M 639 Nonlinear Waves I (Fa 08, T+Th 11-12:15, B245), Sp'12 T+Th 12:30-1:45 EBA 34

passed : mfcif8

~~Lab GMCS-422 ma-student/FA2008~~

Sp'14 new 4-5:15 EGMCS-32

Syllabus :

* Linear Waves

- superposition (RCG p3)
- dissipation (RCG) p5)
- dispersion (RCG) p6)
- Second order linear PDEs & method of charact. (LD p 11-22)
- Waves on a string (RCG) p1-3
- Separation of variable → Read LD p 22-35)
- Read LD chap 1.1 - intro
  - 1.2 - Basic concepts & Defs
  - 1.3 - linear superposition
  - 1.4 - important models

L. Debnath [Nonl. PDEs]

HW # 1 :

- Read LD chaps 1.1 + 1.2 + 1.3 + 1.4 + 1.6
- Read Fourier - Boas
- Do Matlab

Week # 1 : Tu +Th :

- Intro to course
- Waves on a string (RCG p1-3)
- linear wave eq. (RCG p 3-4)
- Dispersion & dissipation (RCG p 4-7)
- Nonlinearity (RCG p-8-9)
- Second order linear PDEs (Classification + characteristics) (LD p11-22)

Syllabus (NEW) :

* Linear Waves

- wave on string, wave eq
- superposition
- dissipation
- dispersion
- PDE coord. transf.
- 2nd order → characteristics

* Nonlinear

- 1st order → Meth. characteristics
- wave breaking

* Water waves

- Euler eqns
- Boussinesq + KdV

* KdV

- elementary sols
- oscill. sols
- rational, breathing sols
- const. of motion
- Bäcklund transf.

* NLS

- envelope wave
- solitons
- gral. DSs
- Modulational stab
- BECs
  - Avoiding IR catastroph
- conservation laws

- Variational approx (VA)
- Perturb. theo. & VA
- soliton-soliton interactions
- perturb. theo. for DSs
- Ring DSs.
- transverse inst DSs
- Vortices
- Numerical techniques
  - Newton
  - Stability

Waves on a string

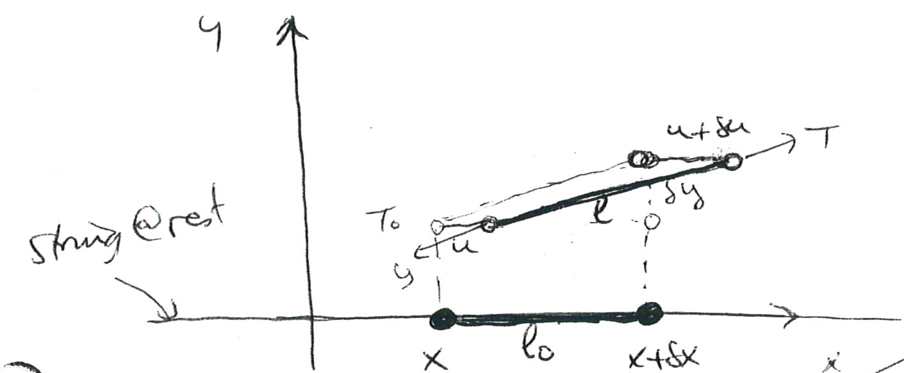
WAVES

["wave theory & applications" D.R. Bland]

12 Nov '05

This is going to be the basis for studying wave propagation. We will derive from first principles (Newton + Hooke's law) the wave eq. on a string and use it to build more general models.

Suppose we have a piece of elastic string and we want to model its deformations:



compression/expansion per unit length.

Let us apply Newton's law for the piece of string.

Def: strain $\boxed{\varepsilon = \lim_{\delta x \to 0} \frac{\Delta \ell}{\ell_0}} = \lim_{\delta x \to 0} \frac{\ell - \ell_0}{\ell_0} = \lim_{\delta x \to 0} \ell/\ell_0 - 1$

but $\ell = \sqrt{(\delta x + \delta u)^2 + (\delta y)^2}$ & $\ell_0 = \delta x$

$$\Rightarrow \lim_{\delta x \to 0} \frac{\ell}{\ell_0} = \lim_{\delta x \to 0} \sqrt{(1+\frac{\delta u}{\delta x})^2 + (\frac{\delta y}{\delta x})^2} = \sqrt{(1+\frac{\partial u}{\partial x})^2 + (\frac{\partial y}{\partial x})^2}$$

$$\therefore \boxed{\varepsilon = \sqrt{(1+\frac{\partial u}{\partial x})^2 + (\frac{\partial y}{\partial x})^2} - 1}$$ compression/expansion per unit length.

Longitudinal motion: $(y=0)$

Newton: $ma = F$

but $m = \rho \delta x$ (ρ: density)

$a = \frac{\partial^2 u}{\partial t^2}$

$F = \delta T = T - T_0$

$$m\,a = F$$

$$\Rightarrow \rho\, \delta x \frac{\partial^2 u}{\partial t^2} = \delta T$$

$$\Rightarrow \rho \frac{\partial^2 u}{\partial t^2} = \frac{\delta T}{\delta x} \simeq \frac{\partial T}{\partial x}$$

?

Hooke's law: $T - T_0 \propto \varepsilon$ Young's modulus

if fact $\frac{T-T_0}{A} = E\,\varepsilon$

A: transverse area

~~$\frac{\partial T}{\partial x} \propto \frac{\delta \varepsilon}{\delta x} \Rightarrow \frac{\partial T}{\partial x} = \frac{\partial \varepsilon}{\partial x}$~~

~~Just for $y=0$ $\Rightarrow$ $\varepsilon = \frac{\partial u}{\partial x}$~~

~~$ma = F \Rightarrow \rho\, \delta x \frac{\partial^2 u}{\partial t^2} \propto \delta \varepsilon$~~

~~$\Rightarrow \rho \frac{\partial^2 u}{\partial t^2} \propto \frac{\delta \varepsilon}{\delta x} = \frac{\partial \varepsilon}{\partial x} = \frac{\partial u}{\partial x}$~~

$$\frac{\partial T}{\partial x} = \frac{\partial T - T_0}{\partial x} = EA \frac{\partial \varepsilon}{\partial x}$$

$$\therefore \rho \frac{\partial^2 u}{\partial t^2} = EA \frac{\partial \varepsilon}{\partial x} = EA \frac{\partial}{\partial x}\Big[\overset{\varepsilon \text{ with } y=0}{\frac{\partial u}{\partial x}} \Big]$$

$$\Rightarrow \rho \frac{\partial^2 u}{\partial t^2} = EA \frac{\partial^2 u}{\partial x^2}$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial t^2} = c_e^2 \frac{\partial^2 u}{\partial x^2}} \quad \text{where} \quad c_e = \sqrt{\frac{EA}{\rho}} > 0$$

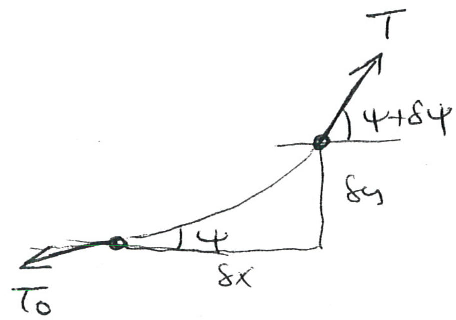
Longitudinal wave eq.

