Water Waves

no viscosity

Euler eqs : • water $\underbrace{\text{inviscid}}$, incompressible fluid.

• conservation of mass + momentum yield:

use my notes on WWW instead of going through details on the board

Euler eqs.

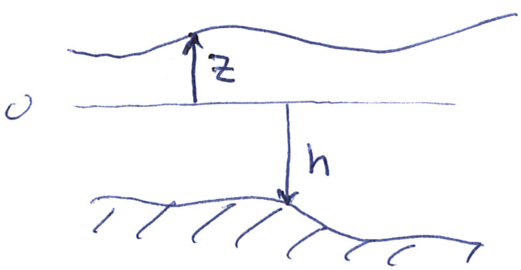
$$\begin{cases} u_x + v_y + w_z = 0 \\ u_t + u u_x + v u_y + w u_z = -\frac{1}{\rho} p_x \\ v_t + u v_x + v v_y + w v_z = -\frac{1}{\rho} p_y \\ w_t + u w_x + v w_y + w w_z = -\frac{1}{\rho} p_z - g \end{cases}$$

g: gravity, ρ: density, $\vec{p} = (p_x, p_y, p_z)$: pressure

$\vec{u} = (u, v, w)$ fluid velocity.

let $z = \eta(x,y,t)$ ~~...~~ : free surface (top)

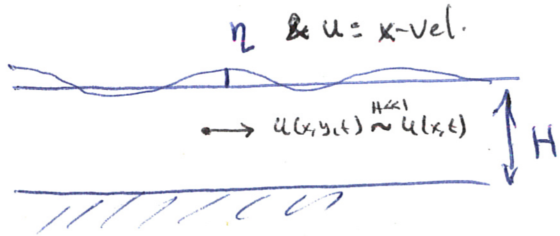
$z = -h(x,y)$: bottom topography.



[S, p 53-69]

→ photocopy to students.

Boussinesq eq + KdV eq



- consider irrotational flow
- 2D
- shallow, flat bottom
- low order nonlinearities:
- adimensionalization

$$\Rightarrow \begin{cases} \eta_t + [(1+\delta\eta)u]_x - \frac{\delta}{6} u_{xxx} = 0 \\ u_t + \delta u u_x + \eta_x - \frac{\delta}{2} u_{xxt} = 0 \end{cases}$$

Boussinesq eqns. (1)

$\delta = \left(\frac{H}{L}\right)^2 \ll 1$, L: typical wave length.

9/18/06 (20)

Linearize eqs: $u = u^e + \varepsilon u^p$, $\eta = \eta^e + \varepsilon \eta^p$ $\boxed{\begin{array}{l} u^e = 0 \\ \eta^e = 0 \end{array}}$ steady state solutions. $|\varepsilon| \ll 1$ perturbation param.

Eul. Eqs

$$\Rightarrow \begin{cases} \varepsilon \eta^p_t + [(1+\delta\varepsilon\eta^p)\varepsilon u^p]_x - \frac{\delta}{6}\varepsilon u^p_{xxx} = 0 \\ \varepsilon u^p_t + \varepsilon^2 u^p u^p_x + \varepsilon \eta^p_x - \frac{\delta}{2}\varepsilon u^p_{xxt} = 0 \end{cases}$$

Suppose u & η variations very soft so that we neglect higher order derivatives: Keeping $O(\varepsilon)$ terms:

$$\begin{cases} \eta^p_t + u^p_x = 0 \\ u^p_t + \eta^p_x = 0 \end{cases} \quad \| \text{ equivalent to } \delta = 0$$

$$\Rightarrow \begin{array}{l} \partial_t : \\ \partial_x : \end{array} \begin{cases} \eta^p_{tt} + u^p_{xt} = 0 \\ u^p_{tx} + \eta^p_{xx} = 0 \end{cases} \quad (-) \quad \boxed{\eta^p_{tt} - \eta^p_{xx} = 0} \quad (2)$$

Linear wave eq.

$$\therefore \quad \eta^p(x,t) = \eta^R(x-t) + \eta^L(x+t)$$

Similar ~~Similar~~ for u^p. Since wave in η and u should be related [and $u^p = \eta^p$ is a solution] in the linear case. ~~Furthermore a possible solution is $u^p = \eta^p$~~

dropped U^p from now on.

~~Then~~, let us take $u = \eta(x,t) + \delta A(x,t) + O(\delta^2)$ (3)

and determine $A(x,t)$ such that (3) is sol. to (1)

(3) in (1) :

$$\begin{cases} \eta_t + [(1+\delta\eta)(\eta+\delta A)]_x - \frac{\delta}{6}[\eta+\delta A]_{xxx} = 0 \\ (\eta+\delta A)_t + \delta(\eta+\delta A)(\eta+\delta A)_x + \eta_x + \frac{\delta}{2}[\eta+\delta A]_{xxt} = 0 \end{cases}$$

now keep only terms $< O(\delta^2)$

$$\Rightarrow \begin{cases} \eta_t + [\eta + \delta\eta^2 + \delta A]_x - \frac{\delta}{6}\eta_{xxx} = 0 \\ \eta_t + \delta A_t + \delta\eta\eta_x + \eta_x + \frac{\delta}{2}\eta_{xxt} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \eta_t + \eta_x + 2\delta\eta\eta_x + \delta A_x - \frac{\delta}{6}\eta_{xxx} = 0 \\ \eta_t + \delta A_t + \delta\eta\eta_x + \eta_x - \frac{\delta}{2}\eta_{xxt} = 0 \end{cases}$$

Now consider only wave traveling to the right ie $\eta_t + \eta_x = 0$

$$\Rightarrow \eta_t = -\eta_x \Rightarrow \delta\,\eta_{xxt} = -\delta\,\eta_{xxx}$$

$$\therefore \begin{cases} \eta_t + \eta_x + 2\delta(\eta\eta_x) + \delta A_x - \frac{\delta}{6}\eta_{xxx} = 0 \\ \eta_t + \eta_x + \delta(\eta\eta_x) + \delta A_t + \frac{\delta}{2}\eta_{xxx} = 0 \end{cases} \quad (4)$$

If the 2 eqns are to be consistent (ie solve for A) ← solvability condition.

$$(4a) - (4b) \Rightarrow \delta\eta\eta_x + \delta(A_x - A_t) - \frac{2}{3}\delta\,\eta_{xxx} = 0$$

$$A_x - A_t = -\eta\eta_x + \frac{2}{3}\eta_{xxx}$$

If $\boxed{A = -\frac{\eta^2}{4} + \frac{\eta_{xx}}{3}}$ (5) $\Rightarrow$

$$A_x - A_t = -\frac{2}{4}\eta\eta_x + \frac{\eta_{xxx}}{3} + \frac{2\eta\eta_t}{4} - \frac{\eta_{xxt}}{3}$$

$\eta_t = -\eta_x$

$$= -\frac{1}{2}\eta\eta_x + \frac{\eta_{xxx}}{3} - \frac{1}{2}\eta\eta_x + \frac{1}{3}\eta_{xxx}$$

$$= -\eta\eta_x + \frac{2}{3}\eta_{xxx} \checkmark$$

since $A_x = -A_t$ $\Rightarrow A_x - A_t = 2A_x = -\eta\eta_x + \frac{2}{3}\eta_{xxx}$

$\Rightarrow 2A = \int_x -\eta\eta_x + \frac{2}{3}\eta_{xxx} = -\frac{1}{2}\eta^2 + \frac{2}{3}\eta_{xx}$

$\therefore A = \frac{1}{2}\left(-\frac{1}{2}\eta^2 + \frac{2}{3}\eta_{xx}\right)$ should work.

$\therefore$ use (4a) or (4b) with (5):

(5) in (4a)

$$\eta_t + \eta_x + 2\delta\eta\eta_x + \delta\left(-2\frac{\eta\eta_x}{4} + \frac{\eta_{xxx}}{3}\right) - \frac{\delta}{6}\eta_{xxx} = 0$$

$$\Rightarrow \eta_t + \eta_x + \left(2 - \tfrac{1}{2}\right)\delta\eta\eta_x + \delta\left(\tfrac{1}{3} - \tfrac{1}{6}\right)\eta_{xxx} = 0$$

$$\Rightarrow \boxed{\eta_t + \eta_x + \frac{3}{2}\delta\eta\eta_x + \frac{\delta}{6}\eta_{xxx} = 0} \quad (6)$$

HW

Now transform :: $\underbrace{t \to \delta^{-1} t}_{\text{rescale time}}$, $\underbrace{x \to x - (1-\delta)t}_{\text{co-moving reference frame @ velocity } (1-\delta)}$

$$\Rightarrow \boxed{\eta_t + \eta_x + \frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{xxx} = 0} \quad (7)$$

Korteweg-deVries Eq. (KdV).

