sqrt{2}\gamma^2/|\epsilon|$.

$$u_{\rm pl}(x,t) \approx (\epsilon/2\sqrt{1-\omega^2})\exp(-\sqrt{1-\omega^2}|x|)\cos(\omega t) \ . \tag{9.101}$$

Let us proceed to evaluation of the absorption rate of external field energy by the semi-infinite Josephson junction. For the plasma-wave solution (9.88), this quantity can be found in two different ways: as the energy absorption rate proper (Olsen and Samuelsen, 1986b),

$$\frac{dE}{dt} = \gamma \int_0^\infty u_t^2(x)dx \ , \tag{9.102}$$

or as the energy flux carried by the traveling wave (9.88) at the point $x=0$ (Malomed, 1987d, 1987i),

$$\frac{dE}{dt} = V_{\rm gr}\langle h \rangle|_{x=0} \ , \tag{9.103}$$

where $V_{\rm gr} = k/\omega$ is the group velocity of the plasma wave and

$$\langle h \rangle = \langle \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + (1-\cos u) \rangle \tag{9.104}$$

is the SG Hamiltonian density, with the angular brackets in Eqs. (9.103) and (9.104) standing for time averaging. In the lowest approximation, inserting Eq. (9.88) into Eq. (9.103) yields

$$\frac{dE}{dt} = \tfrac{1}{2}a_0^2 k \ . \tag{9.105}$$

Equations (9.102) and (9.105) yield the same final expressions for the energy absorption rate: For the solution (9.91) and (9.94),

$$\left[\frac{dE}{dt}\right]_1 = (|\epsilon|/\sqrt{2})(\sqrt{\Omega^2+\epsilon^2/32} - \Omega)^{1/2} \ , \tag{9.106}$$

and for (9.92) and (9.95),

$$\left[\frac{dE}{dt}\right]_{2,3} = (2\gamma/|\epsilon|)(-\Omega \pm \sqrt{\Omega^2 - \epsilon^2/32})^{3/2} \ . \tag{9.107}$$

Note that, when $\gamma=0$, the expression (9.107) becomes zero, while the one (9.106) remains finite. In this case, the absorbed energy is not dissipated, but expanded on the generation of a plasma wave emitted by the edge of the junction. The same pertains to the nonresonant driven plasma oscillations: For the nonwave solution (9.101) ($\omega<1$), Eq. (9.102) yields

$$\frac{dE}{dt} = \gamma(\epsilon\omega/4)^2(1-\omega^2)^{-3/2} \ , \tag{9.108}$$

while for the wavelike one (9.97) ($\omega>1$) both Eq. (9.102) and Eqs. (9.103), (9.104) yield

$$\frac{dE}{dt} = \epsilon^2\omega/8\sqrt{\omega^2-1} \ . \tag{9.109}$$

In the limit $\gamma=0$, the absorption rate (9.108) vanishes, while (9.109) remains finite.

Let us briefly consider a generalization of the above results for the case when the external drive contains both dc and near-resonant ac components, i.e., the driven semi-infinite Josephson junction is described by the equation [cf. Eq. (9.85)]

$$u_{tt}-u_{xx}+\sin u=-\gamma u_t+[\epsilon_0+\epsilon\cos(\omega t)]\delta(x)\ , \qquad (9.110)$$

where, as above, $\omega=1+\Omega$, $|\Omega|\ll 1$. We want to look at the case when the dc component is relatively strong,

$$|\epsilon_0|^3\gg 8\epsilon\ . \qquad (9.111)$$

A plasma-wave solution to Eq. (9.110) is sought in the form

$$u(x,t)=(\epsilon_0/2)e^{-|x|}+a(x)\sin[\omega t-\phi(x)]\ . \qquad (9.112)$$

Inserting Eq. (9.112) into Eq. (9.110) yields a system of two equations,

$$a_{xx}+2\Omega a-a\phi_x^2+\tfrac{1}{8}a^3+\tfrac{1}{8}\epsilon_0^2 ae^{-2|x|}=0\ , \qquad (9.113)$$

$$a\phi_{xx}+2a_x\phi_x+\gamma a=0\ . \qquad (9.114)$$

Due to Eq. (9.111), at $|x|\gg 1$ a relevant solution to Eqs. (9.113) and (9.114) may be taken in the form of Eq. (9.88), where a_0^2 is related to k^2 according to Eq. (9.89). At $|x|\lesssim 1$,

$$a(x)\approx a_0\exp[(\epsilon_0^2/32)e^{-2|x|}]\ ,$$
$$\phi(x)=\phi_0+k|x|\ , \qquad (9.115)$$

with the same a_0 and k. Inserting Eqs. (9.115) and (9.89) into the boundary condition $u_x(x=0)=-\frac{1}{2}[\epsilon_0+\epsilon\cos(\omega t)]$ [cf. Eq. (9.84)] yields, in the lowest approximation, the equation for k^2:

$$\epsilon^2(k^2-2\Omega-\gamma^2/4k^2)^{-1}=\epsilon_0^2/8+32k^2\ , \qquad (9.116)$$

and the expression $\tan\phi_0=-\epsilon_0^2/16k$. In the limit $\gamma^2\to 0$, the physical root ($k^2\geq 0$ of Eq. (9.116) takes the form (Fig. 46)

$$k^2(\Omega)=\begin{cases}\frac{1}{64}\{64\Omega-\epsilon_0^4/8+[(64\Omega+\epsilon_0^4/8)^2+128\epsilon^2]^{1/2}\}\ , & \Omega\geq -4\epsilon^2/\epsilon_0^4\ ,\\ 0, \quad \Omega\leq -4\epsilon^2/\epsilon_0^4\ . & \end{cases} \qquad (9.117)$$

In this limit, the dependence $a_0^2(\Omega)$ resulting from substituting Eq. (9.117) into Eq. (9.89) is (Fig. 46)

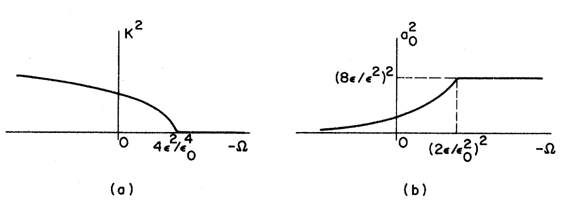

FIG. 46. The dependences (a) $k^2(\Omega)$ and (b) $a_0^2(\Omega)$ for $\gamma\to 0$ in the model (9.110). The condition (9.111) is assumed to hold.

$$a_0^2(\Omega)=\begin{cases}\frac{1}{8}\{[(64\Omega+\epsilon_0^2/8)^2+128\epsilon^2]^{1/2}-(64\Omega+\epsilon_0^2/8)\}\ , & \Omega\geq -4\epsilon^2/\epsilon_0^4\ ,\\ 64\epsilon^2/\epsilon_0^4, \quad \Omega<-4\epsilon^2/\epsilon_0^4\ . & \end{cases} \qquad (9.118)$$

In the same approximation one can readily find the energy absorption rate, substituting Eqs. (9.117) and (9.118) into Eq. (9.105). If one retains corrections $\sim\gamma^2$ to the above expressions, the breaks in the dependences (9.117) and (9.118) (Fig. 46) get smoothed; however, in any case, the presence of the relatively large dc component in the external drive removes the hysteresis, as can be seen by comparing Figs. 46 and 45.

E. Variational approach to propagation of a nonsoliton pulse in the unperturbed nonlinear Schrödinger equation

In the theory of nonlinear optical fibers described by the NS equation, an important problem is the evolution of an initial pulse

$$u(0,x)=N_0\operatorname{sech}x\,\exp(ib_0x^2)\ , \qquad (9.119)$$

where b_0 is the so-called chirp parameter (Anderson, Lisak, and Reichel, 1988). A significant difficulty stems from the fact that the direct scattering problem for the unperturbed NS equation with the initial condition (9.119) cannot be solved exactly, unless b_0 is zero. A "semiclassical" approach to this direct scattering problem was developed by Lewis (1985); her approach aimed at finding a threshold for the birth of a soliton from the chirped pulse (9.119). Anderson, Lisak, and Reichel (1988) have put forward another approach based on variational methods. Although it was applied to the unperturbed NS equation, it seems pertinent to give here their basic ideas.

They look for a solution in a form that ignores separation of the solitonic and radiative components of the wave field:

$$u(t,x)=N(t)\exp[ib(t)x^2]\operatorname{sech}[x/a(t)]\ . \qquad (9.120)$$

Insertion of the assumed wave form (9.120) into the Lagrangian for the unperturbed NS equation and application of a direct variational procedure yield the evolution equation for the parameter $a(t)$,

$$\left[\frac{da}{dt}\right]^2=-\frac{4}{\pi^2}(a^{-2}-1)-\frac{4N_0^2}{\pi^2}(a^{-1}-1)-2b_0^2\ , \qquad (9.121)$$

and three separate equations for other parameters defined by Eq. (9.120):

$$b(t)=\frac{1}{2}\frac{d}{dt}\ln a(t),\quad |N^2(t)|=N_0^2a^{-2}(t)\ ,$$
$$\frac{d}{dt}\arg N=-\tfrac{1}{3}a^{-2}+\tfrac{5}{6}N_0^2a^{-1}\ . \qquad (9.122)$$

Anderson, Lisak, and Reichel (1988) have demonstrated that agreement between the results obtained from Eqs. (9.120)–(9.122) and numerical data is good. They have also proposed a "phenomenological" method to separate solitonic and radiative components contained in the broadening pulse (9.120). It is clear that the variational approach can be readily generalized to take account of possible perturbative terms in the NS equation.

X. SEMICLASSICAL QUANTIZATION OF PERTURBED SOLITONS

As is often emphasized, in the first order in perturbation theory a soliton seems like a classical particle in an external field (Kaup and Newell, 1978a). This analogy can be extended to incorporate the simplest quantum effects. Semiclassical (WKB) quantization of time-periodic solutions of classical wave equations (in particular, of the SG equation) is based on the Bohr-Sommerfeld quantization rule,

$$\int_{-T/2}^{T/2} dt \int_{-\infty}^{+\infty} dx\, u_t^2(x,t) = 2\pi n\gamma \ , \qquad (10.1)$$

where T is a period of the solution, n is a large quantum number, and γ is a small coupling constant (as usual, we set $\hbar = 1$) (Rajaraman, 1982). Following these ideas, in this section we shall consider quantization problems both for a pinned oscillating kink and for a small-amplitude breather with internal oscillations in the presence of a perturbation. Moreover, we shall offer (also from a semiclassical viewpoint) an interpretation of classical energy emission rates (discussed in the preceding sections) as finite widths of quantized solitonic levels.

As far as we know, perturbation-induced corrections to SG soliton quantum spectra have been considered only by the present authors (Malomed, 1987d, 1987e; Kivshar and Malomed, 1988b). However, the problem of calculating a probability for perturbation-induced under-barrier decay of a weakly bound (low-frequency) breather into a kink-antikink pair has been widely studied (e.g., Coleman, 1977; Maki, 1977; Bitar and Chang, 1978; Katz, 1978; Krive and Rozhavskii, 1980). In the lowest approximation, one may solve this problem by considering the two kinks as a pair of bound particles, i.e., the decay probability P is

$$P \sim \exp(-S/\gamma) \ ,$$

where S is an effective classical action for the corresponding trajectory of the under-barrier tunneling. Recently, Krive, Malomed, and Rozhavsky (1989) have considered the problem of creation of kink-antikink pairs by an external field in an *inhomogeneous* SG model, e.g., the one (3.76) (with $a \gg 1$). It has beeen demonstrated that, at $\epsilon \geq 1$, the lattice of inhomogeneities strongly facilitates the pair creation, and at sufficiently large ϵ the creation rate ceases to be exponentially small.

A. Quantization of a pinned kink

Let us now consider quantization of the law of motion (5.47) of a pinned kink oscillating near an attractive inhomogeneity. In the particlelike approximation, a kink is equivalent to a particle moving in the potential wall $U(\xi) = -2\epsilon\,\mathrm{sech}^2\xi$. The quantized levels of the renormalized energy $\tilde{E} \equiv E/\gamma$ are well known (see, for example, Landau and Lifshitz, 1974):

$$\tilde{E}_n^{(0)} = \frac{1}{\gamma}\left[\sqrt{2\epsilon} - \frac{n\gamma}{4}\right]^2 , \qquad (10.2)$$

where γ is the same nondimensional coupling constant as in Eq. (10.1). However, the analogy between a kink and a particle is not exact. It is violated if one takes into account perturbation-induced corrections to the form of the kink,

$$u_k = u_k^{(0)} + u_k^{(1)} \ , \qquad (10.3)$$

where $u_k^{(0)}$ is the kink's wave form (2.61a) valid in the adiabatic approximation. According to Kivshar (1984) (see also Kosevich and Kivshar, 1982), the correction is determined by the general formula

$$u^{(1)} = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda}(\lambda^2 - \tfrac{1}{4} + i\lambda\tanh z)(\lambda^2 + \tfrac{1}{4})^{-1} \times b(\lambda,t)\exp[i(\lambda - 1/4\lambda)x] \ , \qquad (10.4)$$

provided $(d\xi/dt)^2 \ll 1$, where $z = x - \xi$ and $b(\lambda,t)$ is the Jost coefficient defined in Sec. II.D. For the perturbation (1.19), evolution of the function

$$B(\lambda,t) \equiv b(\lambda,t)\exp[i(\lambda + 1/4\lambda)t]$$

is determined by an equation that follows from Eq. (2.72):

$$\frac{\partial B(\lambda,t)}{\partial t} = -\frac{i\epsilon}{4}(\lambda^2 + \tfrac{1}{4})^{-2}\exp\left[i\left[\lambda + \frac{1}{4\lambda}\right]t\right] \times(\tanh\xi/\cosh\xi)(\lambda^2 - \tfrac{1}{4} + i\lambda\tanh\xi) \ . \qquad (10.5)$$

After inserting the law of motion (5.47) into Eq. (10.5), it is natural to single out two qualitatively different parts on the right-hand side. The first, which we shall call the nonresonant component, yields the oscillating part of $b(\lambda,t)$,

$$b(\lambda,t) \approx -\tfrac{1}{2}\epsilon\sigma\lambda(\lambda^2 + \tfrac{1}{4})^{-2}(\lambda^2 - \tfrac{1}{4} + i\lambda\tanh\xi) \times(\tanh\xi/\cosh\xi) \ , \qquad (10.6)$$

which determines the correction $u^{(1)}$ to the kink form via Eq. (10.4) (Kivshar, 1984),

$$u^{(1)}(x,\xi) = -\frac{\epsilon\sigma\sinh\xi}{2\cosh z\cosh^3\xi} \times[\theta(x)(1 - e^{-z}\cosh z - e^{-\xi}\cosh\xi + x) + \theta(-x)(1 - e^{z}\cosh z - e^{\xi}\cosh\xi - x)] \ , \quad z = x - \xi \ , \qquad (10.7)$$

where $\theta(x)$ is the Heaviside step function. The second part of Eq. (10.5) yields a secular term. In fact, it gives rise to the emission of radiation. Inserting the corrected kink wave form with regard to Eqs. (10.7) into (10.4), and then Eq. (10.4) into the semiclassical quantization rule (10.1), we find (Kivshar and Malomed, 1988b)

$$\gamma n = 4(\sqrt{2\epsilon}-\sqrt{E}\,)+E^{3/2}F(\sqrt{2\epsilon/E}\,)\ , \tag{10.8}$$

where

$$F(x)=1-x+x^3/4-\frac{2}{\pi}(4x^2-1)\sqrt{x^2-1}+\frac{6}{\pi}x^3\ln(x+\sqrt{x^2-1})\ . \tag{10.9}$$

Equation (10.8) can be rewritten in a form that explicitly separates the quantum-mechanical spectrum (10.2) and a small field correction to it

$$\tilde{E}_n=\tilde{E}_n^{(0)}+\tilde{E}_n^{(1)}\ ,\qquad \tilde{E}_n^{(1)}=\frac{\gamma}{2}(\tilde{E}_n^{(0)})^2F[(2\epsilon/\gamma\tilde{E}_n^{(0)})^{1/2}]\ . \tag{10.10}$$

From the semiclassical viewpoint, the expression for the partial energy emission rates W_m given in Sec. VI.E [see Eq. (6.93)], with E replaced by $\tilde{E}_n^{(0)}\gamma$, determines the rate $\Gamma(n,m)$ of the radiative transition from the nth quantized level to the level with number $n-2([1/\sqrt{\tilde{E}}\,]+m+1)\equiv n-\Delta n$:

$$\Gamma(n,m)=W_m/2([1/\sqrt{\tilde{E}}\,]+m+1)\omega\approx W_m\ . \tag{10.11}$$

The semiclassical formula (10.11) is applicable provided $\Delta n \ll n$. For the present problem, this condition means $\epsilon \gg \gamma/4$.

B. Quantization of a breather

1. A pinned breather

In Sec. VI.H.2 we described a small-amplitude breather strongly distorted due to the presence of the perturbation (1.19). Inserting the breather's wave form (6.136) and (6.137) into the Bohr-Sommerfeld quantization rule (10.1), we find the spectrum

$$\mu_n=\mu_n^{(0)}+\mu_n^{(1)}\ , \tag{10.12}$$

where $\mu_n^{(0)}=\gamma n/16+\epsilon/2$, and

$$\mu_n^{(1)}=-\tfrac{1}{36}(\mu_n^{(0)})^3\left[\frac{80}{27}+\frac{\epsilon}{2\mu_n^{(0)}}+\frac{\epsilon^3}{24(\mu_n^{(0)})^3}\right]\ . \tag{10.13}$$

The expansion (10.12) is relevant provided $1\ll n\ll\mu^{-1}$.

As to the emission power (6.144), from the semiclassical viewpoint it gives the finite width of the discrete levels (10.12) stipulated by the possibility of radiative transition $n\to n-3$, the transition rate being $\Gamma=W/3\gamma$ [see Eq. (10.11)].

Above we dealt with a quiescent breather. However, in the limiting case (6.145) the asymptotic approach developed in Sec. VI.H.2 can be generalized to describe small oscillations of a breather near an inhomogeneity. These oscillations will be called external to distinguish them from the internal oscillations of the breather. A corresponding solution $u(x,t)$ has the form

$$u(x,t)=4\mu_j\operatorname{sech}(\mu_j|x|+\phi_j)\cos(\omega t),\quad j=1,2\ , \tag{10.14}$$

where $j=1$ for $x>0$, and $j=2$ for $x<0$,

$$\mu_j=\epsilon\cosh\phi_j/(\sinh\phi_1+\sinh\phi_2)\ , \tag{10.15}$$

and the parameters ϕ_j are specified below.