E : Young's modulus
ρ : density
A : ~~Cross~~-sect Area.
$[E] = kg/m^3$

Transverse motion: $(u=0)$

Newton: $ma = F$

$$\rho\, \delta x \frac{\partial^2 y}{\partial t^2} = (\delta T)_y$$



$$\Rightarrow (\delta T)_y = T \sin(\psi + \delta\psi) - T_0 \sin\psi$$

assumptions: ψ small $\Rightarrow$ $\sin\psi \sim \psi$ & $\tan\psi \sim \psi$

string does not move $\Leftrightarrow$ $\Rightarrow$ $|T| \simeq |T_0|$ $\Rightarrow$ $(\delta T)_y \approx$ ~~$T_0 \delta\psi$~~ $T(\psi + \delta\psi) - T_0\psi \overset{|T_0| \approx |T|}{\approx} T_0 \delta\psi$

on the other hand: $\tan\psi = \frac{\delta y}{\delta x} \Rightarrow$ ~~$\tan$~~ $\psi \simeq \psi \simeq \frac{\partial y}{\partial x}$

$$\therefore \rho\, \delta x \frac{\partial^2 y}{\partial t^2} = T_0\, \delta\psi \rightarrow \rho \frac{\partial^2 y}{\partial t^2} = T_0 \frac{\delta\psi}{\delta x} \overset{\lim_{\delta x \to 0}}{=} T_0 \frac{\partial \psi}{\partial x} \overset{\psi = \frac{\partial y}{\partial x}}{=} T_0 \frac{\partial^2 y}{\partial x^2}$$

$$\therefore \boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}} \quad \text{with} \quad c = \sqrt{\frac{T_0 \leftarrow \text{tension}}{\rho \leftarrow \text{density}}}$$

Transverse wave eq.

Transverse & longitudinal equations for waves are the same but with DIFFERENT velocities.

These eqs. where obtained by taking approximations. ~~and Taylor expansions that~~

⚠ because of approximations we obtained LINEAR equations.

Linear: if $A(x,t)$ and $B(x,t)$ are solutions then $\alpha A(x,t)+\beta B(x,t)$ is also a solution: || Superposition Principle!

cf: $\frac{\partial^2 A}{\partial t^2}=c^2\frac{\partial^2 A}{\partial x^2}$ & $\frac{\partial^2 B}{\partial t^2}=c^2\frac{\partial^2 B}{\partial x^2}$

$\Rightarrow \frac{\partial^2(\alpha A+\beta B)}{\partial t^2}=c^2\frac{\partial^2(\alpha A+\beta B)}{\partial x^2} \rightarrow \alpha A+\beta B$ solution.

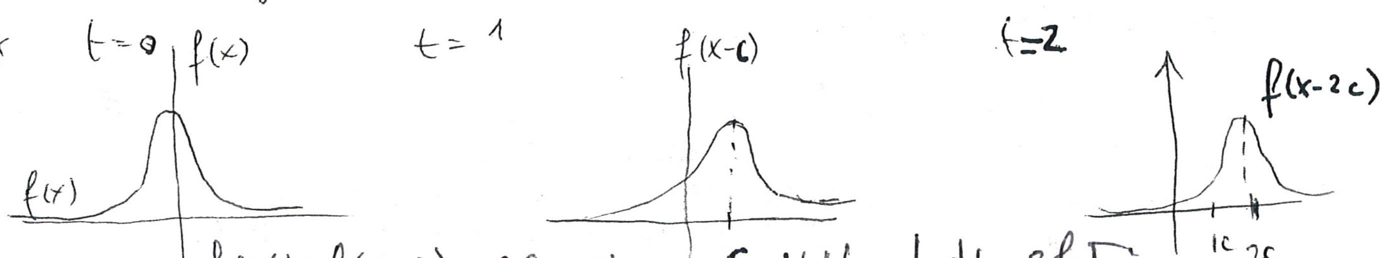
LINEAR WAVE EQUATION

$$\boxed{\frac{\partial^2 u(x,t)}{\partial t^2}=c^2\frac{\partial^2 u(x,t)}{\partial x^2}} \quad (1)$$

- solution $u(x,t)=f(x-ct)$
  $\Rightarrow u_{tt}=c^2 f''$, $u_{xx}=f''$ } $u_{tt}=c^2 u_{xx}$ ✓
- same for $u(x,t)=f(x+ct)$

Let us try a solution that is the combination of one shape travelling to the right $f(x-ct)$ and a shape travelling to the left: $g(x+ct)$

Ex



$h(x,t)=f(x-ct)+g(x+ct)$ ← d'Alembert's solution.

Let us see if ~~$h=f+g$~~ is a solution to (1): ↳ d'Alembertian = $\Box$: $\Box u \equiv \left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}\right)u$

$$\frac{\partial^2 f(x-ct)}{\partial t^2}+\frac{\partial^2 g(x+ct)}{\partial t^2}\overset{?}{=}c^2\frac{\partial^2 f(x-ct)}{\partial x^2}+c^2\frac{\partial^2 g(x+ct)}{\partial x^2}$$

$$c^2 f''+c^2 g''\overset{?}{=}c^2 f''+c^2 g'' \quad ✓$$

$\therefore$ the linear wave eq. can propagate ANY shape at a const. speed in BOTH directions

Initial conditions for linear wave equation ($u_{tt} - c\,u_{xx} = 0$)

9/4/08

Suppose that we know ICs: $u(x,0) = \phi(x)$ & $u_t(x,0) = \psi(x)$

What is the evolution for this wave? (phi) (psi)

Remember d'Alambert's solution:

$$u(x,t) = f(x-ct) + g(x+ct) \quad (1)$$

ICs:
$$u(x,0) = f(x) + g(x) = \phi(x) \quad (a)$$
$$u_t(x,0) = -cf'(x) + cg'(x) = c\,(g'(x) - f'(x)) = \psi(x) \quad (b)$$

$\therefore$ We have to solve the above system for f & g.

$$\int (b) \Rightarrow \quad g(x) - f(x) = \frac{1}{c}\int_0^x \psi(y)\,dy + \underbrace{cte}_{=A} \quad (c)$$

$$(a) + (c) \Rightarrow \quad 2g(x) = \phi(x) + \frac{1}{c}\int_0^x \psi(y)\,dy + A$$

$$\Rightarrow \boxed{\begin{cases} g(x) = \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(y)\,dy + \frac{A}{2} \\ f(x) = \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(y)\,dy - \frac{A}{2} \end{cases}} \quad (a) - (c)$$

now let us use $\begin{pmatrix} x+ct \\ x-ct \end{pmatrix}$ instead of x and add both equations:

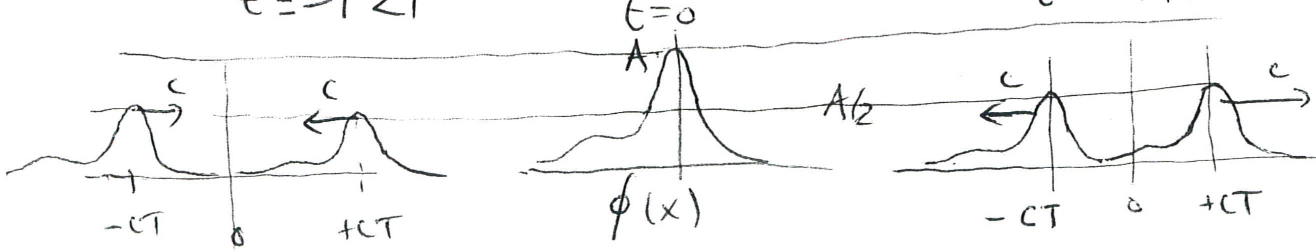
$$u(x,t) = f(x-ct) + g(x+ct) = \frac{1}{2}\phi(x+ct) - \frac{1}{2c}\int_0^{x-ct}\psi(y)\,dy + \frac{1}{2}\phi(x-ct) + \frac{1}{2c}\int_0^{x+ct}\psi(y)\,dy$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy}$$