↑ model for shallow water wave equations in one direction, no dissipation and lowest order dispersion (η_{xxx})

Elementary solutions of KdV

$u_t + \underset{\text{vel.}}{c(u)}\, u_x = 0$

Under, yet another transformation, KdV (7) can be rewritten as } HK

$$\boxed{U_t \mp 6UU_x + U_{xxx} = 0} \quad (8) \text{ KdV [Drazin]}$$

→ no parameters : transf. eliminated all params.

Drazin has ⊖ but this is just $u \to -u$

Scale invariance :

transf: $X = k^a x \Rightarrow \partial_X = k^{-a} \partial_x$

$T = k^b t \Rightarrow \partial_T = k^{-b} \partial_t$

$U = k^c u$

$$\text{KdV} \Rightarrow k^{b-c} U_T - 6\, k^{a-2c} UU_X + k^{+3a-c} U_{XXX} = 0$$

$b-c = a-2c = 3a-c \Rightarrow a=1,\ b=3,\ c=-2$ ✓

$3+2 = 1+4 = 3+2$

∴ if $X = kx,\ T = k^3 t,\ U = u/k^2 \Rightarrow \text{KdV} \Leftrightarrow U_T + 6UU_X + U_{XXX} = 0$

ie: the solution is the same!

∴ if $u = A f(x - vt)$ is sol to KdV

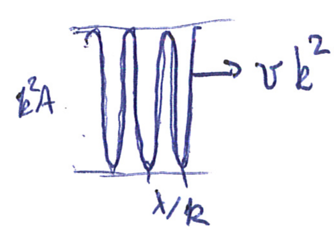
then $U = A f(X - vT)$ is also a sol. to KdV

$$\Rightarrow \frac{u}{k^2} = U = A f(kx - v k^3 t)$$

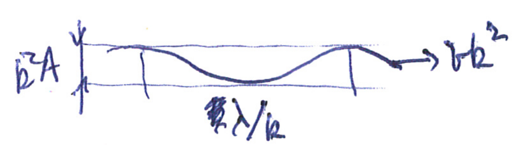
$$\Rightarrow u(x,t) = k^2 A f(k(x - k^2 v t))$$

$$= k^2 A f(k(x - v' t)) \qquad v' = k^2 v$$

Thus if u : 

$k > 1$ ⇒ 

⚠ This is true ∀ $f(x)$!

$k < 1$ ⇒ 

Galilean invariance

If $f(x-vt)$ is a sol $\Rightarrow$ $\lambda + f(x-\bar{v}t)$ is also a solution $\boxed{[v=\bar{v}-6\lambda]}$) HW

do NOT give. Just say: $v=\bar{v}-f(\lambda)$

$\therefore$ one can add/substract water to canal and the shape remains the same and only speed changes.

Traveling solutions

Let us look for traveling solutions:

$$u(x,t) = f(x-ct) = f(\xi), \quad \xi = x-ct$$

$$\text{KdV} \Rightarrow \boxed{-cf' + 6ff' + f''' = 0} \quad (9) \qquad (\cdot)' \equiv \frac{\partial}{\partial \xi}(\cdot)$$

1 parameter family of ODEs

$\therefore$ any solution to ODE (9) is a Sol. to KdV.

Solitary wave:

integrate (9) using $f(\pm\infty) = 0 = f' = f''$

$\div 2$

$$\int (9)\, d\xi \Rightarrow -cf + 3f^2 + f'' = A \leftarrow \text{constant.}$$

but since we require $f(\pm\infty) = 0 \overset{= f' = f''}{\Longrightarrow} A = 0$

$\div 3$

$$\Rightarrow -cf + 3f^2 + f'' = 0$$

$$\times f' \Rightarrow -cff' + 3f'f^2 + f''f' = 0$$

that is why KdV has a 6...

$$\int d\xi \Rightarrow -\frac{c}{2}f^2 + \cancel{3}\,1\, f^3 + \frac{1}{2}(f')^2 = B$$

Again BCs $\Rightarrow B = 0 \Rightarrow (f')^2 = cf^2 - 2f^3 = f^2(c-2f)$

$$\Rightarrow f' = \pm f\sqrt{c-2f}$$

$$\Rightarrow \frac{df}{d\xi} = \pm f\sqrt{} \Rightarrow \int \frac{df}{f\sqrt{c-2f}} = \pm \int d\xi$$

$$\Rightarrow \mp \frac{2}{\sqrt{c}} \tanh^{-1}\left[\frac{\sqrt{c-2f}}{\sqrt{c}}\right] = \xi - x_0$$

$$\Rightarrow \mp\sqrt{c-2f} = \sqrt{c}\tanh \frac{\sqrt{c}}{2}(\xi - x_0)$$

$$\Rightarrow 2f = -c\tanh^2(\;) + c \overset{1-t^2=s^2}{=} c\,\text{sech}^2(\;)$$

$$\Rightarrow f(x-ct) = \frac{c}{2}\,\text{sech}^2\left[\frac{\sqrt{c}}{2}(\xi - x_0)\right]$$

or
$$\boxed{u(x,t) = \frac{c}{2}\,\text{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct-x_0)\right]}$$

Solitary Solution to KdV.

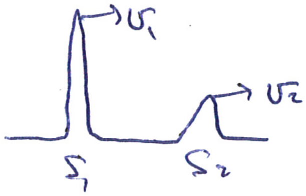
Better use KdV.m

~~$f_0=0$, $f_1=-6u$, $f_2=0$, $f_3=+1$, $f_4=0$~~

~~Matlab PDE integrator~~

Scale invariance

Superposition? → No, eq. is nonlinear but if we start 2 sech^2 solitons very far away it is almost an exact sol:



→ Q: what happens as $t \uparrow$?

- superposition?
- phase shift.

KdV.m 2sol. I.C.

→ Name SOLITON

⚠ KdV is the simplest eq. that incorporates nonlinearity & dispersion: ie. it is a GENERIC eq. that will often show up in nonlinear propagation models.

- Internal gravity waves, stratified fluids
- waves in rotating atmosphere
- ion-acustic waves in plasmas
- pressure waves in liquid-gas bubble mixtures
- nonlinear (anharmonic) lattices
- thermal packets in nonlinear crystals
- ...

General waves of permanent form in KdV

KdV: $u_t + 6uu_x + u_{xxx} = 0$

traveling wave: $u(x,t) = f(\xi) \qquad \xi = x - ct$

$$\Rightarrow \quad -cf' + 6ff' + f''' = 0$$

$$\int d\xi \Rightarrow \quad -cf + 3f^2 + f'' = -A/2$$

(← now we don't require $u(\pm\infty) = u_x(\pm\infty) = 0$) A tb determined by BCs.

$$\Rightarrow \quad f'' = cf - 3f^2 + A/2$$

$$\Rightarrow \quad 2f'' = 2cf - 6f^2 + A$$

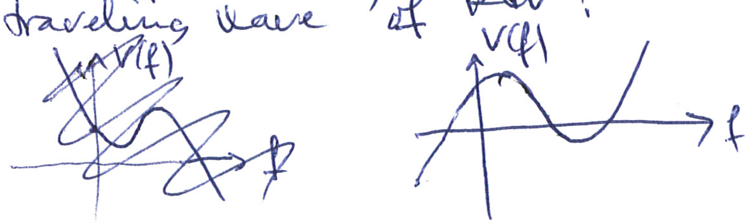
Remember Newton's law: $m\ddot{y} = F(y) = -\frac{dV}{dy}$ ← potential.