Small oscillations of a breather near an inhomogeneity can be described by setting $\phi_{1,2}=\phi\pm\psi(t)$, $\psi\ll\phi$, where ϕ is a constant and ψ is a slowly varying function of time. Inserting this into Eq. (10.15), we obtain expansion for μ_j:

$$\mu_{1,2}-\frac{\epsilon}{2}=\pm\frac{\epsilon}{2}\tanh\psi+\epsilon(1\pm\psi)e^{-2\phi}+O(\epsilon\psi^3e^{-2\phi}) \tag{10.16}$$

[recall that, due to the limit specified by Eq. (6.145), $\exp(-2\phi)\approx(\mu-\epsilon/2)/\epsilon$ is a small quantity].

The underlying SG equation with the perturbation (1.19) corresponds to the following Lagrangian expanded in powers of the wave field up to u^4 terms:

$$L=\tfrac{1}{2}\int_{-\infty}^{+\infty}dx[u_t^2-u_x^2-u^2+\tfrac{1}{12}u^4+\epsilon u^2\delta(x)]\ . \tag{10.17}$$

Inserting Eqs. (10.14), (10.16), and (6.133) into the Lagrangian (10.17) yields the expression for the effective potential energy $U(\psi)$ of the small external oscillations averaged over the rapid internal oscillations:

$$U(\psi)=12\epsilon^3\psi^2e^{-4\phi}\ . \tag{10.18}$$

To find the effective kinetic energy T, we note that differentiating Eq. (10.16) yields, in the lowest approximation, $d\mu/dt\approx(\epsilon/2)d\psi/dt$. Inserting this into the Lagrangian (10.17) brings us to the averaged expression

$$T\approx 4\epsilon e^{-2\phi}\left[\frac{d\psi}{dt}\right]^2\ . \tag{10.19}$$

Using Eqs. (10.18) and (10.19), we immediately find the frequency χ of the small external oscillations:

$$\chi^2=3\epsilon^2e^{-2\phi}\ . \tag{10.20}$$

Since, according to Eq. (10.16), $\exp(-2\phi)\approx(\mu-\epsilon/2)/\epsilon$, where μ stands for the mean value of μ_j, which is the same for $j=1$ and 2, we may finally write Eq. (10.20) in the form

$$\chi^2=3\epsilon\left[\mu-\frac{\epsilon}{2}\right]\ . \tag{10.21}$$

It is important to note that, because Eq. (6.145) is the limiting case, $\chi^2\ll\epsilon^2$. Indeed, the above consideration actually implies that $\mu_j(t)$ and $\phi_j(t)$ vary in time adiabatically slowly over a background of rapid internal oscillations. One can verify that, within the accuracy to which Eqs. (10.18) and (10.19) have been calculated, this assumption is warranted simply by the condition $\chi^2 \ll \epsilon^2$. That is why a consideration of small external oscillations can be performed self-consistently only for the limiting case (6.145).

From the semiclassical viewpoint, the "external" oscillations of a breather as a whole result in splitting of each internal oscillation level (10.12) into a band of "finite-structure" levels,

$$E_{nm} = n + m\chi \ , \tag{10.22}$$

where m is the quantum number of the external oscillations (in the semiclassical case $m \gg 1$). Accordingly, each emission line corresponding to the radiative transition $n \to n-3$ between internal oscillation levels will acquire "fine structure" corresponding to the radiation frequencies $3+\Delta m\chi$, where each line of the "fine structure" is generated by a transition $(n,m) \to (n-3, m-\Delta m)$ between the split levels (10.22). These transitions may be considered semiclassically provided $\Delta m \ll m$.

In conclusion, we note that the analogous quantization problem for external oscillations of a breather with $\mu \gg \epsilon$ was considered by Malomed (1987d).

2. A breather in external fields

Let us consider how a small-amplitude breather's semiclassical spectrum is perturbed by the terms ϵ and $\epsilon \sin(u/2)$ on the right-hand side of the perturbed SG equation. For the former term the calculations are straightforward: using $\mu \ll 1$, we expand the renormalized perturbation (6.16) and the unperturbed part of the SG equation to arrive at

$$U_{tt} - U_{xx} + U - \tfrac{1}{6}U^3 + \cdots = \frac{\epsilon}{2}U^2 + \cdots \ . \tag{10.23}$$

From (10.23) one easily obtains the perturbation-induced correction to the form of the quiescent small-amplitude breather [see Eq. (2.60)]:

$$\begin{aligned} U(x,t) &= 4\mu \sin t \ \mathrm{sech}(\mu x) + 4\epsilon\mu^2[1 - \tfrac{1}{3}\cos(2t)]\mathrm{sech}^2(\mu x) \\ &\quad + \tfrac{25}{3}\epsilon^2\mu^3 \sin t \ \mathrm{sech}^3(\mu x) + \cdots \\ &\equiv U^{(0)} + U^{(1)} + \cdots \ . \end{aligned} \tag{10.24}$$

Inserting Eq. (10.24) into Eq. (10.1) yields, in the first nontrivial approximation,

$$\begin{aligned} \mu_n &= \frac{\gamma n}{16} - (83/27 \times 16^3)\epsilon^2(\gamma n)^3 \\ &\equiv \mu_n^{(0)} + \mu_n^{(1)} \ , \end{aligned} \tag{10.25}$$

where the second term is the perturbation-induced correction to the spectrum of the small-amplitude breather.

In the latter case $[P = \sin(u/2)]$ one should employ the renormalized equation (6.110). The necessary correction is

$$U^{(1)} = 120\epsilon_R \mu^3 \sin t \ \mathrm{sech}^3(\mu x) \ , \tag{10.26}$$

which yields the correction to the spectrum

$$\mu_n^{(1)} = -(5/2^9)\epsilon_R(\gamma n)^3 \ . \tag{10.27}$$

Equations (10.25) and (10.27) are valid provided $n \gg 1$ and $\mu_n \ll 1$. It is clear that distortion of the breather's spectrum by other perturbations, e.g., $P = \sin(2u)$, can be considered quite analogously.

In the absence of perturbations, the breather's energy levels are (Rajaraman, 1982) $\tilde{E}_n^{(0)} = (16/\gamma)\mu_n^{(0)}$, hence the perturbation-induced shifts of the levels are $\tilde{E}_n^{(1)} = (16/\gamma)\mu_n^{(1)}$. There also exist additional contributions to $\tilde{E}_n^{(1)}$ originating directly from the perturbation Hamiltonian, but for both perturbations considered the additional contributions are much smaller than those given above.

Finally, the emission powers (6.109) and (6.112) with $\mu = \mu_n$ may be interpreted from the semiclassical viewpoint as the radiative transition rates Γ [see Eq. (10.11)]. In the case (6.109), $\Gamma_2 = W_2/(2\gamma)$ for the transition $n \to n-2$, and in the case (6.112), $\Gamma_3 = W_3/(3\gamma)$ for the transition $n \to n-3$.

XI. QUASI-ONE-DIMENSIONAL SINE-GORDON SOLITONS

A. Formulation of problems

The multidimensional SG equation

$$u_{tt} - \Delta u + \sin u = 0 \tag{11.1}$$

without extrinsic perturbations, where Δ is the two- or three-dimensional Laplacian, is not integrable. [The stationary version of the two-dimensional SG equation, $-\Delta u + \sin u = 0$, admits the so-called Hirota bilinearization. Due to this fact, the equation mentioned possesses a number of nontrivial solutions. The simplest one is $u = 4\tan^{-1}[(e^x + e^y)/(1 + e^{x+y})]$ that describes a nonlinear superposition of orthogonal quasi-one-dimensional kinks (Hirota, 1973).] At the same time, the time-dependent two-dimensional SG equation is of evident interest from the viewpoint of physical applications. [Two applications, namely, large-area Josephson junctions and axisymmetric magnetic domain walls, were discussed by Maslov (1985).] That is why this equation has been a subject of intensive studies (Olsen and Samuelsen, 1980, 1981; Christiansen and Lomdahl, 1981; Geicke, 1983, 1984; Maslov, 1985; Malomed, 1987c). The three-dimensional case is of less significance for applied physics, but it is interesting as a field-theory model (Bogolyubskii and Makhan'kov, 1976, 1977).

In the rotationally symmetric two-dimensional case and the spherically symmetric three-dimensional case, Eq. (11.1) takes the form

$$u_{tt}-u_{rr}+\sin u=(D/r)u_r \ , \tag{11.2}$$

where r is the radial coordinate and $D+1$ is the spatial dimension. Because the characteristic length L of a spatial pattern described by Eq. (11.2) is small in comparison with its characteristic radius R, the right-hand side of Eq. (11.2) may be treated as a small perturbation, i.e., the pattern itself may be regarded as quasi-one-dimensional. Then it is natural to introduce the local coordinate $x=r-R$, rewriting Eq. (11.2) in the form

$$u_{tt}-u_{xx}+\sin u=(D/R)u_x \ , \tag{11.3}$$

where D/R plays the role of the small parameter ϵ from Eq. (11.1), and $P[u]$ is u_x. Equation (11.3) possesses approximate solutions in the form of kinks (2.61a) and breathers (2.63) with parameters slowly evolving in time under the action of the perturbation. From the viewpoint of the original multidimensional equation (11.1), these solutions describe rotationally or spherically symmetric quasi-one-dimensional solitons. They have been studied both numerically and analytically by Bogolyubskii and Makhan'kov (1976, 1977), Olsen and Samuelsen (1980, 1981), Christiansen and Lomdahl (1981), Geicke (1983, 1984), Maslov (1985), and Malomed (1987c). In particular, it is well known that a kink collapses (shrinks) under the action of "surface tension." According to the numerical results, the ultimate stage of the kink's collapse is accompanied by intensive emission of radiation (Bogolyubskii and Makhan'kov, 1977; Christiansen and Lomdahl, 1981; Geicke, 1983), so that after a few reflections from the center the whole kink's energy is transformed into radiation energy. It is interesting to note that, according to results of numerical simulations, in the two-dimensional case the reflection of a collapsing annular kink from the center is very imperfect. In the three-dimensional case the reflections seem sufficiently perfect, but with the pecularity that the kink is reflected *without the change of polarity*. This fact was qualitatively explained by Geicke (1983).

The decay of a shrinking kink into radiation is considered analytically in Sec. XI.B. It turns out that, somewhat unexpectedly, in the three-dimensional situation the analytical approach based on the perturbation theory for solitons is applicable during the entire collapse time if the initial energy of the kink is large. In the two-dimensional situation, the perturbative approach becomes invalid at the terminal stage of the collapse; however, the duration of that stage is exponentially small. The opposite problem, i.e., expansion of a ringlike kink of small initial radius with sufficiently large initial energy, has been considered by Malomed (1987c).

Section XI.C is devoted to ringlike breathers. As is known, the ultimate stage of their evolution may be either contraction (shrinkage) or expansion, depending on initial conditions [Olsen and Samuelsen (1980, 1981)]. An analytical criterion that enables us to distinguish between the two possibilities has been obtained by Maslov (1985). Here we shall concentrate on a more subtle effect revealed in numerical experiments of Olsen and Samuelsen (1980, 1981), viz., decay of a collapsing breather into a kink-antikink pair. It is important to note that this decay takes place long before the end of the shrinkage process, so that after the decay two distinct kinklike and antikinklike ring waves are well seen in the numerical data, both shrinking independently. The evolution of a kink-antikink pair originating from a decaying breather was also investigated by Malomed (1987c).

As was mentioned above, real physical systems are described by a SG equation with extrinsic perturbations $\epsilon P[u]$, e.g., with the dissipative term $P=-u_t$. It is clear that with decrease of R, i.e., at a sufficiently late stage in the shrinkage of a ringlike soliton, this term is dominated over by the effective perturbation $(D/R)u_x$ from Eq. (11.3). The same pertains to the majority of other extrinsic perturbations listed in Sec. I.B. However, in physical problems one also encounters dissipative terms of the diffusion type [see Eq. (1.16b)]. The SG equation with this term is

$$u_{tt}-\Delta u+\sin u=\epsilon\Delta u_t, \quad \epsilon\ll 1 \ . \tag{11.4}$$

Unlike the usual dissipative perturbation, this one essentially affects the final stage of the shrinkage. The corresponding problem is considered in Sec. XI.D. We infer that a ringlike breather, shrinking in the presence of the dissipative perturbation (11.4), ultimately assumes some value $\mu_0\sim 1$ of the amplitude μ, where μ_0 is independent of ϵ. In Sec. XI.D, we also consider the collapse of a ringlike soliton described by the NS equation with a dissipative perturbation analogous to that in Eq. (11.4):

$$iu_t+\Delta u+2|u|^2u=i\epsilon\Delta u \tag{11.5}$$

[see Eq. (1.8b)]. This problem is of interest in relation to the Langmuir collapse problem in plasma physics (Zakharov, 1972; Budneva *et al.*, 1975).

Extrinsic perturbations that describe rotationally symmetric stationary external fields (in the two-dimensional case), along with the perturbing term from Eq. (11.2), may generate an effective potential well to trap a ringlike kink (Christiansen *et al.*, 1981). This kind of problem was studied in detail by Maslov (1985). In Sec. XI.E we briefly review his results.

Finally, Sec. XI.F is devoted to an analysis of radiative damping of small-amplitude long-wave oscillations of a slightly bent quasi-one-dimensional kink.

The results put forth in Secs. XI.B–XI.D were obtained (except for those explicitly attributed to other authors) by Malomed (1987c).

B. Collapsing ringlike kink

1. Adiabatic approximation

Let us consider the kink (2.61) described by Eq. (11.3) with the initial conditions $V(0)=0$, $R(0)=R_0$, where R_0

is assumed to be large, $R_0 \gg 1$. Using energy conservation and neglecting radiation effects, it is straightforward to obtain a solution describing the shrinking kink (Olsen and Samuelsen, 1980, 1981). In the two-dimensional case ($D=1$),

$$R = R_0 \cos(t/R_0), \quad V = \sin(t/R_0) , \tag{11.6}$$

and in the three-dimensional case ($D=2$),

$$R = R_0 \mathrm{cn}(\sqrt{2}t/R_0, 1/\sqrt{2}), \quad V^2 = 1 - R^4/R_0^4 , \tag{11.7}$$

where $\mathrm{cn}(z, 1/\sqrt{2})$ stands for the elliptic cosine. It is obvious that emission of radiation should become essential at the final stage of the collapse, when $R \ll R_0$. Introducing the shifted time $\tau \equiv t - t_0$, where t_0 is the moment of collapse, one can simplify the solutions (11.6) and (11.7) at the final stage:

$$1 - V^2 \approx (\tau^2/R_0^2)^D , \tag{11.8}$$

where $R \approx -\tau$ (the shifted time τ takes negative values). The solution (2.61a) can provide an approximate description of a ringlike kink if the kink remains quasi-one-dimensional, i.e., if its width $l \sim \sqrt{1-V^2}$ is much smaller than R. According to Eq. (11.8),

$$\epsilon \equiv \sqrt{1-V^2}/R \sim |\tau|^{D-1}/R_0^D . \tag{11.9}$$

It is natural to suppose that the quantity ϵ defined in Eq. (11.9) must be an effective parameter of the perturbation theory, and to base our consideration solely on the condition $\epsilon \ll 1$. Proceeding from the estimate (11.9), we might expect this condition to be valid in the two-dimensional case ($D=1$) provided $R_0 \gg 1$, and even to be "self-improving" in the three-dimensional case ($D=2$). As we shall see below, taking emission into account modifies these estimates.

2. Energy emission from a collapsing kink

In the present case, the general evolution equation (2.72) of the perturbation theory for the emission problem takes the form

$$\frac{dB}{d\tau} = (D/4R) e^{i(\lambda + 1/4\lambda)t} \times \int_{-\infty}^{+\infty} dx\, u_x \{[\psi^{(2)*}(t,x;\lambda)]^2 - [\psi^{(1)*}(t,x;\lambda)]^2\} . \tag{11.10}$$

The spatial integral in Eq. (11.10) can be calculated in a general form; however, it considerably simplifies when we are dealing with the final stage of the shrinkage, i.e., when $1 - V^2 \ll 1$. Indeed, it will be seen below that in this situation the radiation energy is concentrated in the spectral range

$$\lambda \sim \sqrt{1-V^2} . \tag{11.11}$$

The corresponding wave-number range is

$$k \equiv \lambda - 1/4\lambda \sim -1/\sqrt{1-V^2} \tag{11.12}$$

(negativeness of k implies that the energy is emitted primarily backward relative to the kink's direction of motion), and in the range $k^2 \gg 1$ a calculation of the integral in Eq. (11.10) yields

$$\frac{dB}{d\tau} = \frac{\pi \sigma D}{2R} \mathrm{sech}(\pi k \sqrt{1-V^2}) . \tag{11.13}$$

To find the total amplitude of the emitted wave field, which determines the radiation energy according to Eq. (2.69), we should integrate Eq. (11.13) over time. However, prior to performing the integration, we should take into account the diffraction divergence of the circular (or spherical) waves emitted by the ringlike kink. In order to compensate for the divergence, we should integrate over time the renormalized quantity

$$\frac{d\tilde{B}}{d\tau} \equiv R^{D/2} \frac{dB}{d\tau} . \tag{11.14}$$

Next we substitute into Eqs. (11.13) and (11.14) $R \approx -\tau$ and the explicit dependence of $\sqrt{1-V^2}$ on τ. In an approximation disregarding the energy emission, this dependence has the form (11.8). We assume that the radiative energy losses modify it as follows:

$$1 - V^2 = (\tau^2/R_0^2)^D / C^2(\tau) , \tag{11.15}$$

where $C(\tau)$ is a function depending not on τ itself, but solely on $\ln(R_0^2/\tau^2)$. Inserting Eq. (11.15) into Eq. (11.13) and Eq. (11.13) into Eq. (11.14), we calculate

$$\tilde{B}(\tau,k) \equiv \int_{-\infty}^{\tau} \frac{d\tilde{B}(k)}{d\tau'} d\tau' \tag{11.16}$$

in the logarithmic approximation, i.e., regarding $\ln(R_0^2/\tau^2)$ as a large parameter,

$$\tilde{B}(\tau,k) = \frac{\sigma\pi}{2} S \sqrt{C} R_0^{D/2} / \sqrt{k} , \tag{11.17}$$

where

$$S = \sum_{j=0}^{\infty} (-1)^j (2j+1)^{-1/2} \approx 0.86 .$$

Using Eq. (2.69b) for the radiation energy spectral density, one can find the total emitted energy as a function of time:

$$E_{\mathrm{em}}(\tau) = (4/\pi) \int_{k_{\min}}^{k_{\max}} |\tilde{B}(\tau,k)|^2 d|k| . \tag{11.18}$$

As follows from the above, $k_{\min} \sim 1$ and $k_{\max} \sim (1 - V^2)^{-1/2} \sim C(\tau)(R_0/\tau)^D$. These estimates are sufficient for calculating the integral over k with logarithmic accuracy. Inserting Eq. (11.17) into Eq. (11.18) yields

$$E_{\mathrm{em}}(\tau) = 8|z|^2 C R_0^D \ln\left[\frac{k_{\max}}{k_{\min}}\right] \approx \frac{\pi D}{2} S^2 C R_0^D \ln\left[\frac{R_0^2}{\tau^2}\right] , \tag{11.19}$$

where

$$z \equiv \int_0^\infty \frac{dx\,(4x^4-\pi^2)}{(4x^4+\pi^2)\cosh(x^2)} + 4i\pi \int_0^\infty \frac{dx\,x^2}{(4x^4+\pi^2)\cosh(x^2)} .$$

Now we can write the energy balance equation, i.e., that the kink energy plus the emitted energy are equal to the initial energy of the kink. Using Eqs. (2.69) and (11.19), we obtain

$$8CR_0^D + \frac{\pi D}{2} S^2 C R_0^D \ln(R_0^2/\tau^2) = 8R_0^D . \tag{11.20}$$

Within logarithmic accuracy, we may drop the first term on the left-hand side of Eq. (11.20) to arrive at

$$C = (2/D)|z|^{-2}/\ln(R_0^2/\tau^2) . \tag{11.21}$$

Substituting Eq. (11.21) into Eq. (11.15), we obtain the final result:

$$\sqrt{1-V^2} = (D/2)|z|^2(|\tau|/R_0^D)\ln(R_0^2/\tau^2) . \tag{11.22}$$

Accordingly, the share of the initial energy that remains as the energy of the kink is

$$\varepsilon \equiv (R^D/\sqrt{1-V^2})/R_0^D = \left[\frac{D}{2}|z|^2 \ln\left[\frac{R_0^2}{\tau^2}\right]\right]^{-1} . \tag{11.23}$$

The results (11.22) and (11.23) are self-consistent as long as the quantity $\epsilon \equiv l/L \sim \sqrt{1-V^2}/R$ is small. Using Eq. (11.22), we obtain the estimate

$$\epsilon \sim (|\tau|^{D-1}/R_0^D)\ln(R_0^2/\tau^2) . \tag{11.24}$$

Here the difference between the two cases $D=2$ and 1 is evident. In the former case, ϵ is a self-improving parameter, i.e., it decreases when $\tau \to 0$. In the latter case, ϵ ceases being small at exponentially small $|\tau|$, i.e., when $\ln(R_0^2/\tau^2) \gtrsim R_0$. Pursuant to Eq. (11.23), by that time the kink retains a share of the initial energy $\varepsilon \sim R_0^{-1}$. The subsequent evolution of the two-dimensional ($D=1$) shrinking kink cannot be studied within the framework of the perturbation theory, since the kink is no longer quasi-one-dimensional. In this connection, it is pertinent to mention the paper by Geicke (1983) in which the difference between the collapse of three- and two-dimensional ringlike kinks is discussed.