How to get just to the an arbitrary wave going right
$u(x,t) = f(x-ct)$
$u(x,0) = f(x)$
$u_t(x,0) = -cf'(x)$
$\therefore \phi(x) = f(x)$
$\therefore \psi(x) = -cf'(x)$

Ex: $u(x,0) = \phi(x)$ & $u_t(x,0) = 0 = \psi(x)$

$$\Rightarrow \quad u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right)$$



⚠ If you have a shape at a particular instant: it is impossible to know what is thee future/past behavior:

is really … or … ? or … or anything + anything

~~∴ The linear wave equation can propagate ANY shape at constant speed in BOTH directions~~

unidirectional motion:

let us rewrite ①: $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

$$\Rightarrow \underbrace{\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right)}_{\square = \text{d'Alembertian}} u = 0 \qquad \left(\text{Laplacian } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u\right] = 0 \quad \text{or} \quad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u\right] = 0$$

∴ if $\begin{cases} \frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} & (a) \\ \text{or } \frac{\partial u}{\partial t} = +c\frac{\partial u}{\partial x} & (b) \end{cases} \Longrightarrow u$ is a solution of ①

So, instead of using ① we'll use (a) or (b):

Notation: $u_x \equiv \partial u/\partial x$, $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \partial^2 u/\partial x^2$, ...

(a): $u_t + c u_x = 0 \rightarrow f(x-ct) = 0$ ∴ $\overrightarrow{\sim\sim}$

(b): $u_t - c u_x = 0 \Rightarrow f(x+ct) = 0$ ∴ $\overleftarrow{\sim\sim}$

From now on, let us take the right-motion equation:

$$\boxed{u_t + c u_x = 0} \quad ②$$

wave eq. that propagates ANY shape to the right at cte velocity.

⚠ the shape is constant.

~~Jump to page 79 / 10 / 8 For the example of $u_t = c u_{xx}$ vs. $u_t + c u_x = 0$~~

oct 02

~~**DISPERSION & DISSIPATION**~~

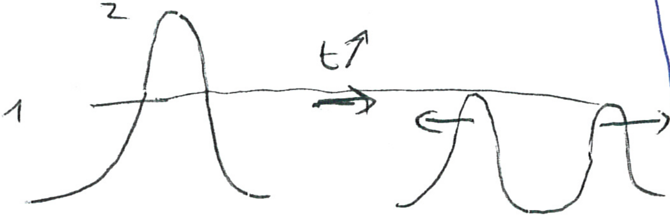
~~Let us try to include some terms into ② in order to have the simplest model for dispersion & dissipation.~~

~~• dispersion: periodic waves of ≠ periods travel at ≠ velocities~~

~~• dissipation: equivalent to damping (loss of energy) ≡ diffusion~~

Ex: $u_{tt} = c^2 u_{xx}$ $u(x,0) = 2e^{-x^2/2}$, $u_t(x,0) = 0$

Show matlab ugauss_utzero.m



$\frac{73}{3}$ (4⅓)

~~What if: $u(x,0) = 2e^{-x^2/2}$ and $u_t(x,0) = ?$~~

~~t↗ ?~~

~~→ See previous lon fits for wave eq.~~

Do this just after doing uni-directional motion p④

For ⚠ the UNIDIRECTIONAL wave eq. $u_t + c u_x = 0$ there is NO ambiguity and $u(x,0)$ suffices to determine future/past behavior.

Ex: $u_t + c u_x = 0$, $u(x,0) = 2e^{-x^2/2}$ $\Rightarrow$ $u(x,t) = 2e^{-(x-ct)^2/2}$ $\Rightarrow$ $u_t(x,0) = \frac{\partial}{\partial t}[u(x,t)]_{t=0}$

$\Rightarrow u_t(x,0) = [c(x-ct)2e^{\cdots}]_{t=0}$

$\underbrace{u_t(x,0) = 2cx\, e^{-(x-ct)^2/2}}_{\text{Not needed since eq. is 1st order}}$

~~⚠ With dispersion: $u_t + c u_x + \delta u_{xxx} = 0$ the wave may travel to the left!~~

~~SOLITONS: Nonlinear wave propagation.~~

~~Read extract of John Scott Russell "Report on waves" → Wikepedia~~

~~How is it possible to have a solitary wave (soliton) with such stability?~~

~~$\boxed{\text{Nonlinearity balances dispersion}}$~~

~~Application: fiber optics.~~

~~Show pictures KdV~~

~~Show wave-integrator with KdV solution~~

DISPERSION & DISSIPATION

let us try to include some terms into (2) in order to have the simplest model for dispersion & dissipation:

- dispersion: periodic waves of ≠ periods travel at ≠ velocities
- dissipation: equivalent to damping (loss of energy). ≡ diffusion

Dissipation (diffusion)

We still want the equation to be linear, so if we add something to the equation we'll like to keep it linear.

We already have u_t, u_x so let us add the next derivative in x u_{xx} (we do not add u_{tt} because we still want 1st order in time)

|w 8/30/06

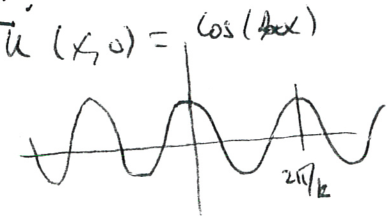
$$u_t + u_x \mp u_{xx} = 0 \qquad (3)$$

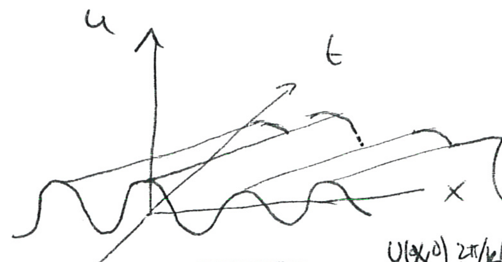
(put c=1 for convenience) Indeed by rescaling time or space we can get rid of c anyway)

Let us see what happens to a sinusoidal wave: $u(x,t) = A\cos(kx-\omega t) = A\cos(k(x-\omega/k\, t))$

No dissipation:

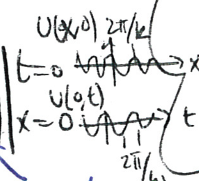
$u(x,0) = \cos(kx)$





Wave travelling at velocity $\boxed{v=\omega/k}$, $\lambda = \frac{2\pi}{k}$

$\boxed{\begin{array}{l} k : \text{wave \#} \\ \omega : \text{frequency} \end{array}}$ λ : wave length; angular



Matlab
~~x = -10:0.05:5~~ (2π, 2·Pi)
~~t = -10:0.05:5~~
~~[X,T] = meshgrid(x,t)~~
~~z = cos(X - 0.5T)~~
~~surfl(X,T,Z)~~
~~shading interp;~~
~~colormap(gray)~~
$\boxed{\text{wave\_eq.m}}$

Let us extend the sinusoidal solution by:

Def: plane wave: $u(x,t) = Ae^{i(kx-\omega t)}$ and if we want to obtain the physical solution we'll take the real part.

∴ plane wave in (3) $\Rightarrow$ $-i\omega Ae^{i(kx-\omega t)} + ikAe^{i(\,)} \pm (-k^2)Ae^{i(\,)} = 0$

Q: what is $\omega = \omega(k)$ for plane wave to be a solution?

$$\Rightarrow \quad -i\omega + ik \pm k^2 = 0$$

$$\xrightarrow{(\times i)} \quad \omega - k \pm ik^2 = 0$$

$$\Rightarrow \quad \boxed{\omega(k) = k \mp ik^2} \quad \leftarrow \text{dispersion relation}$$

Thus for plane wave to be a solution of (3) we need

$$\therefore \quad u(x,t) = Ae^{i(kx - kt \pm ik^2 t)}$$

$\Rightarrow \quad u(x,t) = \underbrace{A e^{ik(x-t)}}_{\text{spatial cosine term}} \underbrace{e^{\boxed{\mp} k^2 t}}_{\text{exponential decay/growth in time}}$

Matlab
$Z = \cos(x - 0.25T) .* \exp(-0.1T)$
$\boxed{\text{dissipation.m}}$

~~75~~
6

$\therefore$ we have a wave $[c = 1]$ $u(x,t) = A\cos k(x-t) e^{\mp k^2 t}$ that decays/growths exponentially.