Here if potential $V(f) = -cf^2 + 2f^3 + Af + B$ $(m=2)$

$$\Rightarrow \quad m f'' = -\frac{dV}{df}$$

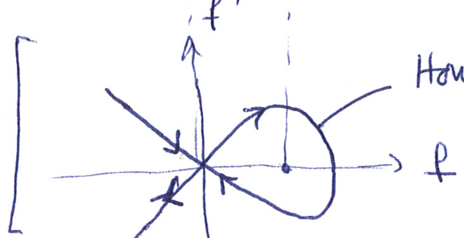
∴ any possible orbit of $m\ddot{y} = -\frac{dV}{dy}$ under influence of V is a valid traveling wave of KdV!

plot potential:



For our previous sech solution : $A = B = 0 \Rightarrow V(f) = 2f^3 - cf^2$
$= f^2(2f - c)$

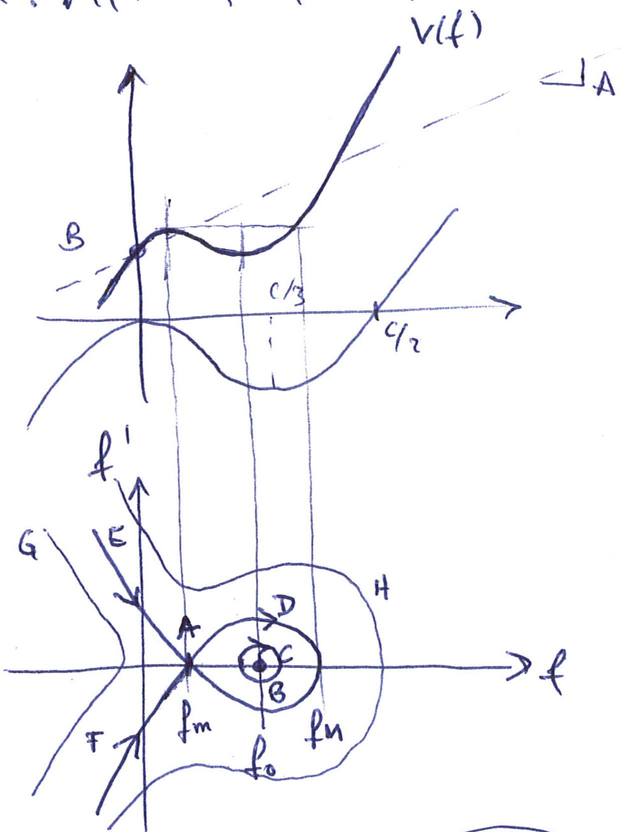
phase plane for 2^{nd} order ODE



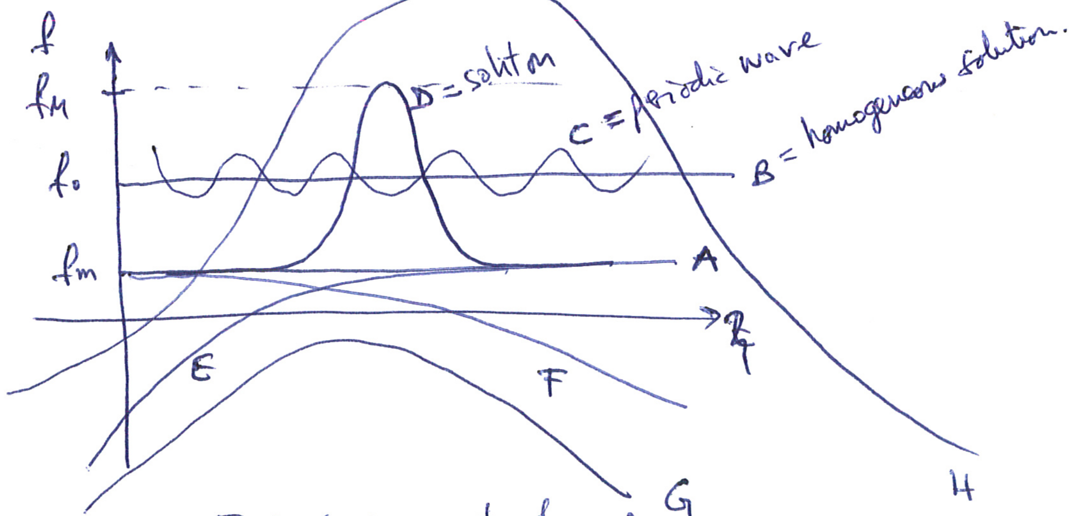
Homoclinic connection
‖
sech solution

In general : $V(f) = 2f^3 - cf^2 + Af + B \Rightarrow mf'' = -\frac{\partial V}{\partial f} = -(6f^2 - 2cf + A)$

$Af + B$



Use pplane.m to show solutions



E, F, G, H : not physical

Oscillatory solutions of KdV

$$u_t + 6uu_x + u_{xxx} = 0$$

[note sign "+" when compared to Drazin. → This changes signs all over the place]

$$u = f(\xi) = f(x - ct)$$

$$\Rightarrow \; -cf' + 6ff' + f''' = 0$$

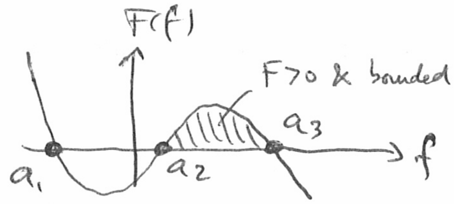
$$\overset{\int}{\Rightarrow} \; -cf + 3f^2 + f'' = -A/2 \quad \overset{\times f'}{\Rightarrow} \quad -cff' + 3f^2f' + f'f'' = -\tfrac{A}{2}f'$$

$$\overset{\int}{\Rightarrow} \; -\tfrac{c}{2}(f^2)' + (f^3)' + \tfrac{1}{2}(f'^2)' = -\tfrac{A}{2}f' \Rightarrow -\tfrac{c}{2}f^2 + f^3 + \tfrac{1}{2}f'^2 = -\tfrac{A}{2}f - \tfrac{B}{2}$$

$$\overset{\times 2}{\Rightarrow} \; (f')^2 = -2f^3 + cf^2 - Af - B \Rightarrow (f')^2 = f^2(c - 2f) - (Af + B)$$

$$(f')^2 = F(f)$$

$F(f)$:



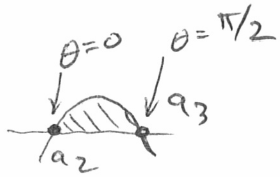
$$\underset{\text{Var}}{\overset{\text{sep}}{\Rightarrow}} \; \pm \int \frac{df}{\sqrt{F(f)}} = \int d\xi = \xi - x_0$$

Now, we will apply some change of variables to bring integral to a "familiar" form → elliptic functions.

- let us factorize $F(f) = -2(f - a_1)(f - a_2)(f - a_3)$

  note: $c = 2(a_1 + a_2 + a_3)$

- change of variables $f = a_2 + (a_3 - a_2)\sin^2\theta$



$$df = (a_3 - a_2)\, 2\cos\theta \sin\theta \, d\theta$$

$$\therefore \; I = \int \frac{df}{\sqrt{F}} = \int \frac{2(a_3 - a_2)\cos\theta\sin\theta}{\sqrt{-2(f-a_1)(f-a_2)(f-a_3)}}\, d\theta$$

$$(f-a_1)(f-a_2)(f-a_3) = \underbrace{(a_2 + (a_3-a_2)s^2 - a_1)}_{a_2(1-s^2) + a_3 s^2 - a_1}(\cancel{a_2} + (a_3-a_2)s^2 - \cancel{a_2})\overbrace{(a_2 + (a_3-a_2)s^2 - a_3)}^{a_2\overbrace{(1-s^2)}^{c^2} - a_3\overbrace{(1-s^2)}^{c^2}}$$