One should bear in mind that the perturbative approach developed here is not completely consistent, as we do not take into account the influence of the perturbation on the emitted waves. Actually, they might form a new soliton. Indeed, reflection of a kink retaining a part of the energy of the collapsing kink, and appearance of a pulson localized near the origin have been observed in numerical experiments by Bogolyubskii and Makhan'kov (1976, 1977), Christiansen and Lomdahl (1981), and Geicke (1983, 1984).[11] However, these phenomena can scarcely be considered from the standpoint of perturbation theory.

[11] It is interesting that, according to those numerical data, in the two-dimensional case the pulson is much more stable against decay into radiation than in the three-dimensional case.

C. Decay of a collapsing ringlike breather

To describe decay of a breather, we need evolution equations for its parameters. In general, we are dealing with a breather moving with velocity V, see Eq. (2.63). Maslov (1985) has derived equations averaged with respect to the breather's internal oscillations for a moving breather in the laboratory reference frame. However, here we find it more convenient to introduce a local frame accompanying the breather. In the accompanying coordinates, the breather is permanently quiescent, while the perturbation on the right-hand side of Eq. (11.3) is Lorenz transformed to

$$u_{tt} - u_{xx} + \sin u = \frac{D}{R\sqrt{1-V^2}} u_x + \frac{VD}{R\sqrt{1-V^2}} u_t \tag{11.25}$$

(we do not introduce new definitions for the transformed coordinates x,t, since in what follows we shall not deal with the laboratory coordinates). It is assumed that $V>0$ in the case of shrinkage, i.e., that there will be inward motion of the ringlike breather. The perturbation-induced evolution equation for the parameter ζ from Eq. (2.66) is

$$\frac{d\zeta^2}{dt} = -2VDR^{-1}(1-V^2)^{-1/2}(1-\zeta^2T^2)(1+T^2)^{-1} \times [1+\ln(T+\sqrt{1+T^2})/T\sqrt{1+T^2}] . \tag{11.26}$$

Here $T(t)$ is the variable defined in Eq. (4.6′). The breather's velocity V (which now is the accompanying frame's velocity relative to the laboratory frame) evolves according to the relativistic velocity composition law:

$$\frac{dV}{dt} = \frac{D}{R}\sqrt{1-V^2}[1-\ln(T+\sqrt{1+T^2})/T\sqrt{1+T^2}] . \tag{11.27}$$

Then, the evolution equation for the mean radius R, which is always assumed to be measured in the laboratory coordinates, is

$$\frac{dR}{dt} = -V/\sqrt{1-V^2} , \tag{11.28}$$

where we have taken into account the relativistic time contraction in the moving reference frame. The half-period of the breather's unperturbed internal oscillations is, according to Eq. (2.66),

$$\theta = \pi/\zeta . \tag{11.29}$$

As can be seen from Eqs. (11.26) and (2.66), during almost the entire half-period (11.29), when $T^2 \sim \zeta^{-2}$ and the two kinks inside the breather are slightly overlapped, the right-hand side of Eq. (11.26) is small, $\sim\zeta^2$. The only time this is not true is during the relatively short period $t \sim 1$, when $T^2 \lesssim 1$ and the internal kinks are strongly overlapped. The evolution of ζ^2 is dominated by that small portion of the half-period (Malomed, 1985). On the other hand, it follows from Eq. (4.13) that, as long as $T^2 \lesssim 1$, we may set $T \approx t$. Thus the change $\delta(\zeta^2)$ of ζ^2 per half-period (11.29) can be calculated as

$$\delta(\zeta^2)\equiv\int_0^{\theta}\frac{d(\zeta^2)}{dt}dt\approx\int_{-\infty}^{+\infty}\frac{d(\zeta^2)}{dt}\Bigg|_{T=t}dt$$
$$=-\pi^2DV/R\sqrt{1-V^2}\ . \quad (11.30)$$

In contrast to ζ^2, the evolution of V is contributed to by the entire half-period, i.e., the change of V during the half-period is

$$\delta V\equiv\int_0^{\theta}\frac{dV}{dt}dt\approx\pi D\sqrt{1-V^2}/R\zeta\ . \quad (11.31)$$

Therefore, designating as V_n,ζ_n the values assumed by the variables V and ζ by the end of the nth half-period, we reduce the breather's dynamics to a discrete map ensuing from Eqs. (11.30) and (11.31):

$$\zeta_{n+1}^2=\zeta_n^2-\pi^2DV_n/R\ , \quad (11.32)$$

$$V_{n+1}=V_n+\pi D/R\zeta_{n+1}\ . \quad (11.33)$$

Note that, proceeding from Eqs. (11.30) and (11.31) to Eqs. (11.32) and (11.33), we presumed $V_n^2\ll 1$; this is the case of most interest.

An estimate based on Eqs. (11.32) and (11.33) shows that the number N of half-oscillations performed by the breather prior to its decay is large provided

$$R\zeta_0^{3/2}\gg 1\ . \quad (11.34)$$

If, on the contrary, $R\zeta_0^{3/2}\lesssim 1$, then $N\sim 1$, and the decay problem can be solved by iterating the transformation (11.32) and (11.33). The resultant formula becomes intricate even for $N=3$. The opposite case, when Eq. (11.34) holds, i.e., $N\gg 1$, is of more interest since it is "typical." In this case, it seems natural to approximate the map Eqs. (11.32) and (11.33) by differential equations treating n as a continuous variable. This procedure eventually recovers the averaged equations of Maslov (1985). Solving these equations is straightforward:

$$\zeta=\zeta_0-(\pi D^2/4R^2)t^2\ , \quad (11.35)$$

$$V=(D/R)t\ . \quad (11.36)$$

As follows from Eq. (11.35), the moment of decay (i.e., that at which ζ becomes zero) is

$$t_D=\sqrt{\zeta_0/\pi}(2R/D)\ . \quad (11.37)$$

Then, according to Eq. (11.36), the velocity at the moment of decay is

$$V_D=V(t=t_D)=2\sqrt{\zeta_0/\pi}\ . \quad (11.38)$$

The change of R during the time (11.37) is negligible:

$$\Delta R\approx-\int_0^{t_D}V\,dt=(2/\pi D)\zeta_0R\ll R\ . \quad (11.39)$$

Finally, the number n of complete half-oscillations treated as a quasicontinuous function of time is

$$n(t)=\frac{1}{\pi}\int_0^t\zeta(t')dt'=\frac{t}{\pi}\left[\zeta_0-\frac{\pi D^2t^2}{12R^2}\right]\ . \quad (11.40)$$

The total number of half-oscillations performed during the time (11.37) is

$$N\equiv n(t_D)=(4/3\pi^{3/2}D)R\zeta_0^{3/2}\ . \quad (11.41)$$

As follows from Eq. (11.41), the change $\delta\zeta_0$ of ζ_0 resulting in one additional half-oscillation is

$$\delta\zeta_0=(\pi^{3/2}D/2)(\sqrt{\zeta_0}R)^{-1}\ . \quad (11.42)$$

However, the breather's dynamics possesses an essential peculiarity that is missed on averaging. According to Eqs. (11.32) and (11.38), the residual value $\delta\zeta_{\rm res}^2$ of the quantity ζ^2 corresponding to the last half-oscillation before decay lies within the interval

$$0<\zeta_{\rm res}^2<(\zeta_{\rm res}^2)_{\max}\equiv(\pi^2D/R)V_D$$
$$=2\pi^{3/2}D(\sqrt{\zeta_0}/R)\ . \quad (11.43)$$

The time of the last half-oscillation $\pi/\zeta_{\rm res}$ [see Eq. (11.29)] is added to the total decay time, and this increment is beyond the scope of the averaging method. Nevertheless, for typical values $\zeta_{\rm res}^2\sim\sqrt{\zeta_0}/R$ the additional time $t_{\rm add}\equiv\pi/\zeta_{\rm res}\sim R^{1/2}\zeta_0^{-1/4}$ is, due to the condition (11.34), much smaller than the time (11.37):

$$t_{\rm add}^2/t_D^2\sim(R\zeta_0^{3/2})^{-1}\sim N^{-1}\ll 1\ . \quad (11.44)$$

On the other hand, the estimate (11.44) is irrelevant for sufficiently small values of $\zeta_{\rm res}^2$. If, for instance,

$$\zeta_{\rm res}^2\leq\tilde{\zeta}_{\rm res}^2\equiv(\pi^3D^2/4)(\zeta_0R^2)^{-1}\ , \quad (11.45)$$

the breather's lifetime is, on account of $t_{\rm add}$, at least two times as large as the mean time (11.37). Thus we conclude that, with the relative "frequency"

$$\Delta\equiv\tilde{\zeta}_{\rm res}^2/(\zeta_{\rm res}^{2^2})_{\max}$$
$$=(\pi^{3/2}D/8)(R\zeta_0^{3/2})=(6N)^{-1}\ , \quad (11.46)$$

initial conditions occur that correspond to long-lived breathers. In other words, inside each interval (11.21), corresponding to a given number N of complete half-oscillations, there is a relatively narrow subinterval of width $\Delta\zeta_0=\Delta\delta\zeta_0$ generating long-lived breathers.

D. Collapsing solitons in a dissipative medium

1. A kink-antikink pair and a weakly bound breather

As was explained in Sec. XI.A, significant physical interest attaches to the dissipatively perturbed sine-Gordon equation (11.4). First of all, let us write the evolution equations for the velocity and mean radius of one kink (in the laboratory coordinates):

$$\frac{dV}{dt}=\frac{D}{R}(1-V^2)-\tfrac{1}{3}\epsilon V\ , \quad (11.47)$$

$$\frac{dR}{dt} = -V \ . \tag{11.48}$$

The simplest way to derive Eqs. (11.47) and (11.48) is to employ the energetic approach (Olsen and Samuelsen, 1981). Here, as in the preceding section, we assume V to be positive for inward motion (shrinkage). The asymptotic form of the solution to Eqs. (11.47) and (11.48) at the final stage of the shrinkage differs essentially from that obtained by Olsen and Samuelsen (1980, 1981), and Maslov (1985) for the unperturbed SG equation:

$$1 - V^2 \approx -\tfrac{2}{3}(2D-1)^{-1}\epsilon\tau \ , \tag{11.49}$$

where we have again introduced the shifted time τ discussed in Sec. XI.B ($\tau < 0$). Note that, according to Eq. (11.49),

$$\sqrt{1-V^2}/R \sim \sqrt{\epsilon/\tau} \ , \tag{11.50}$$

i.e., the quasi-one-dimensional approximation is invalid at very small times $|\tau| \lesssim \epsilon$.

Proceeding to a consideration of a kink-antikink pair, we again find it convenient to employ the accompanying reference frame instead of the laboratory frame. The law of motion for a kink in an accompanying frame can be easily obtained from Eqs. (11.49) and (11.28):

$$1 - V^2 = [\epsilon|\tau|/(2D-1)]^{2/3} \ , \tag{11.51}$$

where, as in the preceding section, we do not introduce new definitions for the coordinates in the moving frames, i.e., τ (which again takes negative values) is hereafter understood as the shifted time measured in the accompanying reference frame.

In what follows, we shall consider a slightly overlapping kink-antikink pair and a slightly bound breather. As follows from Eq. (11.27), in both cases the law of motion of the pair or breather approximately coincides with that of a solitary kink, and the internal dynamics (conversion of the breather into the pair and vice versa) may be considered on the background of the kinklike motion of the breather's or pair's center of mass. Therefore, in what follows we use the law of motion (11.51).

The dissipative terms for right-hand side of Eq. (11.4), being Lorenz transformed to the accompanying frame, generate some new terms. The evolution equations for a pair or a breather are dominated by the terms written below:

$$u_{\tau\tau} - u_{xx} - \frac{D}{R}u_x + \sin u = \epsilon(1-V^2)^{-3/2}[(1+2V^2)u_{\tau xx} + V^2 u_{\tau\tau\tau}] \ . \tag{11.52}$$

As was mentioned in Sec. II.D, a solution of the unperturbed SG equation describing a kink-antikink pair may be obtained from the low-frequency breather solution (2.66) by analytical continuation $\zeta \to i\zeta$. In the present case, the equation for the parameter $Z \equiv -\zeta^2$ derived, as in Eq. (11.47), by means of the energetic approach is

$$\frac{dZ}{d\tau} = \{(2D/4)(V/\sqrt{1-V^2}) - (2\epsilon/3)[(1+2V^2)/(1-V^2)^{3/2}]\}(1+ZT^2)T^{-2} \ . \tag{11.53}$$

It can be verified that, as long as $ZT^2 \lesssim 1$, the dissipative term in Eq. (11.53) is immaterial. Therefore we shall proceed directly to the case $ZT^2 \gg 1$. Simplifying Eq. (11.53) and using Eq. (11.51), we obtain

$$Z^{-1}\frac{dZ}{d\tau} = -\tfrac{2}{3}(4D-3)|\tau|^{-1} \ . \tag{11.54}$$

As can be seen from Eq. (11.54), for both cases $D=1$ and 2, the quantity Z decreases with time. Moreover, due to the divergence of $\int d\tau/\tau$, it is clear that the pair will eventually fuse into a breather.

The next natural step is to investigate the evolution of a low-frequency breather in the presence of dissipation. The dissipative energy loss per half-period of the breather's oscillations can be found directly from Eq. (11.52) and the SG Hamiltonian [see Eq. (2.69a)]:

$$\delta E_{\text{diss}} = -\epsilon(1-V^2)^{-3/2}\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} dx\,(3u_{\tau x}^2 + u_{\tau\tau}^2), \tag{11.55}$$

where we assume $1-V^2 \ll 1$. For the case of a weakly bound breather, we may insert into Eq. (11.55) the solution (2.67). Straightforward calculations yield

$$\delta E_{\text{diss}} = -8\pi^2\epsilon(1-V^2)^{-3/2} \ . \tag{11.56}$$

At the same time, according to Eqs. (11.30) and (4.7), the energy change (per half-period) due to the perturbation (11.25) is

$$\delta E = 8D\pi^2/(R\sqrt{1-V^2}) \ . \tag{11.57}$$

Thus, using Eq. (11.49), we can calculate the ratio

$$\delta E/\delta E_{\text{diss}} = 2D/3(2D-1) \ . \tag{11.58}$$

Clearly, both for $D=1$ and for $D=2$ this ratio is less than one, i.e., the binding energy of the breather increases under the combined action of the perturbations (11.25) and (11.52).

2. A small-amplitude breather

As we have just demonstrated, in presence of the dissipative perturbation (11.4), a collapsing weakly bound breather undergoes "internal collapse" too: its binding energy grows, or, in other words, its amplitude μ [defined in Eq. (2.62)] decreases. Thus there are two alternative possibilities: the breather's internal collapse may develop unrestrictedly, so that $\mu \to 0$ when $R \to 0$, or μ may tend to a finite limit value $\mu_0 < \pi/2$. To treat this problem, we shall consider here the case of $\mu \ll 1$ [a small-amplitude breather; see Eq. (2.65)]. Using an approach based on energy and momentum conservation, one can obtain equations for μ and the breather's velocity V:

$$\frac{d\mu}{d\tau}=D\mu/R-\epsilon\mu/(1-V^2)\ , \tag{11.59}$$

$$\frac{dV}{d\tau}=(D/3R)\mu^2(1-V^2)-\epsilon\ . \tag{11.60}$$

The formal solution to Eqs. (11.59) and (11.60) is

$$1-V^2=-\epsilon\tau/D,\ \ \mu^2=\mu_0^2\equiv(3/2D)(2D+1)\ . \tag{11.61}$$

This solution has no direct meaning, as the value μ_0 does not satisfy the underlying condition $\mu_0 \ll 1$. However, Eq. (11.61) implies that if we take a breather with initially small amplitude $\mu \ll 1$, the amplitude will grow despite the presence of the dissipation. This suggests that, at the final stage of collapse, the amplitude of the dissipatively damped breather assumes an asymptotically constant nonzero value μ_0. Since the ratio (11.58) and the formal value (11.58) of μ_0 do not depend on ϵ, it seems likely that the genuine value of μ_0 does not depend in ϵ either. Therefore, the quasi-one-dimensionality condition $\sqrt{1-V^2} \ll R$ for a dissipatively damped breather of a general form is the same as that for a kink [see Eq. (11.50)]: $|\tau| \ll \epsilon$.