$u_t + u_x - u_{xx} = 0$ decay
$u_t + u_x + u_{xx} = 0$ growth.

$\therefore$ Dissipation can be modeled by:

$$\boxed{u_t + c u_x - \gamma u_{xx} = 0}$$

c: speed.
γ: dissipation constant.
($\gamma > 0$ to have decay)

<u>Dispersion</u>: Let us include even higher order derivatives (still linear)
we do not include u_{xx} because we do not want dissipation here

Let us add u_{xxx}:

$$u_t + u_x \pm u_{xxx} = 0 \quad (4) \qquad (c = 1)$$

Let us use plane waves $u(x,t) = A e^{i(kx - \omega t)}$ into (4):

$$\Rightarrow \quad -i\omega + ik \pm (-ik^3) = 0$$

$$\times i \quad \Rightarrow \quad \omega - k \pm k^3 = 0$$

$$\Rightarrow \quad \omega(k) = k \mp k^3 \quad \leftarrow \text{dispersion relation.}$$

~~Let us take the first sign~~

$$\boxed{\omega(k) = k(1 \mp k^2)}$$

$$\Rightarrow \quad \text{sol}: \quad u(x,t) = A e^{ik(x - (1\mp k^2)t)} \overset{\text{physical}}{=} A\cos k(x - (1\mp k^2)t)$$

$\therefore$ we have a solution that travels at velocity $c = 1 \mp k^2$.

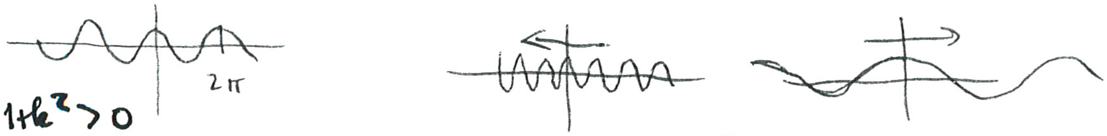
$\therefore$ Each plane wave will travel at $\neq$ velocities depending on its wave number: $\Rightarrow$ DISPERSION

$$\omega(k) = k(1 \mp k^2) \quad \leftarrow \text{dispersion relation.}$$

vel: $v = \omega / k = 1 \mp k^2$

(−) $k=1 \Rightarrow c=0$ $\quad k>1 \Rightarrow c<0$ $\quad 0<k<1 \Rightarrow c>0$

$V_k = 1-k^2$



(+) $V_k = 1+k^2 > 0$

Matlab

dispersion.m

11/9/05
9/9/08

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What happens to general wave shapes?

We recur to the superposition principle and Fourier series:

take $u(x, t=0)$ and decompose it into a Fourier series: $\underline{t=0}$

$$u(x,0) = \sum_k a_k e^{i(kx)} \quad \left[\text{phys.} \Rightarrow \underbrace{a_0}_{u_0} + \underbrace{a_1 \cos x}_{u_1} + \underbrace{a_2 \cos 2x}_{u_2} + \ldots \right]$$

$$\therefore \quad u(x,0) = u_0 + u_1 + u_2 + \ldots$$

$$= \sum_k u_k \quad \text{where } u_k \text{ are plane waves.}$$

Therefore we know $u_k(x,0) = a_k e^{i(kx)}$

$$\Rightarrow \quad u_k(x,t) = a_k e^{i(kx - \omega(k)t)}$$

$$= a_k e^{ik(x-(1\mp k^2)t)}$$

i.e. the k-th mode will travel with velocity $(1 \mp k^2) = \frac{\omega(k)}{k} = V_k$

By superposition:

$$u(x,t) = \sum_k u_k(x,t)$$

$$\boxed{u(x,t) = \sum_k a_k e^{i[kx - \omega(k) t]}}$$

← fourier coefficient of $u(x,0)$

$u(x,0) \dashrightarrow u(x,t)$
$\downarrow \qquad\qquad \uparrow$
Σ plane waves $\xrightarrow{t\uparrow}$ Σ p.w.

or by using the Fourier transform:

$$u(x,t) = \int_{-\infty}^{\infty} a(k)\, e^{i(kx - \omega(k)t)}$$

↑ fourier transform of $u(x,0)$

Matlab
disp_disper.m

$$\therefore \quad \boxed{u_t + c u_x + \delta u_{xxx} = 0}$$

Dispersive wave eq.

$\delta > 0$ or $\delta < 0$

~~Integrate eq. and discuss about~~

Dissipation + dispersion $\quad u_t + c u_x - \gamma u_{xx} + \delta u_{xxx} = 0$

~~Disp: $\omega(k) = k(1 \mp k$~~

dispersion relationship: $\quad u = A e^{i(kx - \omega t)} = A e^{ik(x - vt)}$ $\quad v = \omega / k$

$$\Rightarrow \quad -i\omega + ick + \gamma k^2 + i\delta k^3 = 0$$

$$\Rightarrow \quad \omega(k) = ck - i\gamma k^2 - \delta k^3 = \underbrace{(ck - \delta k^3)}_{Re} + i\underbrace{(-\gamma k^2)}_{Im}$$

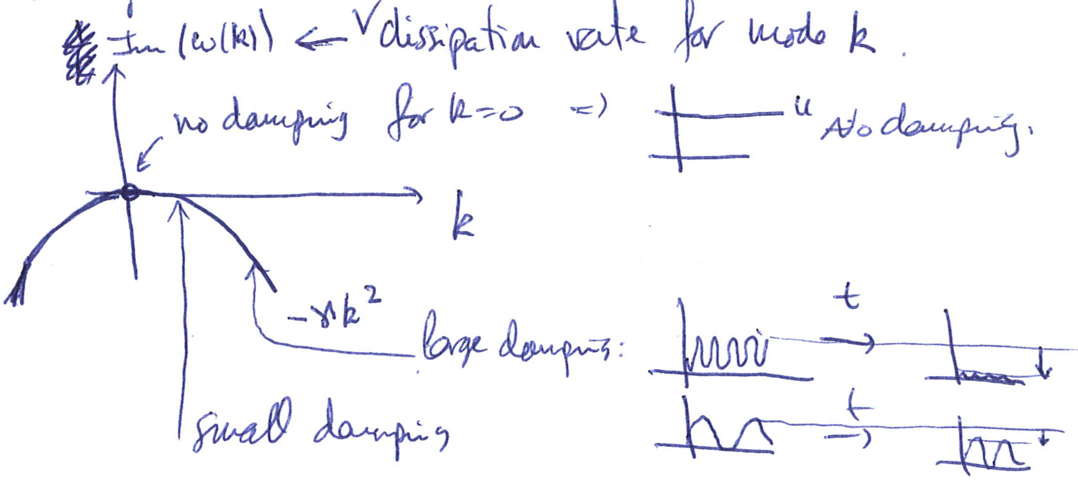
$$\therefore \quad u = A e^{i[kx - (ck - i\gamma k^2 - \delta k^3)t]}$$

$$= A e^{ik[x - (c - \delta k^2)t]} e^{-\gamma k^2}$$

Play with wave_integrator.m

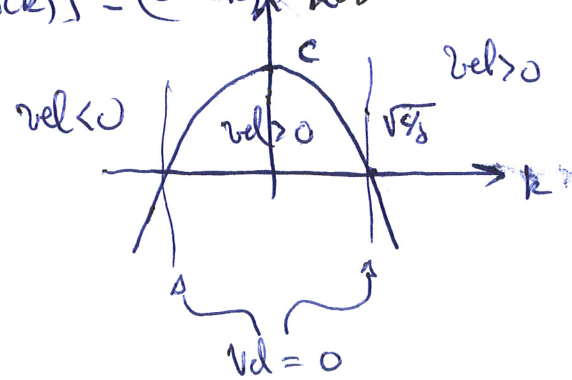
- check velocity for $\delta = 0$
- —"— $\delta \neq 0$
- check damping rate for a single mode
- add many modes and discuss which one prevails

```
view([90 0])
tt=0:0.1:4
plot3(L+tt*0,tt,A*exp(-g*k.^2*tt)
      ,'Linewidth',2)
axis tight
```