~~$= (a_3-a_2)s^2$~~

$$= (a_2c^2 + a_3s^2 - a_1)(a_3 - a_2)s^2(a_2 - a_3)c^2$$

$$= -(a_2c^2 + a_3s^2 - a_1)(a_3-a_2)^2 s^2c^2$$

$$= -(a_2 - a_2s^2 + a_3s^2 - a_1)(a_3-a_2)^2 s^2c^2$$

$$= -[a_2 - a_1 + (a_3-a_2)s^2](a_3-a_2)^2 s^2c^2$$

$$= -(a_2-a_1)[1 + \underbrace{\frac{a_3-a_2}{a_2-a_1}}_{k^2}s^2](a_3-a_2)^2 s^2c^2$$

$$\boxed{k^2 = \frac{a_3 - a_2}{a_2 - a_1}}$$

$$\Rightarrow \; I = \int \frac{2\cancel{(a_3-a_2)}\,\cancel{cs}}{\sqrt{2(a_2-a_1)(1+k^2s^2)\cancel{(a_3-a_2)^2}\cancel{s^2c^2}}}\, d\theta = \sqrt{\frac{2}{a_2-a_1}} \int \frac{1}{\sqrt{1 - k^2\sin^2\theta}}\, d\theta$$

$$\therefore \quad \pm\sqrt{\frac{2}{a_2-a_1}}\int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \zeta - x_0$$

$$\phi: \quad f = a_2 + (a_3-a_2)\sin^2\phi$$

$$\Rightarrow \int_0^{\phi}\frac{d\theta}{\sqrt{\;}} = \pm\sqrt{\frac{a_2-a_1}{2}}(\zeta - x_0) =$$

Similar to $\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1}x$

Now: Jacobi elliptic integral: $u = \int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \iff \mathrm{sn}(u;k) = \sin\phi$

$$\Rightarrow \mathrm{sn}\left[\pm\sqrt{\frac{a_2-a_1}{2}}(\zeta - x_0)\right] = \sin\phi = \sqrt{\frac{f-a_2}{a_3-a_2}}$$

$$\overset{(\;)^2}{\Rightarrow} \mathrm{sn}^2[\pm\sqrt{\;}(\;)] = \frac{f-a_2}{a_3-a_2} \Rightarrow f = (a_3-a_2)\,\mathrm{sn}^2[\;] + a_2$$

$$= (a_3-a_2)(1-\mathrm{cn}^2) + a_2$$

But: $\mathrm{cn}^2 + \mathrm{sn}^2 = 1$

$$= a_3 + (a_3-a_2)\,\mathrm{cn}^2$$

$$\therefore \quad f(\zeta) = a_3 + (a_3-a_2)\,\mathrm{cn}^2\left[\sqrt{\frac{a_2-a_1}{2}}(\zeta - x_0)\right]; k$$

$$\Rightarrow \boxed{u(x,t) = a_3 + (a_3-a_2)\,\mathrm{cn}^2\left[\sqrt{\frac{a_2-a_1}{2}}(x-ct-x_0)\right]}; k$$

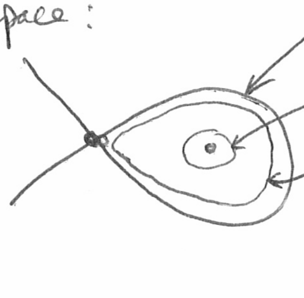
where $c = 2(a_1+a_2+a_3)$
$k^2 = (a_3-a_2)/(a_2-a_1)$

$\boxed{\mathrm{cn}, \mathrm{sn}}$:
- Jacobi elliptic cosine and sine, pronounced — sn: "s"-en, cn: "c"-en
- cn-wave → cnoidal wave

properties:
- many many, ... check online
- cn, sn interpolate between trig. & hyper.
  - k: elliptic modulus [⚠ Matlab use $m = k^2$]
  - $\mathrm{cn}(x; k=0) = \cos x$
  - $\mathrm{cn}(x; k=1) = \mathrm{sech}\,x$
- periodicity ← K ← complete elliptic integral

Show cn.m

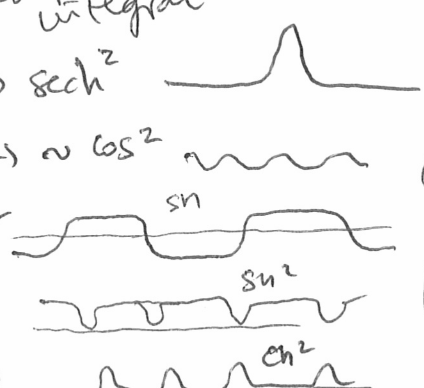
Remember phase space:



$k=1 \rightarrow \mathrm{sech}^2$

$|k| \ll 1 \rightarrow \sim \cos^2$

cn^2



KdV-ellipJ.m

Coefficients of cnoidal wave

Since we know that solution to KdV is cnoidal, let us force $u(x,t) = A + B\,\mathrm{cn}^2[D(x-ct-x_0); k]$ onto KdV and find restrictions/relations between params (A, B, D, c, k)

- We'll need diff. flas. for cn & sn: $\mathrm{sn}' = \mathrm{cn}\,\mathrm{dn}$, $\mathrm{cn}' = -\mathrm{sn}\,\mathrm{dn}$, $\mathrm{dn}' = -k^2\,\mathrm{cn}\,\mathrm{sn}$

$\therefore$ KdV: $u_t + 6uu_x + u_{xxx} = 0$

$$\Rightarrow \quad -cDB2(-\mathrm{sn})(\mathrm{dn})\mathrm{cn} + 6(A+B\mathrm{cn}^2)(BD(-\mathrm{sn})\mathrm{dn}\,\mathrm{cn}) + [BD(-\mathrm{sn})(\mathrm{dn})(\mathrm{cn})]_{xx} = 0$$

... gets messy.

Let us use Maple

kdv_elliptic_solutions

$$\ldots \Rightarrow \begin{cases} B = 2D^2k^2 \\ c = 6A + 6B - 4D^2(1+k^2) \end{cases}$$

<u>Ex</u>: $A=0$ & $k=1$ should recover sech soliton:

$$\begin{cases} B = 2D^2 \\ C = 6B - 8D^2 \end{cases} \Rightarrow C = 12D^2 - 8D^2 = 4D^2$$

$$\Rightarrow D = \frac{\sqrt{c}}{2} \Rightarrow B = 2D^2 = c/2$$

$\therefore$ $A=0$, $B = c/2$, $D = \sqrt{c}/2$, $k=0$ gives:

$$\boxed{u(x,t) = \frac{c}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct-x_0)\right]} \checkmark$$

other solutions (not traveling with const. shape)

~~(A) Similarity solutions~~
~~(B) Rational solutions~~

(A) Similarity solutions: suppose $u(x,t) = t^m f(\eta)$ and $\eta = x t^n$
$\rightarrow$ ie space and time linked
[see Drazin sec 2.6]

$u = -(3t)^{-2/3} f(\eta)$, $\eta = x\cdot(3t)^{-1/3} \overset{KdV}{\Longrightarrow} f''' + (6f-\eta)f' - 2f = 0$ $\leftarrow$ LINEAR!

(B) Rational solutions: assume $u(x,t) = f(x)$ [$c=0 \Rightarrow$ stationary]

$$\Rightarrow 6ff' + f''' = 0$$

$$\int d\xi \Rightarrow 3(f)^2 + f'' = A$$

Suppose u & derivative $\xrightarrow{x=\pm\infty} 0 \quad \Rightarrow A = 0 \Rightarrow 3f^2 + f'' = 0$ (1)

$$\times f' \Rightarrow 3f'f^2 + f'f'' = 0$$

$$\int d\xi \rightarrow (f)^3 + \tfrac{1}{2}(f')^2 = 0$$

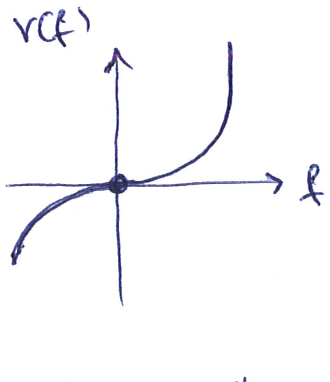
$$\Rightarrow f' = \pm\sqrt{-2f^3}$$

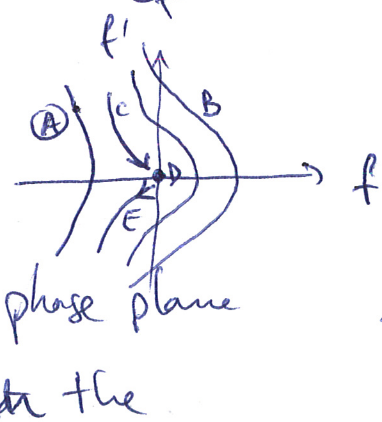
$$\Rightarrow \int \frac{df}{(-f)^{3/2}} = \pm\sqrt{2}\int d\xi \Rightarrow \frac{(-f)^{-1/2}}{-1/2} = \pm\sqrt{2}(\xi - x_0)$$