To conclude this discussion of collapsing breathers, it should be noted that quasi-one-dimensional breather solutions to the SG equation are unstable against transverse perturbations (Ablowitz and Kodama, 1980), while kinks are stable (Scott, 1976). For the decaying breathers considered in Sec. XI.B, this instability is immaterial: it is easy to estimate the characteristic development time of the instability to be larger than the breather's dissociation time. However, for breathers with $\mu \sim 1$, the instability is critical, and it will eventually break the rotational ($D=1$) or spherical ($D=2$) symmetry of the problem. Finite curvature of the ringlike breathers does not remove their transverse instability. At the same time, the dissipation considered above may stabilize the breathers if the parameter ϵ is of order one. Of course, perturbation theory is not, strictly speaking, applicable to such a situation, but we may hope that the perturbative approach is not quite irrelevant, even for $\epsilon \sim 1$, since μ_0 does not depend on ϵ.

3. Collapse of a dissipatively damped nonlinear Schrödinger soliton

A small-amplitude breather in the nonrelativistic case ($V^2 \ll 1$) is equivalent to a NS soliton. In this sense, the dissipatively damped NS equation (11.5) corresponds to the dissipatively perturbed SG equation (11.4). In addition to its relation to the SG equation, the multidimensional NS equation has an even more important application, namely, to the problem of Langmuir collapse in plasma physics (Zakharov, 1972; Budneva *et al.*, 1975).

Evolution equations for the amplitude and velocity of the NS soliton (2.41) resulting from the perturbed equation (11.5) take the form

$$\frac{d\eta}{d\tau}=-\frac{D}{R}V\eta-8\epsilon\eta\left[\frac{V^2}{16}+\frac{\eta^2}{3}\right]\ , \tag{11.62}$$

$$\frac{dV}{d\tau}=-\frac{16D}{3R}\eta^2-\tfrac{14}{3}\epsilon\eta^2V\ . \tag{11.63}$$

The equation for R is obvious:

$$\frac{dR}{dt}=V \tag{11.64}$$

(here the time τ again is negative). It immediately follows from Eqs. (11.62) and (11.63) that

$$\frac{d}{d\tau}(\eta^2-\tfrac{3}{16}V^2)=-\tfrac{16}{3}\epsilon\eta^2(\eta^2-\tfrac{3}{16}V^2)\ ,$$

i.e., the quantity $\eta^2-\frac{3}{16}V^2$ tends to zero, and at the final stage of the collapse we may set

$$\eta^2=\tfrac{3}{16}V^2\ . \tag{11.65}$$

Then Eqs. (11.62) and (11.63) reduce to one equation, which, with regard to Eq. (11.64), can be written as

$$\frac{dV}{dR}=-DV/R-\epsilon V^2\ . \tag{11.66}$$

The solution to Eqs. (11.66) and (11.64) is, for $D=2$,

$$V=-(\epsilon R)^{-1},\ \ R=\sqrt{-2\tau/\epsilon}\ ; \tag{11.67}$$

and for $D=1$,

$$V=-[16\epsilon R\ln(R_0/R)]^{-1}\ ,\qquad R=\sqrt{-4\tau/[\epsilon\ln(\tau_0/\tau)]} \tag{11.68}$$

(it is assumed that $|\tau| \ll |\tau_0|$, $R \ll R_0|$). The underlying quasi-one-dimensional approximation is valid if the size of the soliton is small compared with R. According to Eq. (11.67), in the three-dimensional case the ratio $\eta^{-1}/R \sim \epsilon$ is always small. In the two-dimensional case, according to Eq. (11.68), $\eta^{-1}/R \sim \epsilon\ln(R_0/R)$, i.e., the approximation is violated at exponentially small R, when $\ln(R_0/R) \gtrsim \epsilon^{-1}$.

It is well known that the quasi-one-dimensional NS soliton, like the SG breather, is unstable against transverse "snakelike" perturbations (Zakharov and Rubenchik, 1973). Though in the present problem the collapse takes finite time (in both perturbed and unperturbed cases), one can ascertain that in either case the collapse time is much larger than the characteristic time for development of the instability. However, just as we did when investigating the SG breather, we may expect the dissipatively damped collapsing NS soliton to be stable if $\epsilon \sim 1$.

E. Ringlike solitons in external fields

As we have seen in Sec. XI.A, a ringlike kink described by the pure multidimensional SG equation (11.1) always collapses, and almost all of its energy turns into radiation. The situation may alter due to the presence of extrinsic perturbations in the SG equation. Maslov (1985) has considered the perturbed equation

$$u_{tt}-\Delta u+\sin u=-\frac{\partial W(u)}{\partial u}\ , \qquad (11.69)$$

where $W(u)$ stands for the density of a perturbation Hamiltonian. Particular examples offered by Maslov (1985) are $W'=f$ (a large-area Josephson junction with uniformly distributed thermocurrent) and $W'=-h\sin(u/2)$ (a weak magnetic field acting upon a large-area quasi-two-dimensional weak ferromagnet).

For the general equation (11.69), Maslov (1985) has derived the following adiabatic evolution equations for the kink's velocity V and center-of-mass coordinate R:

$$\frac{dV}{dt}=-\frac{D}{R}(1-V^2)+a(1-V^2)^{3/2}\ , \qquad (11.70)$$

$$\frac{dR}{dt}=V[1+b(1-V^2)]\ , \qquad (11.71)$$

where

$$a\equiv\frac{\sigma}{4}\int_{-\infty}^{+\infty}W'(u_k)\operatorname{sech}z\,dz\ , \qquad (11.72)$$

$$b\equiv\frac{\sigma}{4}\int_{-\infty}^{+\infty}W'(u_k)z\operatorname{sech}z\,dz\ , \qquad (11.73)$$

and $u_k(z)$ is Eqs. (11.72) and (11.73) is the kink wave form (2.61).

Calculation of the kink energy

$$E_k=2^D\pi\int_0^{\infty}r^Ddr\left[\tfrac{1}{2}(u_t^2+u_r^2)+2\sin^2\frac{u}{2}+W(u)\right]$$

yields

$$E_k\approx\frac{2^{D+3}\pi R^D}{(1-V^2)^{1/2}}\left[1-\frac{aR}{D+1}(1-V^2)^{1/2}-b(1-V^2)\right]. \qquad (11.74)$$

Using Eqs. (11.70) and (11.71), it is easy to verify that $dE_k/dt=0$. Therefore the phase trajectory equation may be written as

$$E_k(V,R)=E_0\ , \qquad (11.75)$$

where E_0 is the initial energy of the kink. Following Maslov (1985), we shall give a general qualitative description of the kink's motion ensuing from Eqs. (11.69), (11.70), (11.74), and (11.75).

First let us consider the case $a>0$. In this case the phase plane (V,R) has a singular point of the saddle type, $V=V_s=0$, $R=\rho_s$, where

$$\rho_s=\frac{D}{a}(1-b)\approx\frac{D}{a}\ . \qquad (11.76)$$

The saddle's separatrix is defined by $E_k(V,R)=H_s$, where

$$H_s=2^{D+3}\pi\rho_s^D\frac{1-b}{D+1}\approx\frac{2^{D+3}\pi}{D+1}\left[\frac{D}{a}\right]^D. \qquad (11.77)$$

It lies in the region of positive energies

$$R<\frac{\rho_*}{(1-V^2)^{1/2}}\frac{1-b(1-V^2)}{1-b}\approx\frac{\rho_*}{(1-V^2)^{1/2}}\ , \qquad (11.78)$$

$$\rho_*\equiv\frac{D+1}{D}\rho_s\ . \qquad (11.79)$$

A kink that possesses the positive energy $E_k>H_s$ may either expand monotonically to infinity or shrink monotonically (see Fig. 47). The speed of expansion (shrinkage) takes a minimum value at $R=\rho_{**}$,

$$\rho_{**}\equiv\rho_s\left[\frac{E_k}{H_s}\right]^{1/(D+1)}. \qquad (11.80)$$

If $E_k<H_s$, the kink exhibits a return effect: when R reaches a certain value $\rho_{\min}$ ($\rho_{\max}$), the shrinkage (expansion) stops and changes to expansion (shrinkage). Kinks possessing negative energy cannot collapse: their minimum size is $\rho_{\min}>\rho_*$ [Fig. 47(a)]. As follows from Eq. (11.75), the quantities $\rho_{\min,\max}$ are roots of the equation

$$\left[\frac{R}{\rho_s}\right]^{D+1}-\frac{D+1}{D}\left[\frac{R}{\rho_s}\right]^D+\frac{1}{D}\frac{E_k}{H_s}=0\ . \qquad (11.81)$$

Here the upper and lower signs correspond, respectively, to $\rho_{\min}$ and $\rho_{\max}$. Equation (11.81) yields for $D=1$

$$\frac{\rho_{\min,\max}}{\rho_s}=1\pm\left[1-\frac{E_k}{H_s}\right]^{1/2},\quad E_k<H_s\ ; \qquad (11.82)$$

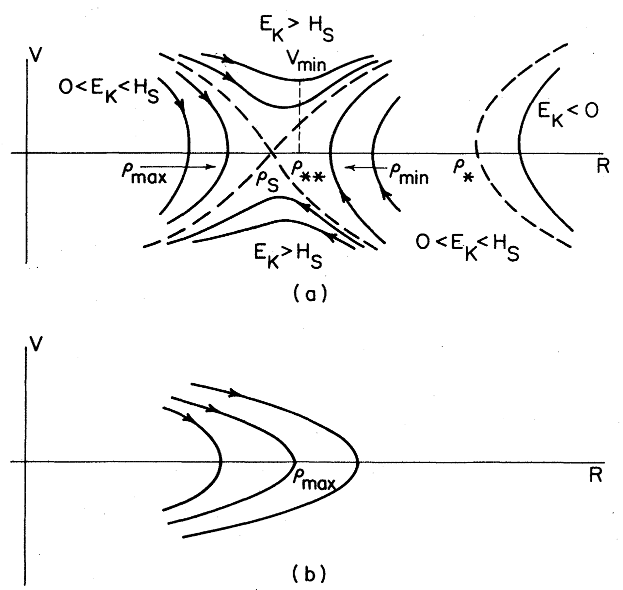

FIG. 47. Phase trajectories of the radial motion of the ringlike SG kink described by Eqs. (11.74) and (11.75): (a) $a>0$; (b) $a<0$. The parameter a is defined by Eq. (11.72).

for $D=2$

$$\frac{\rho_{\min,\max}}{\rho_s}=\frac{1}{2}+\cos\left[\frac{\pi}{3}\mp\frac{2}{3}\cos^{-1}\left[\frac{E_k}{H_s}\right]^{1/2}\right], \quad 0<E_k<H_s, \tag{11.83}$$

$$\frac{\rho_{\min}}{\rho_s}=\frac{1}{2}+\cosh\left[\frac{2}{3}\cosh^{-1}\left[1-\frac{E_k}{H_s}\right]^{1/2}\right], \quad E_k\le 0. \tag{11.84}$$

When $a<0$ the energy E_k is always positive. This time the phase plane has no stationary points [Fig. 47(b)]; however, formally we may again introduce the quantities ρ_s (which is negative now) and H_s by Eqs. (11.76) and (11.77). Then Eq. (11.81) keeps its validity. For $D=1$, the quantity H_s is negative, and $\rho_{\max}$ is defined by Eq. (11.82), where the upper sign should be chosen. For $D=2$ ($H_s>0$), one finds

$$\frac{\rho_{\max}}{\rho_s}=\frac{1}{2}-\cosh\left[\frac{2}{3}\cos^{-1}\left[\frac{E_k}{H_s}\right]^{1/2}\right], \quad 0<E_k<H_s, \tag{11.85}$$

$$\frac{\rho_{\max}}{\rho_s}=\frac{1}{2}-\cosh\left[\frac{2}{3}\cosh^{-1}\left[\frac{E_k}{H_s}\right]^{-1/2}\right], \quad E_k>H_s. \tag{11.86}$$

In the limit $W\to 0$, the stationary point (if any) goes to infinity, and, proceeding from Eq. (11.75), one recovers the earlier result of Samuelsen (1979):

$$\rho^D_{\max}=R_0^D(1-V_0^2)^{-1/2}, \tag{11.87}$$

where R_0 and V_0 are the initial radius and velocity of an expanding ringlike kink, and $\rho_{\max}$ is the maximum radius at which the expansion changes to shrinkage. In the above formulas, Eq. (11.87) corresponds to the limit $E_k/H\to 0$.

In a later paper, Maslov (1988) has considered extrinsic perturbations of a more general form (explicitly dependent of r), which admit, instead of the saddle point in the (V,R) phase plane, a center, i.e., small oscillations of a ringlike kink near a suitable equilibrium position. Such oscillations have been revealed in a numerical experiment by Christiansen *et al.* (1981).

Anisotropic (azimuthal) oscillations of an annular kink in a rotationally symmetric potential well near the stable equilibrium position $R=R_0$ can be analyzed too (Malomed, 1989c). For the azimuthal disturbances $\delta R(\theta)\sim\cos(n\theta)$, where θ is the angular coordinate, and n is an integer, the oscillation frequencies are $\omega_n^2=\omega_0^2+n^2R_0^{-2}$, ω_0^2 being the frequency of the axisymmetric oscillations found by Maslov (1988). Typically, $\omega_0^2\sim R_0^{-1}\gg R_0^{-2}$. The quasi-one-dimensionality requirement imposes on n the restriction $n\ll R_0$. So, the azimuthal contribution to the oscillation frequency may become significant at $R_0\ll n^2\ll R_0^2$.

F. Radiative damping of small long-wave flexural oscillations of a quasi-one-dimensional kink

As was mentioned above, a kink regarded as a quasi-one-dimensional solution to the two-dimensional SG equation

$$u_{tt}-u_{xx}+\sin u=u_{yy} \tag{11.88}$$

is stable against two-dimensional perturbations (see, for example, Scott, 1976; Newell, 1980). Therefore a reasonable problem is to study long-wave oscillations of a quasi-one-dimensional kink with a slightly bent "crest" (Malomed, 1987h). A solution is looked for in the form (2.61a) with $\xi=\xi(t,y)$ and $V=V(t,y)$, where a characteristic scale L, on which the functions $\xi(y)$ and $V(y)$ vary, is sufficiently large, $L\gg 1$. The evolution equations are

$$V_t=(1-V^2)\xi_{yy}+VV_y\xi_y, \qquad \xi_t=V+\frac{1}{2}V\left[\xi_y^2-\frac{\pi^2}{6}(VV_{yy}+V_y^2)\right]. \tag{11.89}$$

A small-amplitude standing-wave solution to Eqs. (11.89) may be sought in the form

$$\xi(t,y)=\sum_{n=0}^{\infty}\Xi_{2n+1}(t)\sin[(2n+1)ky], \qquad V(t,y)=\sum_{n=0}^{\infty}V_{2n+1}(t)\sin[(2n+1)ky], \tag{11.90}$$

where $k^2\ll 1$. Inserting Eq. (11.90) into Eq. (11.89) yields, in the lowest approximation, two equations,

$$\frac{dV_1}{dt}=-k^2\Xi_1+k^2V_1^2\Xi_1, \qquad \frac{d\Xi_1}{dt}=V_1+\frac{1}{8}k^2V_1\left[\Xi_1^2+\frac{\pi^2}{3}V_1^2\right]. \tag{11.91}$$

The linearized equations (11.91) describe the small oscillations

$$\Xi_1=a\cos(kt),\quad V_1=\dot{\Xi}_1, \tag{11.92}$$

where a is an arbitrary small amplitude. The general method developed in Sec. VI.E can be applied in a straightforward manner to obtain the following estimate for the power of the energy emission (per unit length of the kink's "crest") which accompanies the small flexural oscillations (11.90) and (11.92): $W\sim(ka)^{\mathrm{const}/k}$, const ~ 1. Note that, in the present case, the small parameter k plays the role of both ϵ and κ dealt with in Sec. VI.E.

It is pertinent here to mention a similar problem for a quasi-one-dimensional soliton of the exactly integrable Kadomtsev-Petviashvili (KP) equation. Long-wave flexural oscillations of the soliton "crest" were considered in the adiabatic approximation by Ostrovskii and Shrira (1976). Later, Shrira (1980) proposed an effective

Burgers equation to describe the nonlinear long-wave flexural oscillations. Pesenson (1983) evaluated an effective coefficient of dissipation in that equation, accounted for by the radiation losses.

The methods developed in the papers mentioned apply not only to the KP equation, but, as well, to any admitting a stable quasi-one-dimensional soliton. As concerns the KP equation proper, Zakharov (1975) has obtained an exact solution of the problem (including the emission of radiation) by means of the IST.

It is noteworthy that, in contrast to Eq. (11.92) for the SG kink, the emission power for the KP and allied solitons performing flexural oscillations is not exponentially small. The reason is the absence of a gap in the spectrum of linear waves described by the KP and allied equations.

In conclusion, let us mention a recent result of Malomed (1989c) concerning a stationary shape of a curved quasi-one-dimensional kink moving with a constant velocity V under the action of an external drive (1.17) or (1.21) [$f(t)=1$] and dissipation (1.16a). For an arbitrary V from the interval $V_0<V<1$ [V_0 is the equilibrium velocity (3.47) or (3.48) of the driven one-dimensional kink], there exists a stable U-like profile. The angle α between the velocity vector and asymptotes of the profile is related to V as follows: $V^2=V_0^2(1+\cot^2\alpha)$. The interval $V_0\le V<1$ corresponds to $0\le\cot^2\alpha<4\gamma/\pi\epsilon$ [for the drive (1.17)]. For $\cot^2\alpha>4\gamma/\pi\epsilon$, there exists a stable stationary X-shaped profile moving with a velocity $V>1$. The unperturbed SG equation admits a similar "tachyonic" ($V>1$) stable solution produced by the substitution

$$u(x,y,t)=u(y,\tau\equiv(Vt-x)(V^2-1)^{-1/2}) .$$

XII. PERTURBATION THEORY FOR SOLITONS OF THE LANDAU-LIFSHITZ EQUATION

A. Preliminary remarks

In this section we consider a number of problems related to the dynamics of nonlinear magnetization waves in a biaxial ferromagnet. The Landau-Lifshiftz (LL) equation, which describes the nonlinear dynamics in this magnetic system, is completely integrable by means of a specific version of the IST based upon the Riemann problem. Therefore, a perturbation theory for the LL equation can be based upon this technique. A general approach to the analysis of perturbation-induced effects in a biaxial ferromagnet resembles that developed above (see Sec. II). We shall briefly describe the inverse scattering transform for the LL equation and the perturbation theory for it. As examples, we shall present a number of problems that are interesting from a physical viewpoint. These problems are related to the perturbed dynamics of domain walls and magnetic solitons in a biaxial ferromagnet. All the dynamical processes considered in this section find their analogs in the previous sections devoted to perturbed NS and SG equations. It is necessary to mention that, in many particular cases, the LL equation degenerates into the NS and SG equations, as will be shown below.

The perturbation theory for the LL equation describing a ferromagnet with a biaxial anisotropy was elaborated by Kivshar *et al.* (1985a, 1985b), and Kivshar, Kosevich, and Potemina (1986) [see also Potemina (1986) and Kivshar (1989)].