∴ after some time the mode with smallest wave # will dominate!

$$\text{Velocity} = \frac{1}{k} Re[\omega(k)] = (c - \delta k^2)$$



Nonlinearity:

Let us now add nonlinearity to the wave equation $u_t + c u_x = 0$

In most physical situations the linear wave eq is valid for small wave displacements (i.e. $|u| \ll 1$). However, when u becomes large there might be nonlinear effects that have to be considered.

There are ~~a~~ few ways to include a nonlinear term to the wave equation. For example we could add a quadratic or cubic term to the equation.

Let us add the lowest order nonlinearity: a quadratic term.

How to add it? We have to do it in such a way that $|u| \ll 1$

$\Rightarrow$ we recover wave eq.

~~$u_t + (1 + u) u_x$~~

take $u_t + \underbrace{1}_{\uparrow\, |u| \ll 1} u_x = 0$ ($c = 1$ for simplicity)

$$u_t + (u+1) u_x = 0$$

velocity depends on height of the wave.

this is one of the simplest ways to include nonlinearity.

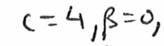
By putting everything together:

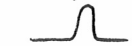
$$\boxed{\underbrace{u_t + c u_x}_{\text{wave eq.}} + \underbrace{\beta u u_x}_{\text{nonlinearity}} - \underbrace{\gamma u_{xx}}_{\substack{\text{dissipation} \\ \text{diffusion}}} + \underbrace{\delta u_{xxx}}_{\text{dispersion}} = 0}$$

$\rightarrow$ shallow water waves

$\rightarrow$

Show matlab code.
wave_integrator.m

• 
$c = 4, \beta = 0, \gamma = 0.7, \delta = 1$

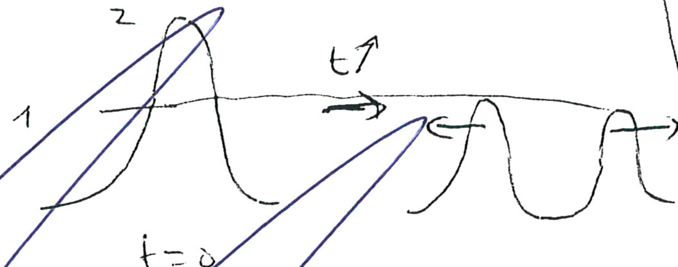
• 
$c = 0, \beta = 6, \gamma = 0, \delta = 1$

31oct02

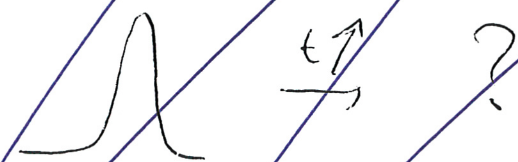
$\longrightarrow$ we'll study the above eq (KdV) in some detail later (p80)

Ex: $u(x,0) = 2e^{-x^2/2}$ $u_t(x,0)=0$

Show matlab ugauss_utzero.m



What if: $u(x,0) = 2e^{-x^2/2}$ and $u_t(x,0) = ?$



See previous [illegible] wave eq.

Do this just after doing uni-directional motion

p(4)

For the UNIDIRECTIONAL wave eq. $u_t + cu_x = 0$ there is NO ambiguity and $u(x,0)$ suffices to determine future/past behavior.

With dispersion: $u_t + cu_x + \delta u_{xxx} = 0$ the wave may travel to the left!

SOLITONS: Nonlinear wave propagation.

Read extract of John Scott Russell "Report on waves" → Wikipedia

How is it possible to have a solitary wave (soliton) with such stability?

$\boxed{\text{Nonlinearity balances dispersion}}$

~~Application: fiber optics~~

Show pictures KdV

Show wave-integrator with KdV solution

Transforming PDEs with coord. transf.

$$PDE(u,x,t) \xrightarrow[\substack{x\to\xi \\ t\to\tau}]{\text{norm}} PDE(u,\xi,\tau)$$

Maple
Change-coord-PDE
tr={t=T, x=X+cT},
dchange(tr, PDE, {X,T})

Ex: $\boxed{u_t + c u_x = 0}$ (1)

transf. to a co-moving reference frame with vel = c

$$\xi = x - ct, \quad \tau = t$$

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \tau}\frac{d\tau}{\partial t} + \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} = u_\tau + u_\xi(-c)$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \tau}\cancel{\frac{\partial \tau}{\partial x}}^{0} + \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = u_\xi$$

$\therefore$ (1) $\Rightarrow$ $u_\tau - \cancel{c u_\xi} + \cancel{c u_\xi} = 0 \Rightarrow u_\tau = 0 \Rightarrow u(\xi,\tau)$ is cte writ

$\therefore$ IC: $u(\xi, 0) = f(\xi) \Rightarrow u(\xi,\tau) = f(\xi)$

$\Rightarrow u(x,t) = f(x-ct)$ ✓

dimensionalization (or non-dimensionalization)

Suppose PDE: $F_{\vec p}(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots, x, t) = 0$

where $\vec p = (p_1, \dots, p_N)$ are N real params. def. F

$\Rightarrow$ transf. $\left\{\begin{array}{l} u = \alpha V \\ x = \beta X \\ t = \gamma T \end{array}\right\}$ could get rid of 3 params.

$\therefore F_{\vec p}(\dots) = 0 \iff F_{\vec \pi}(V, V_X, V_T, \dots) = 0$

where $\vec\pi = (\pi_1, \pi_2, \dots, \pi_{N-3})$

Ex: Wave eq. with params: $\boxed{\rho u_{tt} = EA u_{xx}}$ (2)

A: cross-sect area
ρ = density
E: Young modulus

Write (2) in non-dimensional form.

a) 1rst get rid of some params by dividing through (if all terms have a param).

$\Rightarrow u_{tt} = \frac{EA}{\rho} u_{xx}$ and def: $c^2 \equiv \frac{EA}{\rho}$

$\Rightarrow \boxed{u_{tt} = c^2 u_{xx}}$ (3) $\leftarrow$ one param.