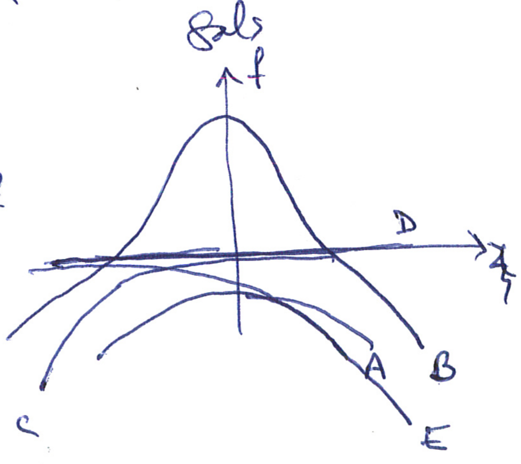
$$\Rightarrow -f = \frac{2}{(\xi-x_0)^2} \Rightarrow \boxed{f = \frac{-2}{(\xi-x_0)^2}} \quad (2)$$

Where does this sol. come from?

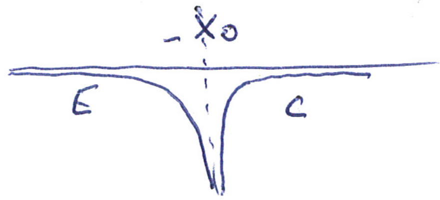
$$(1) \Rightarrow f'' = -3f^2 = -\frac{dV}{df}, \quad V(f) = f^3$$



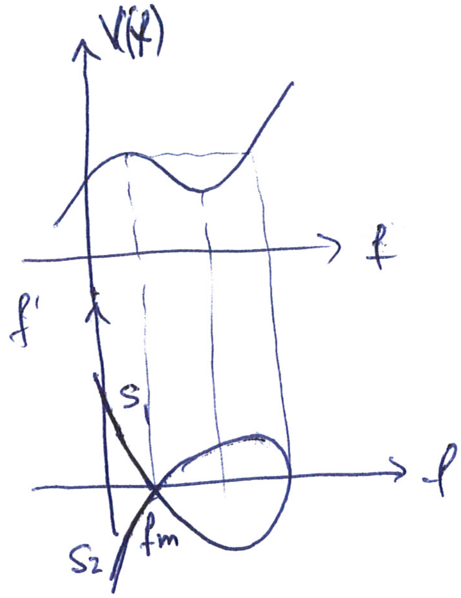


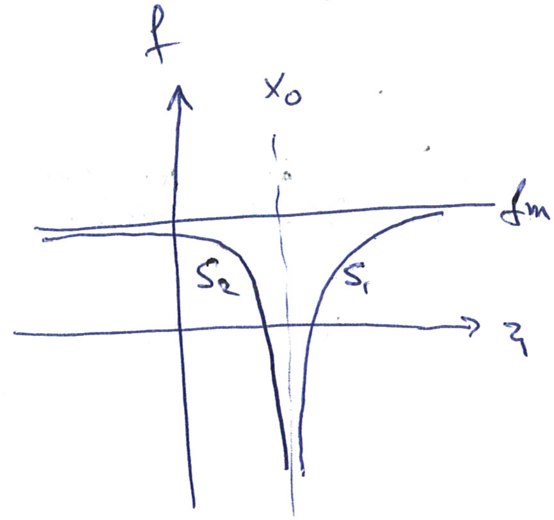


Solution (2) is nothing that the concatenation of E and C



there is a whole family of rational solutions when A & B const. are left:





Q: how can we obtain these genl. solutions using the c=0 solution $f(\zeta) = \frac{-2}{(\zeta-x_0)^2}$?

A: by Galilean invariance! → change background and velocity.

- these rational solutions are not physical due to their singularity.
- there are other rational solutions not included in above description.

skip

Breathing solutions (or bion)

Take the modified KdV eq = mKdV with $N=2$ ($n=1 \Rightarrow$ standard KdV)

$$u_t + (N+1)(N+2)\,u^N u_x + u_{xxx} = 0$$

$u_t + 12u^2u_x + u_{xxx} = 0$
⇒ rescale $u = u/\sqrt{2}$ ⇒

$$u_t + 6u^2u_x + u_{xxx} = 0$$

transform using $u = v_x$ & $\phi = \tan(\frac{1}{2}v)$
⇒ $(1+\phi^2)(\phi_t+\phi_{xxx}) + 6\phi_x(\phi_x^2 - \phi\phi_{xx}) = 0$
⇒ same tricks as in HW7

The following is a solution:

$$u(x,t) = -2\frac{\partial}{\partial x}\Big[\tan^{-1}\Big(\frac{l}{k}\underbrace{\sin(kx+mt+a)}_{\text{traveling sin}}\;\underbrace{\operatorname{sech}(lx+nt+b)}_{\text{traveling sech}}\Big)\Big]$$

$k, l, a, b \in \mathbb{R}$
with $m = k(k^2-3l^2)$ &
$n = l(3k^2-l^2)$

Show
mKdV_breather_m

Two soliton EXACT solution

types of sols

→ more on how to obtain these later.

KdV : $u_t + 6uu_x + u_{xxx} = 0$

⇒ 2 soliton solution

$$u(x,t) = 2\,\frac{k_1^2 E_1 + k_2^2 E_2 + 2(k_2-k_1)^2 E_1E_2 + A_2(k_2^2E_1 + k_1^2E_2)E_1E_2}{(1+E_1+E_2+A_2E_1E_2)^2}$$

where $E_i = e^{(k_i x - k_i^3 t + \alpha_i)}$, $k_i, \alpha_i \in \mathbb{R}$; $A_2 = \dfrac{(k_2-k_1)^2}{(k_1+k_2)^2}$

In particular : $u(x,t) = -12\,\dfrac{3 + 4\cosh(2x-8t) + \cosh(4x-64t)}{[3\cosh(x-28t) + \cosh(3x-36t)]^2}$

$t \to -\infty$: $u(x,t) \to +8\,\mathrm{sech}^2[2(x-16t \boxed{+} \tfrac{1}{4}\log 3)] + 2\,\mathrm{sech}^2(x-4t \boxed{-} \tfrac{1}{2}\log 3)$

$t \to +\infty$: $u(x,t) \to$ $\boxed{-}$ $\boxed{+}$

∴ the fast soliton ($v=16$) is advanced by $2 \times \frac{1}{4}\log 3 = \frac{1}{2}\ln 3$
and the slow soliton ($v=4$) is retarded by $2 \times \frac{1}{2}\log 3 = \ln 3$

Show Matlab
KdV_2sol.m

→ show figs for more solitons from [Shen]

Conservation laws for KdV

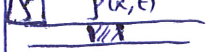
<u>Ex</u> : if $\rho(x,t)$ = density of gas
$u(x,t)$ = velocity of gas (in x-direction)

continuity eq.

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\underbrace{\rho u}_{\text{flux}}) = 0 \quad (1)$$

continuity eq:

ρ $\rho(x,t)$



dx

change $dm_1 = \rho_t\, dt\, dx$

flux u

x x+dx

chang $dm_2 = [u(x+dx)\rho(x+dx) - u(x)\rho(x)]\,dt$

$dm_1 + dm_2 = 0$

$\Rightarrow \rho_t\, dt\, dx + [u(x+dx)\rho(x+dx) - \rho u(x)]\,dt = 0$

$\Rightarrow \rho_t + \dfrac{\rho u(x+dx) - \rho u(x)}{dx} = 0$

$\rho_t + (\rho u)_x = 0$

If gas is confined ⇒ $\rho(x=\pm\infty) = 0$

$\Rightarrow \int (1)\,dx \Rightarrow \dfrac{d}{dt}\int_{-\infty}^{\infty} \rho\, dx + \cancel{[\rho u]_{-\infty}^{+\infty}}^{\,0} = 0 \Rightarrow \int_{-\infty}^{\infty} \rho\, dx = \text{const.}$

conservation of mass!