B. Dynamical equations for the magnetization field in a ferromagnet

In the macroscopic theory of ferromagnetism, the magnetic state of a crystal is described by the magnetization vector $\mathbf{M}=(M_1,M_2,M_3)$, while dynamics and kinetics of a ferromagnet are determined by variations of its magnetization field.

The magnetization of a ferromagnet $M(x,t)$ as a function of space coordinates and time is a solution of the LL equation (Landau and Lifshitz, 1935),

$$\frac{\partial\mathbf{M}}{\partial t}=-\frac{2\mu_0}{\hbar}[\mathbf{M}\times\mathbf{H}_{\text{eff}}] , \tag{12.1}$$

where μ_0 is the Bohr magneton. The effective magnetic field H_{eff} is equal to the variational derivative of the magnetic crystal energy with respect to the vector $\mathbf{M}$: $\mathbf{H}_{\text{eff}}=-\delta E/\delta\mathbf{M}$. The magnetic crystal energy E is assumed to be a functional of $\mathbf{M}(x)$ and its spatial derivatives. Different presentations of the function E for ferromagnets and antiferromagnets are discussed in detail in the book by Akhiezer *et al.* (1967). Equation (12.1) has an integral of motion $\langle\mathbf{M}^2\rangle\equiv\mathbf{M}_0^2=$const, the angular brackets standing for the spatial average. In the ground state, the quantity M_0 coincides with a so-called spontaneous magnetization $M_0=2\mu_0 S/a^3$, where S is the atomic spin and a is the interatomic spacing.

In general, the magnetic energy is $E=E_{\text{ex}}+E_a$, where E_{ex} is an exchange energy and E_a is a magnetic anisotropy energy. Below we shall set

$$E_{\text{ex}}=\tfrac{1}{2}\alpha\int\sum_k\frac{\partial\mathbf{M}}{\partial x_k}\frac{\partial\mathbf{M}}{\partial x_k}d^3\mathbf{x} . \tag{12.2}$$

If $E_a=0$, a crystal is called an isotropic ferromagnet. The anisotropy energy of a biaxial ferromagnet can be written as

$$E_a=-\tfrac{1}{2}\beta_1\int M_1^2d^3\mathbf{x}-\tfrac{1}{2}\beta_3\int M_3^2d^3\mathbf{x} . \tag{12.3}$$

In the limit $\beta_1=0$ we have a uniaxial anisotropic ferromagnet: When $\beta_3>0$, the anisotropy is of the easy-axis type, and when $\beta_3<0$ it is of the easy-plane type. We shall consider one-dimensional ferromagnetic systems only.

If we measure the space coordinate x and time t in units of $l_0\equiv(\alpha/\beta_3)^{1/2}$ and $\omega_0^{-1}=(2\mu_0\beta_3M_0/\hbar)^{-1}$, respectively, we obtain from Eqs. (12.1)–(12.3) the well-known one-dimensional equation

$$\mathbf{S}_t=[\mathbf{S}\times\mathbf{S}_{xx}]+[\mathbf{S}\times\hat{J}\mathbf{S}]+\epsilon\mathbf{R}(\mathbf{S}) \tag{12.4}$$

for the normalized magnetization vector $\mathbf{S}=\mathbf{M}(x,t)/M_0$, where the matrix $J=\mathrm{diag}(J_1,J_2,J_3)$ is related to the anisotropy constants. In particular, we may choose $J=\mathrm{diag}(0,\zeta,1+\zeta)$, where $\zeta\equiv|\beta_1|\beta_3$ $(\beta_1<0)$. Equation (12.4) with $\epsilon=0$ is exactly integrable by means of the IST technique (Borovik, 1979; Sklyanin, 1979). The additional terms $(\sim\epsilon)$ on the right-hand side of Eq. (12.4) describe various perturbations, e.g., magnetic impurities, external fields, dissipative losses, etc.

It is important to note that the SG and NS equations may also describe a magnetic system in a proper limit. The SG equation can be obtained from the LL equation in the limit $J_1 \ll J_2 < J_3$, when the oscillations of the vector $\mathbf{S}$ are localized near the easy plane yz. Let us denote

$$S_1=-p/R,\quad S_2=\sqrt{1-p^2/R^2}\sin\varphi\ ,$$
$$S_3=\sqrt{1-p^2/R^2}\cos\varphi\ ,$$

and $x=\sqrt{R}\,\chi$. Then, as $R\gg1$, Eq. (12.4) with $\epsilon=0$ turns into the SG equation,

$$p_t=\varphi_{\chi\chi}-\tfrac{1}{2}\sin(2\varphi)\ ,\quad \varphi_t=p\ ,$$

if the combinations $R(J_3-J_2)$, $(J_3-J_1)/R$, $(J_2-J_1)/R$ in the limit $R\gg1$ take finite values equal to one.

In a similar way, the NS equation can be obtained from Eq. (12.4) in the limit $J_1\approx J_2\ll J_3$, when the oscillations of the magnetization vector $\mathbf{S}$ are located in the vicinity of the vacuum state $\mathbf{S}(x,t)=(0,0,1)$. Let $\psi(x,\tau)=\sqrt{R}\exp(i\gamma R\tau)(S_1+iS_2)$, $t=\sqrt{R}\,\tau$, and $J_3-J_1=J_3-J_2=\gamma R$. Then, in the limit $R\gg1$, we obtain for the complex function ψ the NS equation

$$i\psi_\tau+\psi_{xx}+(\gamma/2)|\psi|^2\psi=0\ .$$

Thus the LL equation is the most general exactly integrable equation describing a one-dimensional ferromagnet.

C. Inverse scattering transform for the Landau-Lifshitz equation

First of all, let us briefly describe the IST technique for the LL equation based on the Riemann problem. The L-A pair for the LL equation (12.4) with $\epsilon=0$ has been found independently by Borovik (1979) and Sklyanin (1979). They have also constructed a direct scattering problem for the LL equation and have found an infinite series of integrals of motion. The inverse scattering problem was solved with the aid of the Riemann problem by Mikhailov (1982) and Rodin (1983) (see also Rodin, 1984). Recently the inverse scattering problem was formulated by Borisov (1986) [see also Mikhalev (1989)] in its classical form, i.e., in terms of equations of the Marchenko type. To develop the perturbation theory, it is convenient to use the IST based on the Riemann problem (Kivshar *et al.*, 1985a, 1985b).

The LL equation (12.4) with $\epsilon=0$ may be represented as a compatibility condition $\hat{L}_t-\hat{A}_x+[\hat{L},\hat{A}]=0$ of two equations for 2×2 matrices $\Psi(x,t;\lambda)$:

$$\Psi_x=\hat{L}\Psi,\quad \hat{L}=-i\sum_{\alpha=1}^{3}w_\alpha(\lambda)S_\alpha\sigma_\alpha\ ,\qquad \Psi_t=\hat{A}\Psi\ , \tag{12.5}$$

$$\hat{A}=-i\sum_{\alpha,\beta,\gamma=1}^{3}b_\alpha(\lambda)\sigma_\alpha S_\beta(S_\gamma)_x e_{\alpha\beta\gamma} -2i\sum_{\alpha=1}^{3}a_\alpha(\lambda)S_\alpha\sigma_\alpha\ , \tag{12.6}$$

where $e_{\alpha\beta\gamma}$ is the totally antisymmetric tensor, σ_α $(\alpha=1,2,3)$ are the Pauli matrices, and $w_\alpha(\lambda)$, $a_\alpha(\lambda)$, and $b_\alpha(\lambda)$ are certain functions of the spectral parameter λ. They may be taken in the form of elliptic functions (Sklyanin, 1979),

$$w_1(\lambda)=\rho\,\mathrm{ns}(\lambda,k),\quad w_2(\lambda)=\rho\,\mathrm{ds}(\lambda,k)\ ,\quad w_3(\lambda)=\rho\,\mathrm{cs}(\lambda,k)\ , \tag{12.7}$$

where $(0<k<1)$

$$k=\sqrt{(J_2-J_1)/(J_3-J_1)}\ ,\quad \rho=\tfrac{1}{2}\sqrt{J_3-J_1}\ . \tag{12.8}$$

The coefficients w_α include two parameters ρ and k instead of the three J_α, because adding a constant to all the J_α does not change the equations of motion (12.4). Since the coefficients $w_\alpha(\lambda)$ are double-periodic functions of the parameter λ,

$$w_\alpha(\lambda+4mK+4niK')=w_\alpha(\lambda)\ ,$$

it is sufficient to consider λ inside the rectangular

$$|\mathrm{Re}\lambda|\le 2K,\quad |\mathrm{Im}\lambda|\le 2K'\ ,$$

where $K(k)$ is a complete elliptic integral of the first kind, and $K'(k)=K(\sqrt{1-k^2})$.

There are two different types of physical boundary conditions for Eq. (12.4):

$$\mathbf{S}(x,t)\to(0,0,1)\quad\text{at } x\to\pm\infty \tag{12.9a}$$

or

$$\mathbf{S}(x,t)\to(0,0,\pm\kappa)\quad\text{at } x\to\pm\infty\ , \tag{12.9b}$$

where $\kappa=\pm1$. Boundary conditions of the first type correspond to breatherlike solutions usually called magnetic solitons. The second type distinguish kinklike solutions that are usually called domain walls in magnetic models. In order to develop a perturbative technique for domain walls, we present the IST formalism for the boundary conditions (12.9b).

Let us introduce two fundamental eigenfunctions (Jost functions) of the $\hat{L}$ operator (12.5) according to the asymptotics [κ is the same ±1 as in Eq. (12.9b)]

$$\lim_{x\to+\infty}\Psi_+(x,\lambda)=\exp[-i\sigma_3\kappa w_3(\lambda)x]\ ,$$

$$\lim_{x\to-\infty}\Psi_-(x,\lambda)=\exp[i\sigma_3\kappa w_3(\lambda)x]\sigma_2$$

for the contours $\mathrm{Im}\lambda=0$; $\mathrm{Im}\lambda=2K'$. These functions are connected by the transition matrix $\Psi_t(x,\lambda)=\Psi_-(x,\lambda)T(\lambda)$, where [cf. Eq. (2.33)]

$$T(\lambda)=\begin{bmatrix}\alpha(\lambda) & -b(\lambda)\\ b^*(\lambda^*) & a^*(\lambda^*)\end{bmatrix}.$$

The Jost functions may be presented in the form (2.30) and (2.31), where

$$\psi(x,\lambda)\equiv\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}\to\begin{bmatrix}1\\0\end{bmatrix}\exp[-i\kappa w_3(\lambda)x]\ ,\quad x\to+\infty\ ,$$

$$\varphi(x,\lambda)\equiv\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}\to\begin{bmatrix}1\\0\end{bmatrix}\exp[i\kappa w_3(\lambda)x]\ ,\quad x\to-\infty\ .$$

It is easy to prove that the following relations for $\Psi_\pm(x,\lambda)$ are valid:

$$\sigma_2\Psi_\pm(x,\lambda)\sigma_2=\Psi_\pm^*(x,\lambda^*)\ ,$$

$$\sigma_3\Psi_\pm(x,\lambda)\sigma_3=\pm\Psi_\pm(x,\lambda+2K)\ ,$$

$$\sigma_3\Psi_\pm(x,\lambda)\sigma_3=\pm\Psi_\pm^*(x,\lambda^*-2iK)\ ,$$

which entails

$$a(\lambda)=-a(\lambda+2K),\quad a^*(\lambda)=-a^*(\lambda^*+2iK')\ ,$$

$$b(\lambda)=b(\lambda+2K),\quad b(\lambda)=b^*(\lambda^*+2iK')\ .$$

Following Mikhailov (1982) and Rodin (1983), we define the functions

$$f_+(x,\lambda)=\Psi_-(x,\lambda)T_+(\lambda)\exp[i\kappa w_3(\lambda)x\sigma_3]\ , \tag{12.10}$$

$$f_-(x,\lambda)=\exp[-i\kappa w_3(\lambda)x\sigma_3]T_-(\lambda)\Psi_-^{-1}(x,\lambda)\ ,$$

on the contours $\Gamma_1=\{\lambda:|\mathrm{Re}\lambda|\le 2K,\ \mathrm{Im}\lambda=0\}$ and $\Gamma_2=\{\lambda:|\mathrm{Re}\lambda|\le 2K,\ \mathrm{Im}\lambda=2K'\}$. Here

$$T_+(\lambda)=\begin{bmatrix}a(\lambda) & 0\\ b^*(\lambda^*) & 1\end{bmatrix},\quad T_-(\lambda)=\begin{bmatrix}a^*(\lambda^*) & b(\lambda)\\ 0 & 1\end{bmatrix}. \tag{12.11}$$

It is possible to prove that these functions can be analytically continued into the domains $R_\pm$ at $\kappa=1$ or $R_\mp$ at $\kappa=-1$, respectively, where

$$R_+=\{|\mathrm{Re}\lambda|\le 2K,0\le \mathrm{Im}\lambda\le 2K'\}\ ,$$

$$R_-=\{|\mathrm{Re}\lambda|\le 2K,-2K'\le \mathrm{Im}\lambda\le 0\}\ ,$$

and the functions $f_\pm$ are $(4K,4K')$ periodic. Hence the function $a(\lambda)$ admits analytical continuation into $R_+(\kappa=1)$ or $R_-(\kappa=-1)t$.

If $a(\lambda)$ vanishes at the points $\lambda=\lambda_{0k}$ $(0\le\mathrm{Re}\lambda_{0k}\le 2K,\ 0\le\kappa\,\mathrm{Im}\lambda_{0k}\le 2K')$, $a(\lambda)$ also vanishes at the points $\lambda_{1k}=\lambda_{0k}-2K$, $\lambda_{2k}=\lambda_{0k}^*+2iK'$, $\lambda_{3k}=\lambda_{0k}^*-2K+2iK'$. At the zeros of $a(\lambda)$, the matrices $f_\pm$ are degenerate, and their columns are proportional to each other:

$$f_+^{(2)}(x,\lambda_{0k})=b_kf_+^{(1)}(x,\lambda_{0k})e^{-2i\kappa w_3(\lambda_{0k})x}\ ,$$

i.e.,

$$\varphi(x,\lambda_{0k})=b_k\psi(x,\lambda_{0k})\ . \tag{12.12}$$

The set of quantities $\{a(\lambda),\ b(\lambda),\ \lambda_{0k}$, and b_k, where $k=\{1,2,\ldots,N\}$, constitutes a complete and independent set of scattering data. Its dependence on time can be found from Eq. (12.6),

$$\lambda_{0k}(t)=\lambda_{0k}(0),\quad a(\lambda,t)=a(\lambda,0)\ ,$$

$$b(\lambda,t)=b(\lambda,0)\exp[-4i\kappa w_1(\lambda)w_2(\lambda)t]\ , \tag{12.13}$$

$$b_k(t)=b_k(0)\exp[-4i\kappa w_1(\lambda_{0k})w_2(\lambda_{0k})t]\ .$$

The magnetization field may be found from the relations (Kivshar *et al.*, 1985a, 1985b)

$$\sum_\alpha\sigma_\alpha S_\alpha=f_+(x,0)\sigma_3f_+^{-1}(x,0)\quad(\kappa=+1)\ , \tag{12.14a}$$

and

$$\sum_\alpha\sigma_\alpha S_\alpha=-f_-^{-1}(x,0)\sigma_3f_-(x,0)\quad(\kappa=-1)\ . \tag{12.14b}$$

So, the Cauchy problem is solved if the functions $f_+(x,0)$ $(\kappa=1)$ or $f_-(x,0)$ $(\kappa=-1)$ are known.

It is important that a matrix which conjugates the piecewise analytical solutions $f_\pm$ along the contours $\Gamma_{1,2}$ can be represented in terms of the scattering data. It follows from Eqs. (12.10) and (12.11) that $(\mathrm{Im}\lambda=0)$

$$f_-(x,\lambda)f_+(x,\lambda)=G(x,t;\lambda)\ ,$$
$$f_-(x,\lambda-2iK')f_+(x,\lambda+2iK')=G(x,t;\lambda+2iK')\ , \tag{12.15}$$

where

$$G(x,t;\lambda)=\begin{bmatrix}1 & b(\lambda,t)\exp[-2i\kappa w_3(\lambda)x]\\ b(\lambda,t)\exp[2i\kappa w_3(\lambda)x] & 1\end{bmatrix}. \tag{12.16}$$

This is a classical *Riemann boundary problem* on a Riemann surface. In the present case, this surface is a torus, since all the functions in Eqs. (12.15) and (12.16) are double periodic.

The Riemann problem (12.15) and (12.16) may be solved as follows (Mikhailov, 1982; Rodin, 1983; Kivshar *et al.*, 1985a, 1985b, 1986). Let us represent a solution of the Riemann problem as

$$f_+=f^r_+f^R_+,\quad f_-=\ f^R_-f^r_-\ . \tag{12.17}$$

Here f^R denotes a meromorphic function on a torus, and f^r is a solution of the regular Riemann problem ($\det f^r_\pm\neq 0$). Then the problem splits into two separate pairs: construction of $f^R_\pm$ and $f^r_\pm$. The functions $f^R_\pm$ are related to exact N-soliton solutions of the LL equation, and $f^r_\pm$ describe excitations of a continuous spectrum (spin waves). The former functions have been constructed by Mikhailov (1982) and Rodin (1983, 1984) for the boundary condition (12.9a). These functions have a standard polynomial representation. They may also be obtained by the so-called "dressing method" (see, Borisov, 1983).

After determination of $f^R_\pm$, it is necessary to solve the regular Riemann problem ($\det f^r_\pm\neq 0$),

$$f^r_-(\lambda)f^r_+(\lambda)=\widetilde{G}(x,t;\lambda)\ , \tag{12.18}$$

where

$$\widetilde{G}(x,t;\lambda)=f^R_+(\lambda)G(x,t;\lambda)[f^R_+(\lambda)]^{-1}\ .$$

Using the Cauchy-type integral on a torus (see Kivshar *et al.*, 1985a, 1985b, 1986; Kivshar, 1989), we reduce the problem (12.18) and (12.19) to a Fredholm integral equation. For example, at $\kappa=1$

$$\tfrac{1}{2}f^r_+(\lambda)[I+\widetilde{G}^{\,-1}(\lambda)]=f_0+\frac{1}{2\pi i}\int_\Gamma d\mu\{N_1(\mu-\lambda)f^r_+(\mu)[I-\widetilde{G}^{\,-1}(\mu)]+N_2(\mu-\lambda)\sigma_3 f^r_+(\mu)[I-\widetilde{G}^{\,-1}(\mu)]\sigma_3\}\ , \tag{12.19}$$

where $\Gamma\equiv\Gamma_1\cup\Gamma_2$ and I is the unit matrix. The functions $N_1(\lambda)$ and $N_2(\lambda)$ are kernels of the Cauchy representations for the matrices $f_\pm(\lambda)$. Their explicit form is

$$\begin{aligned}N_1(\lambda)&=\tfrac{1}{8}[\mathrm{ns}(\lambda,k)+\mathrm{ds}(\lambda,k)][1+\mathrm{cn}(\lambda,k)]\ ,\\ N_2(\lambda)&=\tfrac{1}{8}[\mathrm{ns}(\lambda,k)+\mathrm{ds}(\lambda,k)][\mathrm{cn}(\lambda,k)-1]\ .\end{aligned} \tag{12.20}$$

A similar integral equation can be obtained for the case $\kappa=-1$.