⚠ Since PDE is linear the transf. $u = \alpha V$ won't change anything:

$$\cancel{\alpha} V_{tt} = \cancel{\alpha} c^2 V_{xx} \Rightarrow V_{tt} = c^2 V_{xx}$$

b) We could get rid of the last param. by rescaling x or t (or both).

b₁) $x = \beta X \Rightarrow \partial_x = \frac{\partial}{\partial x} = \frac{\partial}{\partial(\beta X)} = \frac{1}{\beta}\frac{\partial}{\partial X}$

$$\Rightarrow (\partial_x)^n = \frac{1}{\beta^n}\frac{\partial^n}{\partial X^n}$$

$$\Rightarrow (3) \Rightarrow u_{tt} = c^2 \frac{1}{\beta^2} u_{XX}$$

∴ we can chose $\boxed{\beta \equiv c} \Rightarrow u_{tt} = u_{XX}$

∴ If $x = cX$ eq (2) $\Leftrightarrow u_{tt} = u_{XX}$

b₂) $t = \gamma T \Rightarrow (\partial_t)^n = \frac{1}{\gamma^n}\frac{\partial^2}{\partial T^2}$

∴ (3) → $\frac{1}{\gamma^2} u_{TT} = c^2 u_{xx}$

∴ if $\boxed{\gamma = 1/c} \Rightarrow u_{TT} = u_{xx}$

∴ if $t = \frac{1}{c}\tau$ eq (2) $\Leftrightarrow u_{TT} = u_{xx}$

Maple → modify change_coord PDE laws

b₃) $x = \beta X$ & $t = \gamma T \Rightarrow \frac{1}{\gamma^2} u_{TT} = \frac{c^2}{\beta^2} u_{XX}$

∴ as long as $\boxed{\beta/\gamma = c}$ eq (2) $\Rightarrow u_{TT} = u_{XX}$

⚠ ⊛ Very important to adim. eqs in order to study them: from N params to N-p params.

⊛ Important for numerical studies.

HW: you'll adim. KdV with params:

$$\eta_t + \eta_x + \frac{3}{2}\delta\eta\eta_x + \frac{\delta}{6}\eta_{xxx} = 0 \longrightarrow u_t + 6uu_x + u_{xxx} = 0$$

Debnath: 1.5 2^{nd} order linear eqs & Method of characteristics

general: $A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$ (1)

A, B, C, D, E, F, G are $f(x,y)$'s or constants.

classification: After a coord. change brings (1) to a canonical form.

transf: • $\xi = \phi(x,y)$; $\eta = \psi(x,y)$ (2)

• $\phi, \psi \in C^2$ & $|J(x,y)| = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} = \phi_x \psi_y - \phi_y \psi_x \neq 0$ in the domain of interest so that transf. is invertible.

chain rule: $u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = u_\xi \xi_x + u_\eta \eta_x$

$u_{xx} = \frac{\partial}{\partial x}(u_x) = \partial_x (u_\xi \xi_x + u_\eta \eta_x) = (\partial_x u_\xi)\xi_x + u_\xi \xi_{xx} + (\partial_x u_\eta)\eta_x + u_\eta \eta_{xx}$

P11

$= (u_{\xi\xi} \cdot \xi_x + u_{\xi\eta}\eta_x)\xi_x + u_\xi \xi_{xx} + (u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x)\eta_x + u_\eta \eta_{xx}$

$= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$

subs in (1) $\Rightarrow$ $A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*$ (3)

where $A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$

$D^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$

P11

$\vdots$

$G^* = \ldots$

we want transf (2) s.t. (3) is simpler $\rightarrow$ $A^* = 0$ & $C^* = 0$ & $B^* \neq 0$

[why not $A^* = B^* = C^* = 0$? A $\rightarrow$ cannot reduce degree of PDE!]

$\Rightarrow$ $A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$

$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$

$\Rightarrow$ $A\left(\frac{\gamma_x}{\gamma_y}\right)^2 + B\left(\frac{\gamma_x}{\gamma_y}\right) + C = 0$ (4) $\quad \gamma = \begin{cases} \xi \text{ or} \\ \eta \end{cases}$

level curves: consider level curves for transf:

$\xi = \phi(x,y) = C_1 = \text{const}$

$\eta = \psi(x,y) = C_2 = \text{const}.$

If ξ & η are const. $\Rightarrow u(\xi,\eta) = u(\phi,\psi) = \text{const}.$ $\therefore$ ϕ & ψ give you curve over which sol. [in transf. coord] is const!!!

on these curves: $d\xi = \xi_x dx + \xi_y dy = 0$
$d\eta = \eta_x dx + \eta_y dy = 0$

Slope of these curves $\Rightarrow$ $\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = -\frac{\eta_x}{\eta_y}$ $\left(= \frac{\gamma_x}{\gamma_y}\right)$

$\therefore$ quadratic ④ $\Longrightarrow$ $A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$

$$\Longrightarrow \boxed{\frac{dy}{dx} = \frac{+B \pm \sqrt{\Delta}}{2A}} \quad (5) \qquad \Delta = B^2 - 4AC.$$

(it is an ODE!)

$\uparrow$ characteristic eqns. and their sols are called characteristic sols or characteristics

ODE which gives level curves ~~or which~~ of transformation $\Rightarrow$ In the transf. system new coord. are const $\Rightarrow$ sol. is constant!!!

the 2 solutions for (5) give the 2 $\neq$ (in general) characteristics

Classification:

Case I: $\Delta > 0$: Hyperbolic PDE

Integrating (5) gives 2 $\neq$ characteristics $\phi(x,y) = c_1$ & $\psi(x,y) = c_2$

Thus, the transf. PDE is: $(A^* = 0 = C^*,\ B^* \neq 0)$

$$\Rightarrow \boxed{u_{\xi\eta} = -\frac{1}{B^*}\left(D^* u_\xi + E^* u_\eta + F^* u - G^*\right)} \quad ⑥$$

1rst Canonical form for a hyperbolic PDE.

~~Aft~~ After a further transf: $\alpha = \xi + \eta$, $\beta = \xi - \eta$

$$\Rightarrow \quad u_\xi = u_\alpha \alpha_\xi + u_\beta \beta_\xi = u_\alpha + u_\beta$$
$$u_\eta = \ ---\ = u_\alpha - u_\beta$$
$$u_{\eta\xi} = \ ---\ = u_{\alpha\alpha} - u_{\beta\beta}$$

$$\Longrightarrow \boxed{u_{\alpha\alpha} - u_{\beta\beta} = H_2(\alpha, \beta, u, u_\alpha, u_\beta)} \quad ⑦$$

2nd Canonical form for a hyp PDE

Case II: $\Delta = 0$: Parabolic PDE $(\Delta = 0 \Rightarrow B^2 = 4AC)$

integrating $\frac{dy}{dx} = \frac{-B}{2A}$ gives only one characteristic

- $A^* = 0$ & $B^2 = 4AC$ $\Rightarrow$ $A^* = 0 = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = (\sqrt{A}\xi_x + \sqrt{C}\xi_y)^2$ $\quad B = 2\sqrt{AC}$
- and the $B^* = \ ---\ = 2(\underbrace{\sqrt{A}\xi_x + \sqrt{C}\xi_y}_{=0})(\sqrt{A}\eta_x + \sqrt{C}\eta_y) = 0$

$\therefore A^* = 0 \Rightarrow B^* = 0 \Rightarrow C^* \neq 0$

$\therefore$ PDE $\frac{(3)}{C^*}$ $\Rightarrow$ $\boxed{u_{\eta\eta} = H_3(\xi,\eta,u,u_\xi,u_\eta)}$ (8)

Canonical form of the parabolic Eq.

Case III : $B^2-4AC<0$: elliptic PDE

slope eq $\frac{dy}{dx}$ does not have real sols $\rightarrow$ 2 complex families of charac.