KdV: Let us construct conservation laws for KdV, ie. $\frac{d}{dt}F(u, u_x, u_t, u_{xt}, \dots) = 0$

$$u_t + 6uu_x + u_{xxx} = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\underbrace{\left[3u^2 + u_{xx}\right]}_{\text{flux}} = 0$$

$\therefore$ use same trick as for the gas

$$\int dx \Rightarrow \quad \frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx + \left[3u^2 + u_{xx}\right]_{-\infty}^{\infty} = 0$$

$\therefore$ if solution decays @ $\pm\infty$, $\pi\,(\partial_x)^n u(\pm\infty) = 0$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} u(x,t)\,dx = \text{const.}}$$ conservation of mass (water)

⚠ if $u(x,t)$ is ~~...~~ periodic in x then the integral for mass can be done on a single period (otherwise $\int_{-\infty}^{\infty} = \infty$)

More const. of motion: massage KdV and integrate.

KdV: $\quad u_t + 6uu_x + u_{xxx} = 0$

$\times u \quad uu_t + 6u^2u_x + \underset{f}{u}\,\underset{g'}{u_{xxx}} = 0$

parts: $\int fg' = \int[(fg)' - f'g] = \int[(uu_{xx})_x - u_xu_{xx}] = uu_{xx} - \frac{u_x^2}{2}$

$$\int \Rightarrow \int \frac{(u^2)_t}{2}\,dx + \left[2u^3 + uu_{xx} - \frac{u_x^2}{2}\right]_{-\infty}^{+\infty} = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} u^2(x,t)\,dx + \left[2u^3 + \cancel{uu_{xx}} - \frac{u_x^2}{2}\right]_{-\infty}^{\infty} \to 0 = 0$$

[if u decays to zero; if not, just subtract the constant background: $\Leftrightarrow$ regularizing the $\int$'s]

$$\therefore \boxed{\int_{-\infty}^{\infty} u^2(x,t)\,dx = \text{const}}$$ conservation of momentum.

physics: $\underbrace{p}_{\text{momentum}} = mv$, here $v \propto u$ and $u \propto$ mass $\quad \therefore p \propto u^2$

Another: Energy: $3u^2(\text{KdV}) + u_x\frac{\partial}{\partial x}(\text{KdV}) = 0 \Rightarrow$ HW (because $u_t + 6uu_x + u_{xxx} = 0$)

$$\boxed{\int_{-\infty}^{\infty}\left(\underset{\uparrow \text{ potential}}{u^3} - \underset{\uparrow \text{ kinetic}}{\frac{1}{2}u_x^2}\right)dx = \text{const}}$$ conservation of Energy

⚠ It is possible to prove that KdV has an ∞ # of independent constants of motion/conservation laws

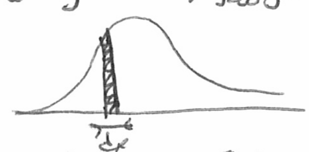
⚠ conservation laws very useful:

* project solutions onto trial functions
* check that numerics are doing ok.

10/2/08

Center of Mass (not a const. of motion)

- Center of mass of distribution $\rho(x)$: $x_c = \int_{-\infty}^{\infty} x\, \rho(x)\, dx \Big/ \int_{-\infty}^{\infty} \rho(x)\, dx$
- for us: $x_c(t) = \int_{-\infty}^{\infty} x\, u(x,t)\, dx \Big/ \int_{-\infty}^{\infty} u\, dx$



$\sum x \underbrace{f(x)\, dx}_{m_i} = \sum x\, m_i$ = center of mass

- let us follow x_c for KdV:

$$u_t + 6uu_x + u_{xxx} = 0$$

$$\Rightarrow u_t + 3(u^2)_x + u_{xxx} = 0$$

$$\Rightarrow u_t + \partial_x [3u^2 + u_{xx}] = 0$$

$(uv)' = u'v + uv'$
$uv' = (uv)' - u'v$

$$\cdot x \quad \Rightarrow x u_t + \underset{u}{x}\, \underset{v'}{\partial_x [\;\;]} = 0$$

$$\Rightarrow (xu)_t + [3xu^2 + x u_{xx}]_x - [3u^2 + u_{xx}] = 0$$

$$\int_{-\infty}^{\infty} dx \Rightarrow \frac{d}{dt}\int_{-\infty}^{\infty} x u\, dx + \overset{0}{\cancel{[3xu^2 + x u_{xx}]_{-\infty}^{\infty}}} - 3\int u^2 - \overset{0}{\cancel{[u_x]_{-\infty}^{\infty}}} = K_1^{*}$$

If u decays fast enough $xu^2, x u_{xx} \xrightarrow{x \to \pm\infty} 0$ $\Rightarrow \frac{d}{dt}\int xu\, dx = \underbrace{K_1 + 3\int u^2}_{\text{const.}}$

$\Rightarrow \frac{d}{dt}\left[\frac{\int xu}{\int u}\right] = K_2$

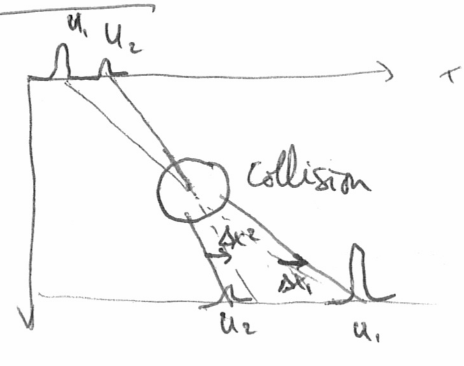
$$\rightarrow \frac{d}{dt}[x_c(t)] = K_2^{*} \quad \therefore \quad x_c(t) = K_2 t + K_3$$

$$\therefore \boxed{x_c(t) = V_c\, t + x_0} \leftarrow \text{CoM moves @ const. speed!}$$

9/27/08

Application to ~~phys~~ soliton collisions

2-soliton solution:



Before collision:
$$u_1 \simeq \frac{v_1}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{v_1}}{2}(x - v_1 t - x_1)\right]$$
$$u_2 \simeq \frac{v_2}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{v_2}}{2}(x - v_2 - x_2)\right]$$

After collision: suppose u_1 is displaced by Δx_1:
$$u_1 \simeq \frac{v_1}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{v_1}}{2}(x - v_1 t - x_1 - \Delta x_1)\right]$$
$\Rightarrow$ how much is u_2 displaced? ie $\Delta x_2 = \Delta x_2(\Delta x_1)$
$$u_2 \simeq \frac{v_2}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{v_2}}{2}(x - v_2 t - x_2 - \Delta x_2)\right]$$

center of mass:

before: $\int x u\,dx = At + B \Rightarrow \int x u_1 + \int x u_2 = At + B$ ① $(u \simeq u_1 + u_2)$

but: $\int_{-\infty}^{\infty} x\,\mathrm{sech}^2 a(x-b)\,dx \xrightarrow[a>0,\,b\in\mathbb{R}]{\text{Maple}} 2b/a$ {show assume trick in Maple}

$\therefore \int x u_1 = \frac{v_1}{2}\, 2\,(v_1 t + x_1)\cdot\frac{2}{\sqrt{v_1}} = 2\sqrt{v_1}\,(v_1 t + x_1)$

① $\Rightarrow$ $2\sqrt{v_1}(v_1 t + x_1) + 2\sqrt{v_2}(v_2 t + x_2) = At + B$

After: $2\sqrt{v_1}(v_1 t + x_1 + \Delta x_1) + 2\sqrt{v_2}(v_2 t + x_2 + \Delta x_2) = At + B$

$\ominus$ $-\cancel{2}\sqrt{v_1}\,\Delta x_1 - \cancel{2}\sqrt{v_2}\,\Delta x_2 = 0 \Rightarrow \boxed{\frac{\Delta x_2}{\Delta x_1} = -\sqrt{\frac{v_1}{v_2}}}$

For the 2-sol. solution that we saw:

$v_1 = 16$, $\Delta x_1 = \frac{1}{2}\ln 3$

$v_2 = 4$, $\Delta x_2 = -\sqrt{\frac{v_1}{v_2}}\,\Delta x_1 = -\sqrt{4}\,\frac{1}{2}\ln 3 = -\ln 3$ ✓

⚠ center of mass need to move at const. speed even DURING collision.