For subsequent calculations, it is important to relate the physical quantities, e.g., the energy, to the scattering data. According to the IST formulas (Sklyanin, 1979), the energy of the LL equation can be presented as follows:

$$E=-4i\rho\kappa a^{-1}(\lambda=0)\left[\frac{\partial}{\partial\lambda}a(\lambda)\right]_{\lambda=0}\ .$$

Using the expression for $a(\lambda)$,

$$\ln a(\lambda)=\ln a_s(\lambda)+\frac{\kappa}{2\pi i}\int_0^{2k}d\mu\,\mathrm{cs}(\mu-\lambda-i\kappa 0)\ln|a(\mu)|^2$$

[where $a_s(\lambda)$ is the soliton part of $a(\lambda)$], one can present the radiation part of the energy $E_{\rm em}$ in the form [cf. Eq. (6.4)]

$$E_{\rm em}\simeq\frac{2}{\pi\rho}\int_0^{2k}d\lambda\,w_1(\lambda)w_2(\lambda)|b(\lambda,t)|^2\ . \tag{12.21}$$

Equation (12.21) is used for calculation of radiation-induced effects in the perturbed LL equation.

First-order corrections $S^{(1)}_\alpha$ ($\sim\epsilon$) to the magnetization field can be found from Eqs. (12.14). After straightforward calculations, one can obtain (Kivshar *et al.*, 1985a, 1985b; Kivshar, 1989)

$$\sum_\alpha S^{(1)}_\alpha\sigma_\alpha=\kappa\left[f_1(x,0),\sum_\alpha S^{(0)}_\alpha\sigma_2\right]\ , \tag{12.22}$$

where [,] stands for the matrix commutator, $S^{(0)}_\alpha$ stands for an unperturbed solution of the LL equation at $\epsilon=0$, and the function $f_1(x,0)\equiv f^r_\pm(x,\lambda)-I$ can be found from the Fredholm integral equation (12.19) and (12.20):

$$f_1(x,0)=-\frac{1}{2\pi i}\int_r d\mu[N_1(\mu)G_1(\mu)+N_2(\mu)\sigma_3G_1(\mu)\sigma_3]\ .$$

$N_1(\mu)$ and $N_2(\mu)$ are defined in Eq. (12.21), and $G_1(\mu)\equiv\widetilde{G}^{\,-1}(\mu)-I$ [see Eq. (12.18)]. The functions $G_1(\mu)$ and f_1 are of order ϵ.

D. Perturbation-induced evolution equations for scattering data

According to the general ideas of the perturbation theory for solitons developed in Sec. II, for the perturbed LL equation (12.4) we shall use the scattering-data representation. The evolution equations for the scattering data in the presence of perturbations can be presented as follows [cf. Eqs. (2.7) and (2.8)]:

$$\frac{d\mathcal{L}}{dt}=\{\mathcal{L},\mathcal{H}_0\}+\epsilon\int_{-\infty}^{+\infty}dx\sum_{\alpha=1}^{3}R_\alpha(\mathbf{S})\frac{\delta\mathcal{L}}{\delta\mathbf{S}}\ , \tag{12.23}$$

where $\mathcal{L}$ stands for any functional of scattering data, $\mathcal{H}_0$ is the Hamiltonian of the unperturbed LL equation [Eq. (12.4) with $\epsilon=0$], and { , } stands for the functional Poisson brackets,

$$\{F,G\}=-\int_{-\infty}^{+\infty}dx\,e_{ijk}\frac{\delta F}{\delta S_i(x)}\frac{\delta G}{\delta S_j(x)}S_k(x)\ ,$$

e_{ijk} being the totally antisymmetric tensor.

Let us consider the full set of scattering data for the LL equation. By means of direct calculations, one can obtain the relation $\{\mathcal{L},\mathcal{H}_0\}=(d\mathcal{L}/dt)_{\epsilon=0}$, which means that the first term on the right-hand side of Eq. (12.23) describes the unperturbed evolution of the scattering data [see Eqs. (12.13)]. Calculating the variational derivatives with the aid of the IST technique, we obtain (Kivshar *et al.*, 1985a, 1985b, 1986; Kivshar, 1989)

$$\frac{\partial a(\lambda,t)}{\partial t}=-i\epsilon\int_{-\infty}^{+\infty}dx\ W\{\psi(x,\lambda),\mathcal{R}(\lambda)\varphi(x,\lambda)\}\ , \tag{12.24}$$

$$\frac{\partial b(\lambda,t)}{\partial t}=-4i\kappa w_1(\lambda)w_2(\lambda)b(\lambda,t)-i\epsilon\int_{-\infty}^{+\infty}dx\ W\{\mathcal{R}(\lambda)\varphi(x,\lambda),\tilde{\psi}(x,\lambda)\}\ , \tag{12.25}$$

$$\frac{d\lambda_k}{dt}=\frac{i\epsilon}{a^1(\lambda_k)}\int_{-\infty}^{+\infty}dx\ W\{\psi(x,\lambda_k),\mathcal{R}(\lambda_k)\varphi(x,\lambda_k)\}\ , \tag{12.26}$$

$$\frac{db_k}{dt}=-4i\kappa w_1(\lambda_k)w_2(\lambda_k)b_k+\frac{i\epsilon}{a'(\lambda_k)}\int_{-\infty}^{+\infty}dx\ W\{\varphi'(\lambda_k)-b_k\psi'(\lambda_k),\mathcal{R}(\lambda_k)\varphi(x,\lambda_k)\}\ , \tag{12.27}$$

where we have introduced the notation

$$\mathcal{R}(\lambda)\equiv\sum_{\alpha=1}^{3}w_\alpha(\lambda)R_\alpha(\mathbf{S})\sigma_\alpha \tag{12.28}$$

for the components of the perturbation vector. In Eqs. (12.24)–(12.27), $W\{\psi,\varphi\}$ stands for the Wronskian

$$W\{\psi,\varphi\}=\det\{\psi,\varphi)$$
$$=\psi^{(1)}(x,\lambda)\varphi^{(2)}(x,\lambda)-\psi^{(2)}(x,\lambda)\varphi^{(1)}(x,\lambda)\ , \tag{12.29}$$

and

$$\varphi'(\lambda_k)\equiv[\partial\varphi(x,\lambda)/\partial\lambda]_{\lambda=\lambda_k}\ .$$

To calculate the variational derivatives in Eqs, (12.24)–(12.27), we have employed the relations $a(\lambda)=W\{\psi,\varphi\}$ and $b(\lambda)=W\{\varphi,\tilde{\varphi}\}$.

Equations (12.24)–(12.29) are the perturbation-induced evolution equations for the scattering data. They constitute the basis of the soliton perturbation theory [cf. Eqs. (2.23)–(2.26), (2.46)–(2.50), and (2.71)–(2.74) in Sec. II for the KdV, NS, and SG equations].

E. Dynamics of a domain wall in the presence of perturbations

1. Adiabatic equations for domain-wall parameters and a correction to the domain-wall shape

The one-soliton solution of the LL equation describes a domain wall in a biaxial ferromagnet [$\mathbf{S}=(S_1,S_2,S_3)$],

$$S_1+iS_2=e^{i\varphi}\mathrm{sech}z,\quad S_3=\kappa\tanh z\ , \tag{12.30}$$

where

$$z=\xi(\varphi)x-\kappa\tau(\varphi)t=\xi(\varphi)[x-X(t)]\ , \tag{12.31}$$

$$\xi(\varphi)\equiv\sqrt{1+\zeta\cos^2\varphi}\ , \tag{12.32a}$$

$$\tau(\varphi)\equiv-\zeta\sin\varphi\cos\varphi \tag{12.32b}$$

[in Eq. (12.32) we choose $J=\mathrm{diag}(0,\zeta,1+\zeta)$]. The parameter $\kappa=\pm1$ is the polarity of the domain wall, and the angle φ (φ=const in the case $\epsilon=0$) determines the orientation of the magnetization vector $\mathbf{S}$ in the plane orthogonal to the anisotropy axis (0,0,1). According to Eq. (12.31), the width of the domain wall is $\sim\xi^{-1}$, and its velocity is

$$V(\varphi)=\frac{\kappa\tau(\varphi)}{\xi(\varphi)}=-\frac{\kappa\zeta\sin\varphi\cos\varphi}{\sqrt{1+\zeta\cos^2\varphi}}\ . \tag{12.33}$$

Simple calculations yield the energy $E_0=2\xi(\varphi)$ of the domain wall. The dependences of the velocity $V(\varphi)$ and energy $E_0(\varphi)$ are shown in Figs. 48(a) and 48(b), respec-

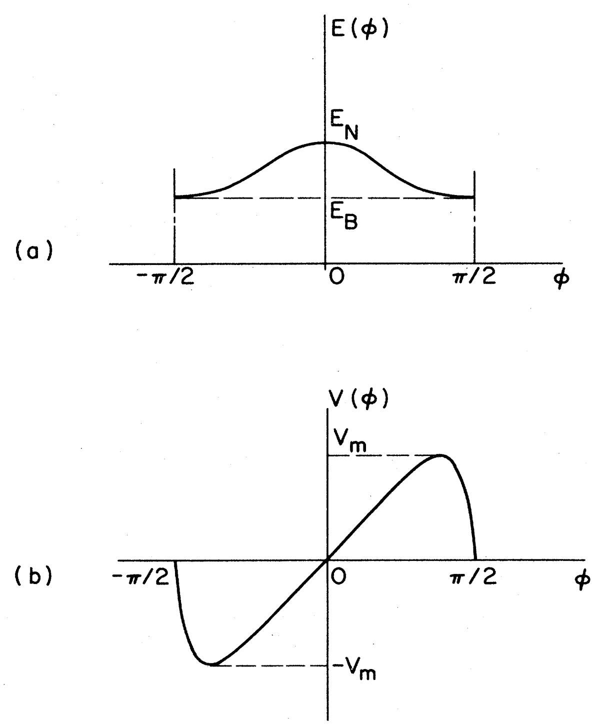

FIG. 48. The energy (a) and velocity (b) of a domain wall [described by Eqs. (12.30)–(12.33)] vs the orientation angle φ.

tively. It is important to note that the energy has a maximum $E_N=2\sqrt{1+\xi}$ at $\varphi=0$, which corresponds to the energy of the Néel-type domain wall. A domain wall with $\varphi=\pm\pi/2$ is usually called a Bloch wall. Its energy equals 2 in our notation. As is depicted in Fig. 48(b), the function $V(\varphi)$ is monotonic.

The one-soliton solution (12.30) to Eq. (12.32) gives rise to the Jost functions

$$\psi_s(x,\lambda)=\frac{\exp[-iw_3(\lambda)\kappa x]}{\sqrt{2}\cosh z}\begin{Bmatrix} e^{z/2} \\ -a(\lambda)e^{-z/2}\end{Bmatrix}, \quad (12.34)$$

$$\varphi_s(x,\lambda)=\frac{\exp[iw_3(\lambda)\kappa x]}{\sqrt{2}\cosh z}\begin{Bmatrix} e^{-z/2} \\ a(\lambda)e^{z/2}\end{Bmatrix}, \quad (12.35)$$

where $a(\lambda)$ is the Jost coefficient

$$a(\lambda)=\frac{w_3(\lambda)\kappa+i\xi(\varphi)/2}{-w_1(\lambda)\cos\varphi+iw_2(\lambda)\sin\varphi} . \quad (12.36)$$

The function $a(\lambda)$ has two complex zeros, λ_k $(k=1,2)$ lying on the line $\lambda=2iK'$ and separated by the distance $2K$. The zeros are roots of the equation

$$w_3(\lambda_k)=-i\kappa\xi(\varphi)/2, \quad k=1,2 . \quad (12.37)$$

At $\lambda=\lambda_k$ the Jost functions are

$$\psi_s(x,\lambda_k)=\frac{\exp[-\xi(\varphi)X/2]}{\sqrt{2}\cosh z}\begin{bmatrix}1\\0\end{bmatrix}, \quad \varphi_s(x,\lambda_k)=\frac{\exp[\xi(\varphi)X/2]}{\sqrt{2}\cosh z}\begin{bmatrix}1\\0\end{bmatrix}, \quad (12.38)$$

and Eq. (12.12) is valid, $b_k=\exp[\xi(\varphi)X]$ being the Jost coefficient of the discrete spectrum.

According to the basic idea of the perturbative technique based on the IST, we consider the domain wall's parameters φ and dX/dt as slowly varying functions of time. Inserting Eqs. (12.34)–(12.38) into Eqs. (12.26) and (12.27) yields the adiabatic equations for these parameters:

$$\frac{d\varphi}{dt}=-\frac{\epsilon}{2}\int_{-\infty}^{+\infty}dz\,R_-(z)\operatorname{sech}z , \quad (12.39)$$

$$\frac{dX}{dt}=V(\varphi)-\frac{\epsilon\tau(\varphi)}{2\xi^3(\varphi)}\int_{-\infty}^{+\infty}dz\frac{zR_-(z)}{\cosh z}+\frac{\epsilon}{2\xi(\varphi)}\int_{-\infty}^{+\infty}dz\frac{R_+(z)}{\sinh z} , \quad (12.40)$$

where

$$R_\mp(z)=\begin{bmatrix}\sin\varphi\\ \cos\varphi\end{bmatrix}R_1(z)\mp\begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix}R_2(z) , \quad (12.41)$$

and the physical relation $\mathbf{R}\cdot\mathbf{S}=0$ is used.

The adiabatic equations (12.39) and (12.40) may also be derived by means of direct perturbation theory similar to that developed by McLaughlin and Scott (1978) for the SG equation. For a ferromagnet with anisotropy of the easy-axis type, the derivation has been performed by Borovik *et al.* (1984) (see also Potemina, 1986, and Kivshar, 1989).

Equation (12.39) governs the evolution of the domain-wall energy. Indeed, Eq. (12.4) gives rise to the evolution equation

$$dE_0/dt=\epsilon\int_{-\infty}^{+\infty}dx\{\mathbf{R}(\mathbf{S})\cdot[\mathbf{S}\times d\mathbf{S}/dt]\} ,$$

which is equivalent to Eq. (12.39). To obtain corrections to the adiabatic approximation, one should use the perturbation-induced evolution equations (12.24) and (12.25). For example, for the one-soliton solution, Eq. (12.25) can be represented as follows (with accuracy up to terms $\sim\epsilon$):

$$\frac{\partial b(\lambda,t)}{\partial t}=-4i\kappa w_1(\lambda)w_2(\lambda)b(\lambda,t)+\epsilon U(\lambda,\varphi)e^{2iw_3(\lambda)\kappa X} , \quad (12.42)$$

where

$$U(\lambda,\varphi)=\frac{1}{2\xi(\varphi)}\int_{-\infty}^{+\infty}dz\{a^*(\lambda)e^{-z}[iw_1(\lambda)R_1(z)-w_2(\lambda)R_2(z)]-a(\lambda)e^{z}[iw_1(\lambda)R_1(z)+w_2(\lambda)R_2(z)]-2iw_3(\lambda)R_3(z)\}e^{2i\kappa z w_3(\lambda)/\xi(\varphi)}\operatorname{sech}z . \quad (12.43)$$

Kivshar *et al.* (1985a, 1985b, 1986) have calculated a correction $\mathbf{S}^{(1)}$ to the domain-wall shape (12.30) with the help of the Riemann problem. The result follows from the general formula (12.22) and may be written

$$\mathbf{S}^{(1)}=[\mathbf{A}\times\mathbf{S}^{(0)}] , \quad (12.44)$$

where

$$A_\alpha=\frac{\kappa}{2\pi}\int_{-2k}^{2k}d\mu\, w_\alpha(\mu)g_\alpha(\mu) , \quad (12.45a)$$

$$g_1+ig_2=\tfrac{1}{2}a(\lambda)\{b^*(\lambda,t)e^{z}\exp[2i\kappa w_3(\lambda)x]-b(\lambda,t)e^{-z}\exp[-2i\kappa w_3(\lambda)x]\}\operatorname{sech}z , \quad (12.45b)$$

$$g_3=\operatorname{Re}\{b(\lambda,t)\exp[-2i\kappa w_3(\lambda)x]\} . \quad (12.45c)$$

The evolution of $b(\lambda,t)$ is described by Eq. (12.42) written in the same approximation.

2. Interaction of a domain wall with an inhomogeneity

Let us consider interaction of the domain wall (12.30) with a local magnetic inhomogeneity in the presence of a magnetic field $\mathbf{h}=(0,0,h)$ and dissipation. This problem is described by the perturbation

$$\epsilon\mathbf{R}(\mathbf{S})=[\mathbf{S}\times\mathbf{h}]+\gamma[\mathbf{S}\times\mathbf{S}_t] + \epsilon_0[\mathbf{S}\times(f_1(x)\mathbf{S}_x)_x]+f_2(x)[\mathbf{S}\times\hat{\imath}\mathbf{S}] , \quad (12.46)$$

where γ is the dissipation constant, $f_1(x)$ and $f_2(x)$ characterize the inhomogeneity, and the diagonal matrix $\hat{\imath}=\mathrm{diag}(i_1,i_2,i_3)$ characterizes the influence of the inhomogeneity on the anisotropy constants. The case of a slowly varying inhomogeneity was considered by Kivshar *et al.* (1985a, 1985b; Kivshar, 1989), who showed that energy emitted in the form of spin waves (linear magnetic excitations) by a moving domain wall is exponentially small.

For a local inhomogeneity one should take $f_1(x)=f_2(x)=\delta(x)$. Inserting the perturbation (12.46) into the right-hand side of Eqs. (12.39) and (12.40) yields a system of two adiabatic equations:

$$\frac{d\varphi}{dt}=h-\gamma\tau(\varphi)-\kappa\xi(\varphi)G_1(\varphi)\frac{\tanh[\xi(\varphi)X]}{\cosh^2[\xi(\varphi)X]} , \quad (12.47)$$

$$\frac{dX}{dt}=-V(\varphi)[1-G_2(\varphi,X)] , \quad (12.48)$$

where

$$G_1(\varphi)=\tfrac{1}{2}[(i_3-i_2)\sin^2\varphi+(i_3-i_1)\cos^2\varphi] ,$$
$$G_2(\varphi,X)=\frac{i_2-i_1}{2\xi}\frac{\xi(\varphi)}{\cosh^2[\xi(\varphi)X]} - G_1(\varphi)X\frac{\tanh[\xi(\varphi)X]}{\cosh^2[\xi(\varphi)X]} \quad (12.49)$$

[for simplicity, we present results for the case $\epsilon_0=0$ only; more general equations can be found in the paper of Kivshar (1989)].