Since the corresponding roots (for ξ, & η) are complex conjugates let us introduce the real variables:

$$\alpha = \frac{1}{2}(\xi+\eta) \quad \text{(Real part)}$$
$$\beta = \frac{1}{2i}(\xi-\eta) \quad \text{(imag part)}.$$

$\therefore$ $\xi = \alpha + i\beta$ & $\eta = \alpha - i\beta$

thus PDE (3) transf. into:

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} = H_4(\alpha,\beta,u,u_\alpha,u_\beta) \quad (9)$$

and ~~$A^{**} = A^*$~~ these new ~~coef~~ have a similar ~~same~~ form as the $A^*, B^*, \ldots$

thus $A^* = 0 = C^*$ $\Rightarrow$ $A^{**} - C^{**} \pm iB^{**} = 0$

$$\Rightarrow \begin{cases} A^{**} = C^{**} \\ B^{**} = 0 \end{cases}$$

$\therefore$ (9) $\Rightarrow$ $\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \frac{H_4}{A^{**}} = H_5(\alpha,\beta,u,u_\alpha,u_\beta)}$ (10)

Canonical for of Elliptic PDE

Summary:
- the PDE (1) is called hyp., parab, elliptic @ (x_0,y_0) if respectively, $B^2(x_0,y_0) - 4A(x_0,y_0)C(x_0,y_0)$ is >0, $=0$, <0
- If the imag. ~~eq.~~ is true $\forall (x_0,y_0)$ in Domain the PDE is called hyp, parab, ellip.
- Any PDE of the form (1) can be cast, under appropriate transf, into one of the 3 canonical forms.

Ex1.5.1 : (a) wave eq: $u_{tt} - c^2 u_{xx} = 0$

$A = -c^2$, $B = 0$, $C = 1$ $\Rightarrow$ $\Delta = B^2 - 4AC = 0 + 4c^2 > 0 \Rightarrow$ Hyp.

(b) diff. eq: $u_t - k u_{xx} = 0$

$A = -k$, $B = 0$, $C = 0$ $\Rightarrow$ $\Delta = B^2 - 4AC = 0 \Rightarrow$ parab.

(c) Laplace eq: $u_{xx} + u_{yy} = 0$

$A = 1,\ B = 0,\ C = 1 \Rightarrow \Delta = -1 < 0 \Rightarrow$ elliptic PDE.

⋮

Ex 1.5.2: find charac. and charac. eq. and reduce eq:

$$x u_{xx} + u_{yy} = x^2$$

$A = x,\ B = 0,\ C = 1,\ \Delta = -4x \Rightarrow \begin{cases} x<0 & \text{hyp} \\ x=0 & \text{parab} \\ x>0 & \text{ellip.} \end{cases}$

Charact. eq: $\dfrac{dy}{dx} = \dfrac{B \pm \sqrt{\Delta}}{2A} = \dfrac{0 \pm \sqrt{-4x}}{2x} = \pm \dfrac{2\sqrt{-x}}{2\sqrt{-x}\sqrt{-x}} = \pm \dfrac{1}{\sqrt{-x}}$

$\boxed{x<0 \Rightarrow}$ hyp.

$\Rightarrow \int dy = \int \pm \dfrac{1}{\sqrt{-x}}\, dx \Rightarrow y = \pm 2\sqrt{-x} + c$

$[\,(y-c)^2 = -4x\,]$

$\Rightarrow y \mp 2\sqrt{-x} = \text{const} \leftarrow$ charac.

$\therefore\ \xi = y + 2\sqrt{-x} = \text{const}.$

$\eta = y - 2\sqrt{-x} = \text{conts}.$

We apply the coord. transf. (see Debnath p 15) and find canonical eqns for the 3 cases:

$\underline{x<0}$: hyp

$\underline{x=0}$: parab

$\underline{x>0}$: ellip.

⚠ The method of characteristic to find solutions can only be applied to physical problems that are hyperbolic (or parabolic)

- characteristic → curves over which the solution is constant.
- charac: $\dfrac{dy}{dx} = \dfrac{B \pm \sqrt{\Delta}}{2A}$

$\therefore$ they only correspond to real curves if $\Delta \geq 0 \Rightarrow$ Hyp. or parabolic

Ex: Solve the wave eq. using characteristics:

Ex 1.5.3: $u_{tt} - c^2 u_{xx} = 0 \qquad A = -c^2,\ B = 0,\ C = 1$

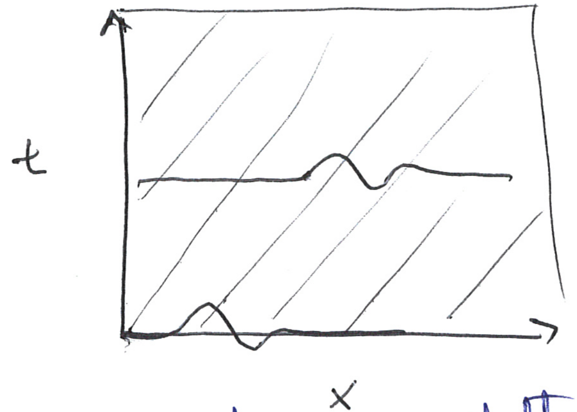
$\Rightarrow \Delta = 4c^2 > 0 \Rightarrow$ hyp.

charac : $\frac{dt}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \frac{0 \pm 2c}{-2c^2} = \mp \frac{1}{c}$

$\Rightarrow \quad \mp c\,dt = dx \quad \to \quad x = \mp ct + \text{const}$

$\Rightarrow \quad \boxed{x \pm ct = \text{const}}$ charact.

$\therefore$ over lines $x \pm ct = \text{const}$ the sol. to wave. eq. is const !



$\therefore$ const shape prop. in <u>both</u> directions.

First order equations $\to$ Method of characteristics

<u>Ex</u> : (Not in Debnath) $\to$ wave breaking.

consider $\quad u_t + \underbrace{c(u)}_{\text{speed depends on height.}} u_x = 0$

⚠ this eq. is No longer linear but we can still apply method of charac.

<u>Classic ex</u> : $c(u) = u \to$ wave breaking.

<u>Sols</u> :

wave breaking $\Rightarrow$ no longer a function (singularity)

Method of charact. $\Rightarrow$ 1) will give breaking pt time
2) will allow to follow wave after breaking.

Method of characteristics for 1st order quasi-linear PDEs

<u>Def</u> : quasi-linear : PDE is linear in its highest order derivatives.

- $f \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$ quasi-linear
- $\left(\frac{\partial f}{\partial x}\right)^2 + \frac{\partial f}{\partial y} = 0$ not quasi linear.

Gral. 1st order quasi-linear PDE (homogeneous).

$$P(u;x,t)\frac{\partial u}{\partial x} + Q(u;x,t)\frac{\partial u}{\partial t} = 0$$

→ this includes the wave eq. with generalized speed of the above example: $u_t + c(u)u_x = 0$

- Method of charac. → follow ~~path~~ $u(x,t)$ over a path $S = \{(x,t) \text{ s.t. } u = \text{const}\}$

$\therefore\ u(x,t) = u(s)$ and $u(s) = \text{const} \Rightarrow \frac{du}{ds} = 0$

$$\Rightarrow \frac{du}{ds} = u_x\frac{dx}{ds} + u_t\frac{dt}{ds} = 0$$

$u_x = -\frac{Q}{P}u_t$ $\Rightarrow$ $-\frac{Q}{P}\cancel{u_t}\frac{dx}{ds} + \cancel{u_t}\frac{dt}{ds} = 0 \Rightarrow \frac{dx/ds}{dt/ds} = P/Q$

$\therefore\ \boxed{\frac{dx}{dt} = \frac{P}{Q} \Leftrightarrow \text{path } S}$ $\left[x(t) = \int \frac{P}{Q}dt \text{ if } P \,\&\, Q \text{ do not depend on } x \text{ or } u\right]$