5.1.2 More on Conservation Laws : ∞ # !

KdV: $u_t - 6uu_x + u_{xxx} = 0$ [note I am using "−" as in Drazin's book]

+ Miura transf: $u = v^2 + v_x$
+ another transf: $v = \frac{1}{2}\varepsilon^{-1} + \varepsilon w$ $\qquad \varepsilon \in \mathbb{R}$

$$\Rightarrow \quad u = \underbrace{\tfrac{1}{4}\varepsilon^{-2}}_{} + w + \varepsilon^2 w^2 + \varepsilon w_x$$

eliminate since we can always add through Galilean invariance

$$\therefore \quad \boxed{u = w + \varepsilon^2 w^2 + \varepsilon w_x} \quad \text{Gardner transf.}$$

$$\text{KdV} \xrightarrow[\cdots]{\text{Gardner}} \left(1 + \varepsilon \frac{\partial}{\partial x} + 2\varepsilon^2 w\right)\underbrace{\left[w_t - 6(w + \varepsilon^2 w^2) w_x + w_{xxx}\right]}_{\Downarrow = 0} = 0$$

$\therefore$ u KdV sol $\Rightarrow$ w sol of $\Downarrow = 0$

⚠ $\varepsilon = 0 \Rightarrow w = u$ and recover KdV

+ Now search for conserv. quant: $\varepsilon \neq 0$

$$\frac{\partial}{\partial t}(w) + \frac{\partial}{\partial x}\left(-3w^2 + 2\varepsilon^2 w^3 + w_{xx}\right) = 0$$

$$\int_{-\infty}^{\infty} dx \quad \Rightarrow \quad \frac{d}{dt}\left[\int_{-\infty}^{\infty} w\, dx\right] + \left[\;\right]_{-\infty}^{\infty} = 0$$

decay for w.

$\therefore$ $\int_{-\infty}^{\infty} w\, dx$ = const of motion.

+ since $w \xrightarrow{\varepsilon \to 0} u$ we can expand w by

$$w = \sum_{n=0}^{\infty} \varepsilon^n w_n \qquad \text{when } |\varepsilon| \ll 1$$

~~and~~ we need $\int w\, dx = \text{const} \Rightarrow \int w_n = \text{const}$ since ε is arbitrary.

+ use Gardner transf: $u = w + \varepsilon^2 w^2 + \varepsilon w_x$

+ expansion $\quad u = \sum \varepsilon^n w_n + \varepsilon^2 \left(\sum \varepsilon^n w_n\right)^2 + \varepsilon \sum \varepsilon^n w_{nx}$

$$\Rightarrow \sum \varepsilon^n w_n = u - \sum \varepsilon^{n+1} w_{nx} - \varepsilon^2\left(\varepsilon^0 w_0 + \varepsilon^1 w_1 + \varepsilon^2 w_2 + \cdots\right)^2$$

$$\Rightarrow \sum \varepsilon^n w_n = u - \sum \varepsilon^{n+1} w_{nx} - \varepsilon^2\left(w_0^2 + \varepsilon^2 w_1^2 + \cdots + 2\varepsilon w_0 w_1 + 2\varepsilon^3 w_1 w_2 + \cdots + 2\varepsilon^2 w_0 w_2 + \cdots\right)$$

ε^0 : $w_0 = u$

ε^1 : $w_1 = -w_{0x}$

ε^2 : $w_2 = -w_{1x} - w_0^2$

ε^3 : $w_3 = -w_{2x} - 2w_0 w_1$

ε^4 : $w_4 = -w_{3x} - w_1^2 - 2w_0 w_2$

$\vdots$

& since each $\int w_n = \text{const}$ we can solve progressively:

ε^0: $\boxed{\int u = ct}$ (Mass)

ε^1: $\int -u_x = ct$

ε^2: $\int + u_{xx} - u^2 = ct$ ~~(Energy)~~

ε^3: $\int -(u_{xx}-u^2)_x - 2u(-u_x) = ct$

ε^4: $\int [(u_{xx}-u^2)_x + 2uu_x]_x - (u_x)^2 - 2u(u_{xx}-u^2) = ct$

$\vdots$

- Note that all ε^{odd} are made up of exact differentials

$$\varepsilon^1: \int_{-\infty}^{\infty} u_x \, dx = [u]_{-\infty}^{\infty} = 0$$

$$\varepsilon^{odd}: \int_{-\infty}^{\infty} \frac{\partial}{\partial x}[f(u)]\, dx = [f(u)]_{-\infty}^{\infty} = 0$$

$\therefore$ ε^{odd} do NOT give any conservation laws

u decays fast enough.

- the others:
  - ε^2: $\int \frac{\partial}{\partial x}[u_x] - u^2 = ct \Rightarrow \cancel{[u_x]}_{-\infty}^{\infty} \overset{0}{} - \boxed{\int u^2 = ct}$ (Momentum)
  - ε^4: $\cancel{[-(u_{xx}-u^2)_x \dots]}_{-\infty}^{\infty} \overset{0}{} + \int u_x^2 + 2uu_{xx} + 2u^3 = ct$

$$\Rightarrow \int u_x^2 - 2\frac{\partial}{\partial x}(uu_x) + 2u_x^2 + 2u^3 = ct$$

$$\Rightarrow \cancel{[uu_x]}_{-\infty}^{\infty} \overset{0}{} + \int u_x^2 + 2u^3 = ct$$

$$\Rightarrow \div 2 \quad \boxed{\int \frac{u_x^2}{2} + u^3 = ct} \text{ (Energy)}$$

It can be shown that ε^{2k} has terms up to u^{k+1} and thus indep. from lower order $[\varepsilon^{2(k-1)}]$ $\Rightarrow$ ∞ # of terms $\Rightarrow$ ∞ conservation laws !!!

Recurrence fla for conservation laws

[D: p94-95]

~~$P^0 = u \Rightarrow \int_{-\infty}^{\infty} p^n dx = cte.$~~

~~$P^n = u + \varepsilon^2 [p^{n-1}]^2$~~

~~$n=0$: $\int u = ct$~~

~~$n=1$: $P^1 = u + \varepsilon^2 [P^0]^2 = u + \varepsilon^2 [u]^2$~~

~~$\therefore \int u + \varepsilon^2 \int u^2 = ct \Rightarrow \int u^2 = cte$~~

~~$n=2$: $P^2 = u + \varepsilon^2 [P^1]^2 = u + \varepsilon^2 [u + \varepsilon^2 u^2]^2$~~

~~$= u + \varepsilon^2 u^2 + 2\varepsilon^4 u^3 + \varepsilon^6 u^4$~~

~~$\therefore \int u + \varepsilon^2 \int u^2 + \varepsilon^4 \int 2u^3 + \varepsilon^6 \int \dots$~~

5.4 Bäcklund transformation (BT)

- consider 2 uncoupled, nonlinear, PDEs:

$$\begin{cases} P(u) = 0 \\ Q(v) = 0 \end{cases}$$

~~i.e. If v sol. of $R_i = 0$ ... $P(u) = 0$ & $Q(v) = 0$~~

If u, v such that $R_i = 0 \iff \begin{cases} P(u) = 0 \\ Q(v) = 0 \end{cases}$

- let R_1 & R_2: $\quad R_i(u, v, u_x, v_x, u_t, v_t, \dots ; x, t) = 0$
  be 2 relations such that if R_i is integrable for v when $P(u) = 0$ and v is a sol. to $Q(v) = 0$ and vice-versa $\Longrightarrow R_i = 0$ is the Bäcklund transf. of Eqs $P(u)$ & $Q(v)$.

[i.e. If u sol. to $P(u) = 0 \overset{R_i}{\Longrightarrow} v$ is a sol. to $Q(v) = 0$]

- If the 2 eqs are same: $P = Q$ then $R_i = 0$ are called auto-Bäcklund transf.
- Use: find one solution $\xrightarrow[\text{Bäcklund}]{\text{auto}}$ find another!