When $h=0$ and $i_1=i_2=i_3=0$, the dynamics of the domain wall are rather simple: under the action of dissipation, the parameter φ changes from the initial value $\varphi=0$, which corresponds to the maximum energy [Fig. 48(a)], to a final value $\varphi=\pm\pi/2$ corresponding to the minimum energy. According to Fig. 48(b), the corresponding initial and final values of the domain-wall velocity are zero, while at $0<t<\infty$ the velocity is not zero.

The stable solution in a dissipative system is a Bloch domain wall corresponding to $\varphi=\pm\pi/2$. The Néel wall is unstable. Under the action of a constant magnetic field $\mathbf{h}=(0,0,h)$, the steady-state motion of a domain wall is characterized by the equilibrium velocity [cf. Eq. (3.48)]

$$V_W=V(\varphi_W)=-\frac{\kappa\xi}{2}\left[\frac{h}{h_c}\right]\left[1+\frac{\xi}{2}\left[\frac{h}{h_c}\right]^2\left[1+\left\{\left[1-\left[\frac{h}{h_c}\right]^2\right]^{1/2}\right\}^{-1}\right]^{-1/2}\right] , \quad (12.50a)$$

where

$$\tan\varphi_W=-\frac{1}{h}[h_c+(h_c^2-h^2)^{1/2}] ,$$

and

$$h_c=\gamma\xi/2 . \quad (12.50b)$$

Equation (12.30) with the velocity (12.50a) is usually called a Walker domain wall.

It is interesting that for $h>h_c$ the motion of the domain wall is oscillating,

$$\tan\varphi=\frac{1}{h}\{(h^2-h_c^2)^{1/2}\tan[(h^2-h_c^2)^{1/2}(t-t_0)]-h_c\} .$$

In the limit $\xi\ll 1$ but for $h\sim 1$, similar oscillations were investigated by Słonczewski (1972) within the framework of another approach.

In the presence of an inhomogeneity, Eqs. (12.47) and (12.48) take a more complicated form, and it is difficult to find their exact solutions. However, one can gain considerable information about the dynamics of the domain wall by looking at the phase plane of the dynamical system (12.47) and (12.48) (Figs. 49–51). Stable stationary points of this system correspond to states in which the domain wall is pinned by a local inhomogeneity. The

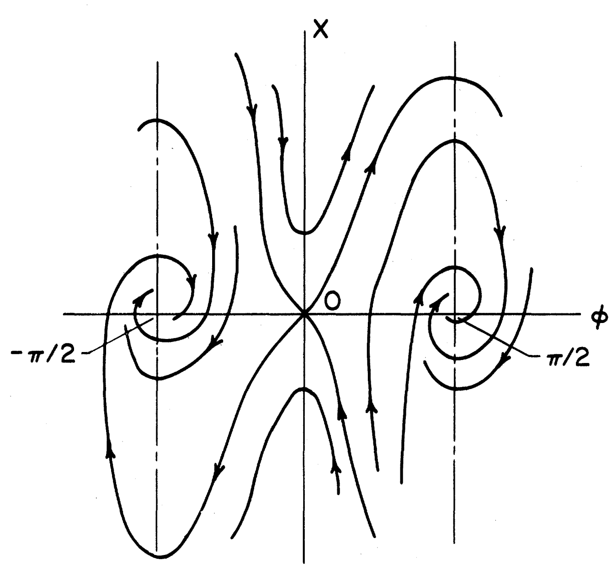

FIG. 49. The phase plane of the dynamical system (12.46)–(12.48) for $h=0$.

pinned states exist provided

$$|h|<h_0\equiv(2/3^{3/2})\epsilon,\quad \epsilon\equiv\tfrac{1}{2}(i_3-i_2)\ . \tag{12.51}$$

The phase plane is shown in Fig. 49 for the case $h=0$, in Fig. 50 for the case $0<h<h_c$, where h_c is defined by Eq. (12.50b), and in Fig. 51 for $h>h_c$ (we assume $h_c<h_0$).

A full analysis of the phase plane has been carried out by Kivshar (1989). In his paper, the threshold value of the external magnetic field admitting the capture by a local inhomogeneity of a domain wall moving with velocity (12.50a) has been calculated in the McLaughlin-Scott approximation (McLaughlin and Scott, 1978):

$$h_{\rm thr}=2h_c\sqrt{A-1}/A\ ,\quad A=\zeta[(1+\epsilon/2)^2-1]^{-1}$$

[cf. Eq. (3.57)].

The perturbation-induced evolution of the Jost coefficient $b(\lambda,t)$ governed by Eq. (12.42) gives rise to radiative effects accompanying the scattering of the domain wall by the local inhomogeneity. The spectral density of the energy emitted by the domain wall has been calculated with the use of Eq. (12.21) by Kivshar (1989) (the full expression is very ponderous). In particular, for a small domain-wall velocity, $V=V_w\approx h/\lambda$, the total emitted $E_{\rm em}$ energy is exponentially small [cf. a similar result (6.63) for a SG kink]. This means that the threshold velocity for radiative capture of the domain wall by a local inhomogeneity (in the case $h=0$, $\gamma=0$) should be exponentially small in $\sqrt{\epsilon}$. The calculations of Kivshar (1989) yield

$$E_{\rm em}=(2\sqrt{\pi}/\zeta)(1+\zeta)^{3/2}(2+\zeta)^{-1}\times\exp\{-[(1+\zeta)/\zeta\epsilon]^{1/2}\}$$

[cf. Eq. (6.67) obtained by Malomed (1985b) for a SG kink interacting with an attractive local inhomogeneity (1.19)].

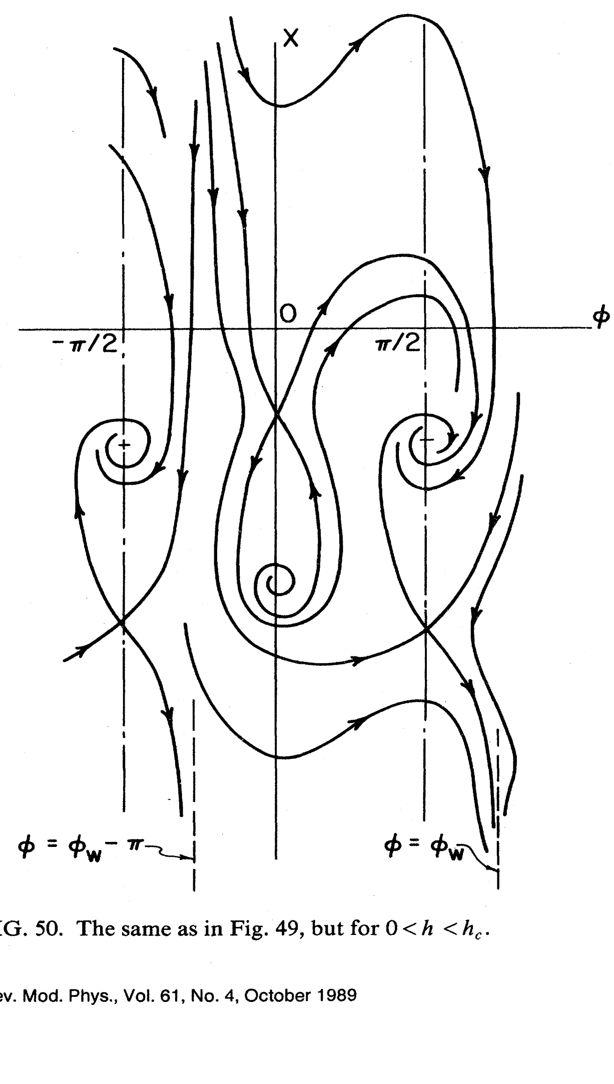

FIG. 50. The same as in Fig. 49, but for $0<h<h_c$.

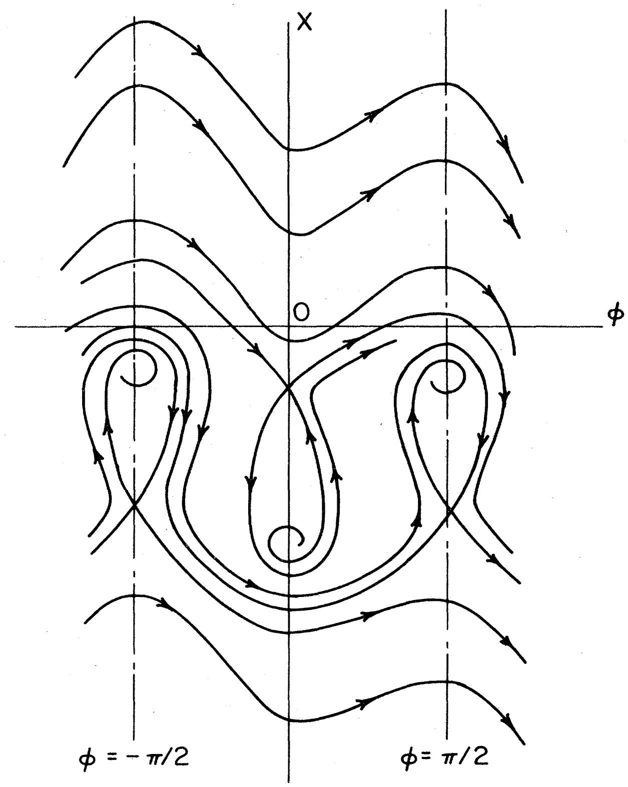

FIG. 51. The same as in Fig. 49, but for $h_c<h<h_0$.

3. Emission from a domain wall oscillating under the action of an ac magnetic field

Let us consider the emission of spin waves from a domain wall under the action of an additional variable magnetic field, $\mathbf{h}=(0,0,h(t))$, $h(t)=h_0+h_1\sin(\Omega t)$. In the limit $V\gg\sqrt{h_1}$, the perturbation-induced change in the mean velocity of the domain wall is small, and it may be neglected. The ac component of the field gives rise to oscillations in the domain wall with the frequency Ω. If $\Omega>\omega_0(V)$, where the function $\omega_0(V)=\omega+kV$ is shown in Fig. 52, the emission of spin waves takes place (see the shaded range in Fig. 52). This problem was first considered by Kivshar *et al.* (1985b) and treated in further detail by Potemina (1986).

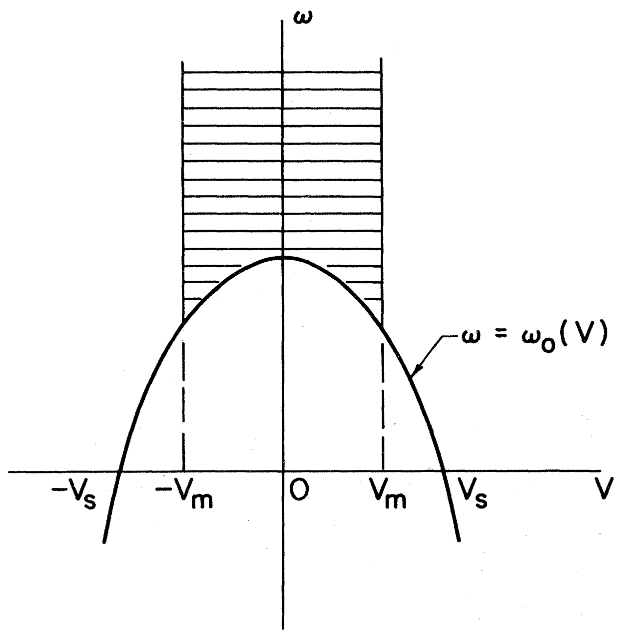

FIG. 52. The spin-wave dispersion law in terms of the wave frequency ω and group velocity V. Emission of spin waves takes place in the shaded range.

In the region $|z| \gg 1$, an asymptotic form of the emitted wave field can be presented as $[v_g^{\pm} \equiv \partial\omega_0(k)/\partial k|_{k=k_\pm}]$ as

$$\begin{bmatrix} S_1^{(1)} \\ S_2^{(1)} \end{bmatrix} \approx \pm \frac{2D_\Omega \operatorname{sech}(\pi k_\pm/2\xi)}{\omega_0(k_\pm)|V-v_g^{\pm}|} \times \begin{bmatrix} -(k_\pm^2+I_3-I_2)^{1/2}\cos\Phi_\pm \\ (k_\pm^2+I_3-I_2)^{1/2}\sin\Phi_\pm \end{bmatrix}, \quad (12.52)$$

where

$$D_\Omega = -\frac{\pi h \Omega V}{4\xi(\Omega^2+\omega_0^2)^{1/2}},$$

$$\delta_\Omega = \tan^{-1}(\Omega/\omega_0),$$

$$\Phi_\pm = \pm[\omega_0(k_\pm)t - k_\pm x + \delta_\Omega] - \arg(ik_\pm - \xi) + \arg[-\cos\varphi(k_\pm^2+I_3-I_1)^{1/2} + i\sin\varphi(k_\pm^2+I_3-I_2)],$$

and $k_\pm$ are defined by

$$k_\pm V + \Omega - \omega_0(k_\pm) = 0 .$$

The emission power $W(\Omega) = dE_{\rm em}/dt$ can be calculated with the help of Eq. (12.21):

$$W(\Omega) \approx 2D_\Omega^2 \sum_\pm |V-v_g^{\pm}|^{-1}\operatorname{sech}^2\frac{\pi k_\pm}{2\xi} \quad (12.53)$$

[cf. Eq. (6.43) obtained by Malomed (1987d) for an analogous problem formulated in terms of the SG equation]. The influence of radiative energy losses on the experimentally measurable dependence $V(h_0)$ (the domain-wall mobility) is also discussed by Kivshar (1989).

The case of $h_0=0$ (a pure ac field) required special consideration. In this case the result is zero in the lowest approximation. To calculate the emission power, one needs to take into account small oscillations of the domain wall under the action of the ac field. Kivshar (1989) has obtained an expression for the emission power which is of the fourth order in ϵ instead of the common second order [see Eq. (6.33) for a SG kink oscillating under the action of a pure ac field (1.21)].

F. Inelastic interaction of domain walls

The interaction of domain walls of opposite polarities may be inelastic (see the analogous problem for SG kinks formulated in Sec. IV.A.2). This process is characterized by the maximum (threshold) value $h_{\rm thr}$ of the external field h allowing the fusion of two domain walls into a magnetic soliton. A natural way to find $h_{\rm thr}$ is to employ the energy balance. Indeed, the LL equation (12.4) gives rise to the energy equation

$$\frac{dE}{dt} = \epsilon\int_{-\infty}^{+\infty} dx\{\mathbf{R}(\mathbf{S})\cdot[\mathbf{S}\times\mathbf{S}_t]\}, \quad (12.54)$$

where E is the magnetic energy,

$$E = \tfrac{1}{2}\int_{-\infty}^{+\infty} dx[\mathbf{S}_x^2 + (I_3-I_1)S_1^2 + (I_3-I_2)S_2^2] .$$

The expression for the dissipative energy loss $E_{\rm diss}$ can be obtained directly from Eq. (12.54) with regard to the form of the dissipative perturbation in Eq. (12.46):

$$E_{\rm diss} = \gamma\int_{-\infty}^{+\infty} dt\int_{-\infty}^{+\infty} dx[\mathbf{S}\times\mathbf{S}_t]^2 . \quad (12.55)$$

Inserting an unperturbed expression that describes the pair of domain walls with opposite polarities into Eq. (12.55) [see, e.g., Kosevich *et al.* (1983)], one can find (Kivshar and Kosevich, 1986)

$$E_{\rm diss} = \frac{2\pi\gamma\xi}{\sqrt{1+\xi}} E\left[\frac{1}{\sqrt{1+\xi}}\right], \quad (12.56)$$

where $E(z)$ is a complete elliptic integral of the second kind. The threshold magnetic field can be calculated by equating Eq. (12.55) to the kinetic energy $E_{\rm kin}$ of the pair, which is

$$E_{\rm kin} \approx 2V^2/\xi \simeq 2h^2/\xi\gamma^2$$

(it is assumed that h/λ is small). The final result is (Kivshar and Kosevich, 1986)

$$h_{\rm thr} = \gamma\xi\left[\frac{2\pi\gamma}{\sqrt{1+\xi}}E\left[\frac{1}{\sqrt{1+\xi}}\right]\right]^{1/2}. \quad (12.57)$$

G. Perturbation-induced damping of a magnetic soliton

As has been demonstrated above, a pair of domain walls with opposite polarities may annihilate into a magnetic soliton under the action of dissipative losses. It is clear that a magnetic soliton undergoes dissipative damping. General formulas describing magnetic solitons in biaxial ferromagnets are rather cumbersome (see Kosevich *et al.*, 1983). That is why we shall present results only for a small-amplitude magnetic soliton. A small-amplitude quiescent magnetic soliton can be represented in the following form:

$$\mathbf{S}=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)\ ,$$

where

$$\tan^2\frac{\theta}{2}=\frac{\beta^2}{1+\zeta}[1+\zeta\sin^2(\omega_0 t+\psi_0)]\operatorname{sech}^2(\beta x)\ , \quad (12.58)$$

$$\beta^2=\Delta\sqrt{1+\zeta}/(1+\zeta/2)\ , \quad (12.59)$$

$$\tan\varphi=\sqrt{1+\zeta}\tan(\omega_0 t+\psi_0)\ . \quad (12.60)$$

The small parameter $\Delta\equiv\omega_0-\omega$ is proportional to the soliton amplitude and also characterizes its width $\sim\beta^{-1}\sim\Delta^{-1/2}$. In this approximation, the magnetic-soliton frequency $\omega_0\equiv\sqrt{1+\zeta}$ coincides with the limiting frequency of spin waves, which have the dispersion law

$$\omega^2=(1+k^2)(1+\zeta+k^2)\ .$$

Straightforward calculations using Eq. (12.54) yield the law of dissipative damping for the amplitude of a magnetic soliton:

$$\Delta=\Delta(0)\exp[-4\gamma\omega_0(1+\zeta/2)t]\ . \quad (12.61)$$

This result was obtained by Bar'yakhtar (1985) (see also Bar'yakhtar *et al.*, 1986) for the case of a uniaxial ferromagnet ($\zeta=0$).

Dissipative damping of a moving magnetic soliton was studied by Bar'yakhtar *et al.* (1986) for $\zeta=0$. The main difference between this problem and the analogous one for a SG breather is that the magnetic soliton may be accelerated by a dissipative force. This is due to the nonmonotonic dependence $V(E)$ for the biaxial ferromagnet.