Ex: Wave eq. with const speed:

$$u_t + cu_x = 0,\quad P = c,\ Q = 1$$

$\Rightarrow x(t) = \int c\,dt \Rightarrow x(t) = ct + \text{const}$

$\Rightarrow \boxed{x - ct = \text{const}}$ ✓

Ex: generalized speed wave eq:

$$u_t + c(u)u_x = 0 \quad \text{with ICs} = u_0(x) = u(x,t=0)$$

charact: S: $\frac{dx}{dt} = c(u)$ & $x(t=0) = \xi$ ← pt where characteristic starts

and along S: $\frac{du}{dt} = 0 \Rightarrow u = \text{const} = \text{const}(\xi)$
$\overset{IC}{\Rightarrow} u(\xi) = u_0(\xi)$

$\frac{dx}{dt} = c(u_0(\xi)) \Rightarrow x = c(u_0(\xi))\,t + \text{const}.$

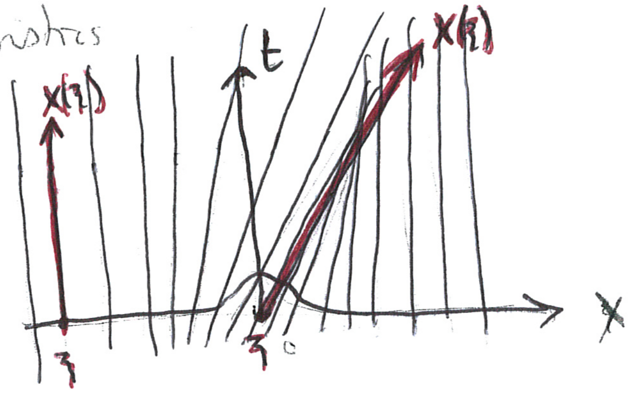
$\Rightarrow$ IC $x(0) = \xi \Rightarrow \boxed{x = c(u_0(\xi))t + \xi}$

Thus, given $c(u)$ & $u_0(x)$ we can draw curves where $u = \text{const.}$ starting @ $x(t=0) = \xi$

∴ going back to $u_t + \overbrace{u}^{c(u)} u_x = 0$ with IC: $u_0 = \alpha e^{-x^2}$

$x = c(u_0(\xi))t + \xi \Rightarrow \boxed{x = \xi + (\alpha e^{-\xi^2})t \quad \text{and} \quad u_0 = \alpha e^{-\xi^2}}$ we have solved the IVP.

plot characteristics



characteristic_m ← theory
+
charac_example_pde ← numerics
[α = 0.101]

Numerical integrator: PDE_integrator_higher-order

$$u_t = f_0(u,x) + f_1(u,x)\, u_x + f_2(u,x)\, u_{xx} + f_3(u,x)\, u_{xxx} + f_4(u,x)\, u_{xxxx}$$

Wave breaking:

consider: $u_t + \underbrace{c(u)}_{\text{characteristic speed}} u_x = 0$

IC: $u(x,0) = u_0(x) \qquad x \in \mathbb{R}$.

characteristics: $\begin{cases} \dfrac{dx}{dt} = c(u_0(\xi)) & x(t=0) = \xi \\ \dfrac{du}{dt} = 0 & \text{and } x = \xi + c(u_0(\xi))t \end{cases}$

Breaking occurs when two nearby characteristics collide
Let us look for earliest time when two nearby characteristics collide $\rightarrow t_B$

characteristic: $\begin{cases} x = \xi + c(u_0(\xi))t & C1 \\ x = \xi + d\xi + c(u_0(\xi + d\xi))t & C2 \end{cases}$:nearby characteristic

t_B is when both C1 & C2 hold simultaneously for $d\xi \rightarrow 0$

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$$\Rightarrow \cancel{\xi} + c(u_0(\xi))t_B = \cancel{\xi} + d\xi + c(u_0(\xi+d\xi))t_B$$

$$\Rightarrow t_B\left[c(u_0(\xi+d\xi)) - c(u_0(\xi))\right] = -d\xi$$

$$\Rightarrow t_B = -\frac{1}{\dfrac{c(u_0(\xi+d\xi)) - c(u_0(\xi))}{d\xi}}$$

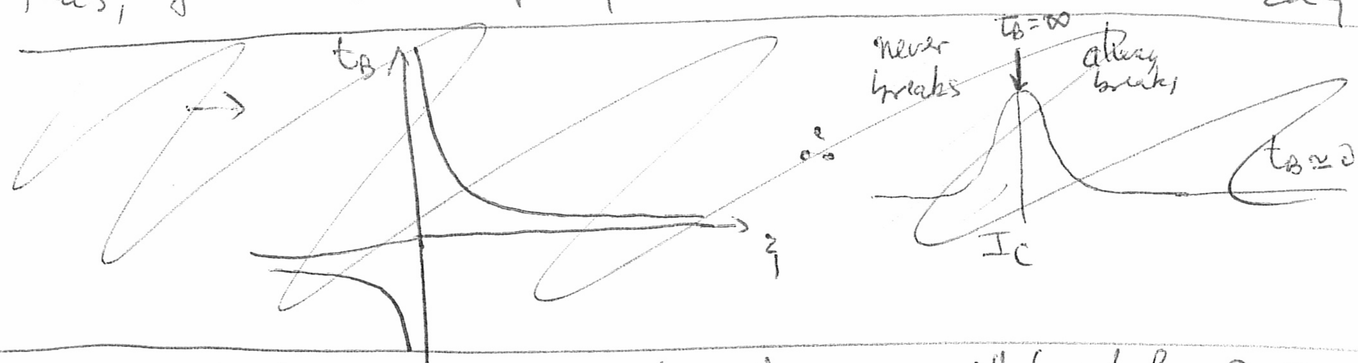
$\lim d\xi \to 0$: $\boxed{t_B = -\dfrac{1}{u_0'(\xi)\cdot c'(u_0(\xi))}}$

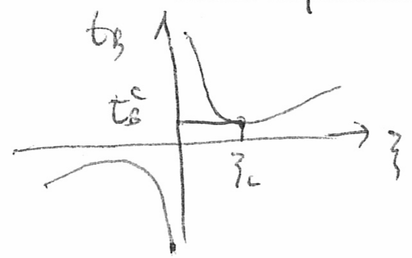
For our example: $u_0(\xi) = \alpha e^{-\xi^2} \Rightarrow u_0'(\xi) = -2\xi\alpha e^{-\xi^2}$

$c(u) = u \Rightarrow c'(u) = 1$

$$\therefore \quad t_B = -\frac{1}{-2\xi\alpha e^{-\xi^2}\cdot 1} = \frac{e^{+\xi^2}}{2\alpha\xi}$$

thus, for the starting pt ξ the wave breaks @ $t_B(\xi) = \dfrac{e^{+\xi^2}}{2\alpha\xi}$





thus the wave will break for $\xi = \xi_c$ corresponding to $\min\limits_{\xi, t_B > 0}(t_B) = t_B^c$

$$t_B'(\xi) = \frac{\cancel{2}\xi e^{\xi^2} 2\cancel{\alpha}\xi - \cancel{2\alpha} e^{\xi^2}}{\cancel{4}_2\alpha^{\cancel{2}}\xi^2}$$

$$= \frac{(2\xi^2 - 1)e^{\xi^2}}{2\alpha\xi^2}$$

$t_B'(\xi_c) = 0 \Rightarrow \xi_c = 1/\sqrt{2}$ and $t_B^c = t_B(\xi_c) = \dfrac{e^{-1/2}}{\alpha\, 2/\sqrt{2}} = \dfrac{e^{-1/2}\sqrt{2}}{\alpha} \simeq \dfrac{1.1658}{\alpha}$

In our matlab example $\alpha = 0.1 \Rightarrow t_B^c = 11.658$ ✓

Shocks: follow path of discontinuity created by multivaluedness
$\longrightarrow$ photocopies of [S, § 2.2.3 + 2.2.4 + 2.2.5]] reading material] better do it from Debnath