The parameter γ in Secs. XII.D and XII.F described the relaxation term in the Gilbert form. As was pointed out by Bar'yakhtar (1984), generalized phenomenological equations for the magnetization field should include relaxation terms generated by both relativistic and nonuniform-exchange interactions. The latter interaction gives rise to the dissipative constant $\gamma_e a^2$, where a is the lattice spacing. In certain cases, a full system of dynamical equations for the magnetization vector $\mathbf{M}$, allowing for variation of M_0^2, reduces to the LL equation (12.4) for the unit-length vector $\mathbf{S}$ with perturbation terms (12.45) and

$$\epsilon\mathbf{R}(\mathbf{S})=\gamma_e a^2[\mathbf{S}\times[\mathbf{S}\times(\mathbf{S}_{xx}-\zeta S_1\mathbf{e}_1+S_3\mathbf{e}_3)_{xx}]]\ , \quad (12.62)$$

where $\mathbf{e}_1$ and $\mathbf{e}_3$ are unit vectors directed along the anisotropy axes.

The dynamics of domain walls and magnetic solitons in the presence of additional exchange dissipation (12.62) have been investigated by Bar'yakhtar (1985), Bar'yakhtar, Ivanov, and Sukstanskii (1986), Bar'yakhtar *et al.* (1986), and Kivshar and Soboleva (1988). In particular, the maximum value of the external magnetic field admitting steady motion of a domain wall is [cf. Eq. (12.50a)]

$$h_c=\frac{\zeta}{2}\left[\gamma+\frac{\gamma_e a^2}{3}\left[1+\frac{\zeta}{2}\right]\right]\ .$$

In the presence of conservative perturbations, e.g., weak inhomogeneities, a magnetic soliton will be damped by radiative losses. According to Eq. (12.21), the spectral density of the emission power may be represented in the form [cf Eq. (6.5) for a perturbed SG equation]

$$\mathcal{W}(\lambda)=\frac{d\mathcal{E}(\lambda)}{dt}$$
$$\equiv\frac{4}{\pi\rho}w_1(\lambda)w_2(\lambda)\mathrm{Re}\left[B(\lambda,t)^*\frac{dB(\lambda,t)}{dt}\right]\ ,$$

where

$$B(\lambda,t)\equiv b(\lambda,t)\exp[4i\kappa w_1(\lambda)w_2(\lambda)t]\ .$$

The spectral parameter λ $(0\le\lambda\le 2K)$ is related to the spin-wave wave number, $k=w_3(\lambda)$.

Let us consider the radiative damping of a magnetic soliton pinned by a local magnetic impurity corresponding to the perturbation $\epsilon\mathbf{R}(\mathbf{S})=\delta(x)[\mathbf{S}\times\hat{i}\mathbf{S}]$, where the matrix $\hat{i}=\mathrm{diag}(i_1,i_2,i_3)$ describes a local perturbation of the magnetic anisotropy. Straightforward calculations based upon the general ideas outlined in Sec. VI yield the following law of radiative damping for a small-amplitude magnetic soliton (Kivshar, 1989):

$$\Delta(t)=\Delta(0)[1+\tfrac{3}{2}\Delta^{2/3}(0)Ct]^{-3/2}\ ,$$

where Δ is the same as in Eq. (12.61) and

$$C\equiv(i_2-i_1)^2\zeta^2/2\pi(1+\zeta)(1+\zeta/2)\ .$$

Some other problems in the perturbed dynamics of magnetic solitons have also been considered by Kivshar (1989).

XIII. CONCLUSION

To conclude this presentation of the results of IST perturbation theory, it is fitting to discuss feasible directions that further development of the theory may take. First of all, one many consider one-dimensional systems described by perturbed equations close to exactly integrable models that are distinct from the KdV, modified KdV, NS, SG, and LL equations. For instance, the important phenomenon of self-induced transparency in nonlinear optics is described by an exactly integrable Maxwell-

Bloch system, which degenerates into the SG equation in a certain particular case (McCall and Hahn, 1969; Lamb, 1971). A perturbation theory for this system has been elaborated by Kaup (1977b). Perturbative treatment was also developed by Ablowitz and Kodama (1979), who have demonstrated the instability of a quasi-one-dimensional kinklike soliton of this system against transverse long-wave perturbations.

Another important exactly integrable system which finds applications in plasma physics and nonlinear optics is the coupled system of dispersionless equations describing the interaction of N ($N \geq 3$) simple waves. A perturbation theory for it based upon the IST technique was worked out by Kaup (1986) for the case $N=3$. As a particular physical problem, he considered the influence of a relaxational (dissipative) term. The same problem was solved by Drühl, Carlsten, and Wentzel (1985) within the framework of a simpler approach based on the energy balance equation (modified energy conservation, in terms of Sec. I). An interesting version of a perturbed three-wave system was derived by Verheest (1987) to describe the simultaneous generation of third and second harmonics in nonlinear optics. Another nonintegrable perturbation of the N-wave system was studied numerically by Bol'shov *et al.* (1985).

One more example is the so-called derivative NS equation

$$u_t+(|u|^2u)_x+iu_{xx}=\epsilon P(u) ,$$

which has applications in plasma physics. Wyller and Mjølhus (1984) developed an adiabatic variant of IST perturbation theory for this equation and considered dissipative damping of one soliton.

The exactly integrable Boussinesq equation

$$u_{tt}=u_{xx}+\tfrac{1}{3}u_{xxxx}-\tfrac{4}{3}(u^2)_{xx}$$

occurs in hydrodynamics and plasma physics (see, for example, Whitham, 1974), one-dimensional solid-state physics (Flytzanis, Pnevmatikos, and Remoissenet, 1985), models of long molecules DNA (Scott, 1985); nonlinear elasticity (Soerensen *et al.*, 1984), and so on. The usual KdV equation can be derived from the Boussinesq equation in an approximation that takes account of waves traveling in one direction only. A perturbation theory for this equation was developed by Grimshaw (1970) and Scott (1985).

The exactly integrable Benjamin-Ono equation,

$$u_t+2uu_x+\pi^{-1}\int_{-\infty}^{+\infty}u_{xx}(x')(x'-x)^{-1}dx'=0 ,$$

plays an important role in the hydrodynamics of a deep stratified liquid (Benjamin, 1967; Ono, 1975). An important distinctive feature of this equation is the fact that its one-soliton solution,

$$u_{\text{sol}}=2A\{1+[A(x-At)]^2\}^{-1} , \tag{13.1}$$

where A is an arbitrary soliton amplitude, is of an algebraic type, in contrast with the exponential KdV soliton. A perturbation theory for a soliton described by the Benjamin-Ono equation with variable coefficients [analogous to the variable-coefficient KdV and NS equations (3.8) and (3.40)] has been worked out by Grimshaw (1980). In the same work, damping of a soliton under the action of friction or radiation losses (the latter being caused by a special perturbation) was also analyzed. An evolution equation for the amplitude A of the Benjamin-Ono soliton (13.1) damped by various dissipative terms has been derived by Ostrovskii, Stepanyants, and Tsimring (1984b). More general rational solutions to a perturbed Benjamin-Ono equation have been studied by Birnir and Morrison (1985).

With regard to the system of coupled KdV equations (1.3) and (1.4), it is relevant to mention the papers of Liu, Kubota, and Ko (1980) and Liu, Pereira, and Ko (1982). They studied leapfrogging dynamics of solitons in a system of two coupled Joseph (intermediate-depth) equations describing resonant interaction of two internal waves in a finite-depth stratified liquid (the Joseph equation is exactly integrable, and it comprises, as limiting particular cases, the KdV and Benjamin-Ono equations).

An important example of an exactly integrable discrete system is the Toda lattice, which consists of particles interacting through exponential pairwise potentials. Perturbation theory for this system has been developed by many authors. A general formalism of IST perturbation theory has been elaborated by Yajima (1979), who has also considered, as a particular example, the scattering of a soliton by a mass impurity in the lattice. Reflection of a soliton from an impurity was considered by Klinker and Lauterborn (1983). In a series of papers, Kaup (1984, 1987), Kaup and Neuberger (1984), and Kaup and Hansen (1987) developed a semi-analytical approach to a forced Toda lattice that has much in common with IST perturbation theory. As another example of the perturbation theory for nearly integrable discrete systems, we can mention the paper of Vakhnenko and Gaididei (1986). With the aid of the Lagrangian formalism, they studied the evolution of a Davydov soliton in a molecular chain described by equations close to the exactly integrable discrete Ablowitz-Ladik model (Ablowitz and Ladik, 1975, 1976).

As is well known, exact methods that are akin to the IST have been developed for various integrable equations with periodic boundary conditions (see, for example, Zakharov *et al.*, 1980). Some work towards constructing a perturbation theory for these systems has also been done. In particular, Krichever (1983, 1988) has elaborated a general mathematical basis of perturbation theory for the KdV equation with periodic boundary conditions [see also Maslov and Dobrokhotov (1982), who studied the perturbation-induced evolution of parameters of a finite-band solution to the KdV equation with the aid of Whitham's averaging method]. Geist and Lauterborn (1986) investigated numerically the decay of a soliton in a periodic Toda lattice containing a heavy-mass impurity.

Another feasible way of extending IST perturbation theory is to apply it to two-dimensional integrable systems. Two well-known two-dimensional equations amenable to solution by the IST are the Kadomtsev-Petviashvili (KP) equations,

$$(u_t-6uu_x-u_{xxx})_x=\pm 3u_{yy} .$$

They are as general in the two-dimensional case as the KdV equation is in one dimension (see, for example, Zakharov *et al.*, 1980). The KP equation with the upper sign on its right-hand side has stable quasi-one-dimensional solitons of the KdV type (see also the end of Sec. XI.F). The equation with the lower sign has two-dimensional algebraic solitons (so-called lumps), while its quasi-one-dimensional solitons are unstable. For the latter equation, an IST perturbation theory was developed by Benilov and Burtsev (1986). As an application, they have considered dissipation-induced damping of a lump (of course, this particular problem can be solved by means of simpler methods).

It would be interesting to develop an analog of IST perturbation theory for exactly integrable quantum systems. It is well known that the quantum NS and SG models are exactly integrable (see, for example, Rajaraman, 1982). In Sec. X we have given an account of perturbation theory for the semiclassical SG system. There are few papers in which analysis of perturbations goes beyond the framework of the semiclassical approximation. de Martino, de Siena, and Sodano (1985) have demonstrated that, in certain cases, a fully quantum model based upon the double SG equation (1.21) (with $f\equiv 1$) is exactly equivalent to the unperturbed SG quantum model. Yamamoto (1986) considered effects produced by a local perturbation of the type (1.19) in the quantum SG system at a special value of the coupling constant (when the unperturbed system is equivalent to a free Fermion gas). Konishi and Wadati (1986) considered perturbation-induced decay of a soliton in a fully quantum NS system. Finally, it is pertinent to mention the paper of Robnik (1986), who considered, in a general form, the problem of eliminating a small perturbation in a nearly integrable quantum system (with a purely discrete spectrum) by means of canonical transformations. He has demonstrated that this is possible in the first order of perturbation theory.

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APPENDIX A: ONE-SOLITON SCATTERING DATA AND JOST FUNCTIONS

KdV equation. The scattering data corresponding to the one-soliton solution (2.18) are

$$\begin{aligned}&a(k)=(k-k_1)(k+k_1)^{-1} ,\\ &b(k,t)=0 \quad (k \text{ is real}) ,\\ &k_1=i\kappa,\quad b(k_1,t)=\exp(2\kappa\zeta) ,\end{aligned} \tag{A1}$$

and the corresponding Jost functions are

$$\begin{aligned}&\Phi^{(1)}(x,k)=(k+i\kappa)^{-1}(k-i\kappa\tanh z)\exp(-ikx) ,\\ &\Psi^{(2)}(x,k)=(k+i\kappa)^{-1}(k-i\kappa\tanh z)\exp(ikx) ,\end{aligned} \tag{A2}$$

where ζ and z are the same as in Eq. (2.19).

NS equation. The scattering data corresponding to the one-soliton solution (2.41) are

$$\begin{aligned}&a(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_1^*)^{-1} ,\\ &b(\lambda,t)=0 \quad (\lambda \text{ is real}) ,\\ &\lambda_1=\xi+i\eta,\quad b(\lambda_1)=\exp(2\eta\zeta+i\phi) ,\end{aligned} \tag{A3}$$

and the corresponding Jost functions are

$$\begin{aligned}&\psi(x,\lambda)=(\lambda-\xi+i\eta)^{-1}\\ &\qquad\times\exp(i\lambda x)\begin{bmatrix}\lambda-\xi+i\eta\tanh z\\ i\eta\exp(-2i\xi x-i\phi)\operatorname{sech} z\end{bmatrix} ,\\ &\varphi(x,\lambda)=(\lambda-\xi+i\eta)^{-1}\\ &\qquad\times\exp(-i\lambda x)\begin{bmatrix}i\eta\exp(2i\xi x+i\phi)\operatorname{sech} z\\ \lambda-i\xi-i\eta\tanh z\end{bmatrix} .\end{aligned} \tag{A4}$$

In Eqs. (A3) and (A4), ϕ and ζ are the same as in Eq. (2.42), and $z\equiv 2\eta(x-\zeta)$.

An important generalization of Eqs. (A3) and (A4) has been obtained by Satsuma and Yajima (1974). They have taken the initial condition

$$u_0(x)=ia\operatorname{sech}(bx)\exp(ikx) ,$$

which is more general than that corresponding to the soliton (2.41), and (2.42) at $t=0$ (the parameters a, b, and k are independent). The scattering data corresponding to this generalized wave-field configuration are

$$\begin{aligned}&a(\lambda)=\Gamma^2(q)/[\Gamma(q-N)\Gamma(q+N)] ,\\ &b(\lambda)=iqN^{-1}\Gamma(q)\Gamma(-q)/[\Gamma(N)\Gamma(-N)] ,\\ &q\equiv\tfrac{1}{2}+(i/2b)(k-2\lambda),\quad N=a/b ,\end{aligned}$$

where $\Gamma(q)$ is the gamma function. The zeros of the function $a(\lambda)$ lie at the points $q-N=n$ ($n=0,\pm1,\pm2,\ldots$). The number of zeros, i.e., the number of solitons contained in the initial wave-field configuration, is $[N+\frac{1}{2}]$, where the brackets indicate the integer part.

SG equation. The scattering data for the one-kink solution (2.61) are

$$a(\lambda)=(\lambda-\lambda_1)(\lambda+\lambda_1)^{-1}\,,$$
$$b(\lambda,t)=0\quad(\lambda \text{ is real})\,, \tag{A5}$$
$$\lambda_1=i\nu,\quad b(\lambda_1)=i\sigma\exp[\zeta(1-V^2)^{-1/2}]\,,$$

and the corresponding Jost functions are

$$\psi(x,\lambda)=(\lambda+i\nu)^{-1}\exp[\tfrac{1}{2}ik(\lambda)x]\begin{bmatrix}\lambda+i\nu\tanh z\\ \sigma\nu\,\mathrm{sech} z\end{bmatrix}, \tag{A6}$$

$$\varphi(x,\lambda)=(\lambda+i\nu)^{-1}\exp[-\tfrac{1}{2}ik(\lambda)x]\begin{bmatrix}-\sigma\nu\,\mathrm{sech} z\\ \lambda-i\nu\tanh z\end{bmatrix}. \tag{A7}$$

Here $k(\lambda)\equiv\lambda-1/4\lambda$, ζ is the same as in Eq. (2.61b), and $z\equiv(x-\zeta)(1-V^2)^{-1/2}$.

APPENDIX B: SCATTERING DATA AND JOST FUNCTIONS FOR A SINE-GORDON BREATHER

For the breather solution (2.63) and (2.64),

$$a(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)/(\lambda-\lambda_1^*)(\lambda-\lambda_2^*)\,, \tag{B1}$$

where $\lambda_{1,2}$ are defined in Eq. (2.62). The following expressions for the quantities $b_1\equiv b(\lambda_1)$, $b_2=b(\lambda_2)=-b^*(\lambda_1)$, and for the corresponding Jost functions are presented, for example, by Karpman, Maslov, and Solov'ev (1983) (see also Kosevich and Kivshar, 1982):

$$b_1=2i\lambda_1\tanh\mu\, a'(\lambda_1)\exp[it\cos\mu\,(1-V^2)^{-1/2}-i(\lambda_1-1/4\lambda_1)\xi_0]\,, \tag{B2}$$

$$\psi^{(1)}(x,\lambda)=\sin(2\mu)|\lambda_1|^3\{i|\lambda_1|\cosh(z\sin\mu)\sin(\Psi\cos\mu)$$
$$+\lambda[\cos\mu\cosh(z\sin\mu)\cos(\Psi\cos\mu)+\sin\mu\sinh(z\sin\mu)\sin(\Psi\cos\mu)]\}/\Delta\,, \tag{B3}$$

$$\psi^{(2)}(x,\lambda)=|\lambda_1|^2\{\cos^2\mu\,(\lambda^2-|\lambda_1|^2)\cosh^2(z\sin\mu)+\sin^2\mu\,(\lambda^2+|\lambda_1|^2)\sin^2(\Psi\cos\mu)$$
$$+\tfrac{1}{2}i\lambda\sin(2\mu)[\cos\mu\sinh(2z\sin\mu)-\sin\mu\cos(2\Psi\cos\mu)]\}/\Delta\,, \tag{B4}$$

where

$$\Delta\equiv(\lambda+\lambda_1)(\lambda-\lambda_1^*)[\cos^2\mu\cosh^2(z\sin\mu)+\sin^2\mu\sin^2(\Psi\cos\mu)]\exp[-i(\lambda-1/4\lambda)x/2]\,,$$
$$|\lambda_1|=\tfrac{1}{2}[(1+V)/(1-V)]^{1/2}$$

[see Eq. (2.62)], and where z and Ψ are the same as in Eq. (2.64).

APPENDIX C: FUNCTIONS $F_{1,2}$ AND $G_{1,2}$ FOR EQ. (4.74)

The functions $F_{1,2}$ and $G_{1,2}$ that enter Eq. (4.74) are

$$F_1(T_0)=32\int_{-\infty}^{+\infty}dz\,(T_0+z)^2\cosh^2z[\cosh^2z-(T_0+z)^2]^2[\cosh^2z+(T_0+z)^2]^{-4}\,, \tag{C1}$$

$$F_2(T_0)=T_0\int_{-\infty}^{+\infty}dz\,\cosh z[\cosh^4z+(T_0+z)^4-6(T_0+z)^2\cosh^2z][\cosh^2z-(T_0+z)^2][\cosh^2z+(T_0+z)^2]^{-4}\,, \tag{C2}$$

$$G_1(T_0)=T_0\int_{-\infty}^{+\infty}dz\,\cosh^2z[\cosh^2z-(T_0+z)^2][\cosh^4z+(T_0+z)^4-6(T_0+z)^2\cosh^2z][\cosh^2z+(T_0+z)^2]^{-5}\,, \tag{C3}$$

$$G_2(T_0)=\int_{-\infty}^{+\infty}dz\,\cosh z[\cosh^8z+(T_0+z)^8+70(T_0+z)^4\cosh^4z-28(T_0+z)^2\cosh^6z$$
$$-28(T_0+z)^6\cosh^2z][\cosh^2z+(T_0+z)^2]^{-5}\,. \tag{C4}$$

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