Ex:
- consider $\quad R_1: \quad u_x = v_y$
  $\qquad\qquad\quad R_2: \quad u_y = -v_x$
- and Laplace eq: $\quad u_{xx} + u_{yy} = 0$
  $\qquad\qquad\qquad\quad\; v_{xx} + v_{yy} = 0$

Bäcklund?: u sol: $u_{xx} + u_{yy} = 0$
~~$R_1, R_2 \Rightarrow v_{xx} + v_{yy} = 0$~~
then $v_{xx} + v_{yy} = (-u_y)_x + (u_x)_y = 0$ ✓

$\therefore$ yes it is a $\underbrace{\text{BT}}_{\text{auto-}}$-

- one solution $v(x,y) = xy$

$$\Rightarrow \begin{matrix} u_x = v_y = x & \Rightarrow & u = \frac{x^2}{2} + f(y) \\ u_y = -v_x = -y & \Rightarrow & u = -\frac{y^2}{2} + g(x) \end{matrix} \Bigg\} \; u = \underbrace{\tfrac{1}{2}(x^2 - y^2)}_{u_{xx} + u_{yy} = 0 \checkmark}$$

is another solution to Laplace Eq.

5.4.2 KdV: use KdV and Miura transf:

- KdV: $u_t - 6uu_x + u_{xxx} = 0 \quad (1)$
- Miura transf: $\quad u = v^2 + v_x$
  gral —"—: $\quad u = \lambda + v^2 + v_x \; (2)$ (Miura + Galilean invariance)

$$\to \text{modified KdV}: \boxed{v_t - 6(v^2 + \lambda)v_x + v_{xxx} = 0} \quad (3)$$

$\therefore$ if v sol to mKdV $\overset{\text{Miura}}{\iff}$ u sol to KdV

- Note that if v sol to (3) then $-v$ is also sol

$\therefore$ take (2): $\begin{cases} u_+ = \lambda + v^2 + v_x \\ u_- = \lambda + v^2 - v_x \end{cases}$

$$\Rightarrow u_+ + u_- = 2(\lambda + v^2) \; ; \; u_+ - u_- = 2v_x$$

~~• transf: $u_\pm = (w_\pm)_x \Rightarrow \begin{cases} (w_+ + w_-)_x = 2\lambda \\ w_+ - w_- = 2v + c_2 \end{cases}$~~

55 ~~54~~ 32 4/5

• transf: $U_\pm = (W_\pm)_x$ (3.5) $\Longrightarrow \begin{cases} W_+ - W_- = 2v + C_2 \\ \text{and } (W_+ + W_-)_x = 2\lambda + 2v^2 \end{cases}$

$\Rightarrow (W_+ + W_-)_x = 2\lambda + \frac{1}{2}(W_+ - W_- - C_2)^2$

choose $C_2 = 0$ $\begin{cases} W_+ - W_- = 2v & (4a) \\ (W_+ + W_-)_x = 2\lambda + \frac{1}{2}(W_+ - W_-)^2 & (4b) \end{cases}$ (4)

• transf modified KdV using $(U_\pm) = (W_\pm)_x$ & (4)

(3) with (2.5) & (4) $\Longrightarrow (W_+ - W_-)_t - 3(W_{+x}^2 - W_{-x}^2) + (W_+ - W_-)_{xxx} = 0$ (5)

(4b) and (5) ~~(4)~~ are the auto-bäcklund transf. of the mKdV (5) [(4b) is the "x" transf. & (5) is the "t" transf.]

• Find sols of KdV   10/21/08

– give a 1st sol: trivial sol: $W = 0$ $u=0 \Rightarrow W_x = 0 \rightarrow W = \text{cont}$

– use $W_- = 0 \overset{(4b)}{\Longrightarrow} (W_+ + 0)_x = 2\lambda + \frac{1}{2}(W_+ - 0)^2$

$\Longrightarrow W_{+x} = 2\lambda + \frac{1}{2}W_+^2$ (6) $k \equiv \sqrt{-\lambda}$

~~$\Rightarrow W_+ = 2\lambda(1+(W_+)^2)$~~

$\int dx$ $\Longrightarrow W_+(x,t) = -2k \tanh[kx + f(t)]$ $\left[|W_+| < 2k \right.$ If $|W_+| > 2k \Rightarrow \tanh \to \coth \left. \right]$

– For $f(t)$ we need (5) to be satisfied:

~~$W_{+t} - 3W_{+x}^2 + W_{+xxx} = 0$~~

~~but $W_{+xxx} \overset{(6)}{=} \frac{\partial}{\partial x}(W_{+xx}) = \frac{\partial}{\partial x}(W_+ W_{+x}) = W_{+x}W_{+x} + W_{+x}W_{+xx}$~~

~~$\Rightarrow W_{+t}' - 3W_{+x}^2 + W_{+x}^2 + W_x W_{+xx} = 0$~~

~~$\Rightarrow W_{+t} - 2W_{+x}(W_{+x} - \frac{1}{2}W_{+xx}) = 0$~~

plug $W_+ = \tanh$ into (5) and solve for $f(t)$ ...

$\Longrightarrow f(t) = -4k^3 t - kx_0$ $k \equiv \sqrt{-\lambda}$

$\therefore W_+(x,t) = -2k \tanh[k(x - x_0 - 4k^2 t)]$

is another solution to mKdV.

– remember $U_+ = (W_+)_x$ $(\tanh' = \operatorname{sech}^2)$

$\Rightarrow \boxed{U_+ = -2k^2 \operatorname{sech}^2[k(x - x_0 - 4k^2 t)]}$ soliton sol. of KdV

$\therefore$ Vacuum $\xrightarrow[BT]{\text{Bäcklund}}$ 1 soliton solution   vel $= 4k^2$

This process could be applied as many times as we want to produce more solutions. Everytime we do this we need one x-integration and one t-integration $\longrightarrow$ there is a more elegant way:

5.4.3 the KdV Bäcklund transf: an algebraic relation

- Suppose that one starts with a solution w_0 and one builds two solutions using the BT for $2 \neq \lambda$'s: $\lambda_1 \to w_1$, $\lambda_2 \to w_2$
- One can write:

$$\text{4b} \Rightarrow \begin{cases} (w_1+w_0)_x = 2\lambda_1 + \frac{1}{2}(w_1-w_0)^2 \\ (w_2+w_0)_x = 2\lambda_2 + \frac{1}{2}(w_2-w_0)^2 \end{cases} \quad (6)$$

- now construct another 2 solutions: w_{12} from w_1 and λ_2, w_{21} from w_2 and λ_1

$$\begin{cases} (w_{12}+w_1)_x = 2\lambda_2 + \frac{1}{2}(w_{12}-w_1)^2 \\ (w_{21}+w_2)_x = 2\lambda_1 + \frac{1}{2}(w_{21}-w_2)^2 \end{cases} \quad (7)$$

- It turns out, Bianchi's theo of permutability, that $w_{12} = w_{21}$ this is because

$$\begin{array}{ccccc} & & w_1 & & \\ & \overset{\lambda_1}{\nearrow} & & \overset{\lambda_2}{\searrow} & w_{12} \\ w_0 & & & & \parallel \\ & \underset{\lambda_2}{\searrow} & & \underset{\lambda_1}{\nearrow} & w_{21} \\ & & w_2 & & \end{array}$$

$\text{vel} = 4k^2 = -4\lambda$

$\lambda_1 \neq \lambda_2$

(6) $w_0 \,\&\, \lambda_1 \to w_1$, $w_0 \,\&\, \lambda_2 \to w_2$ $\longrightarrow w_{12} \neq 0$

- Do $[(7a)-(7b)] - [(6a)-(6b)]$ and solve for w_{12}:

$$\boxed{w_{12} = w_0 - \frac{4(\lambda_1-\lambda_2)}{w_1-w_2}}$$

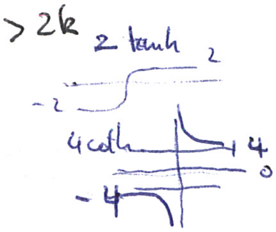
this gives a sort of nonlinear superposition principle if one takes ~~w_0 and λ_1 and λ_2 to be the desired velocities~~ $w_0 = 0$ & $\underbrace{\text{vel}_1 = 4k_1^2}_{w_1}$, $\underbrace{\text{vel}_2 = 4k_2^2}_{w_2}$ $\Longrightarrow w_{12}$

- Ex: $\text{vel}_1 = 4 \Rightarrow k_1 = 1 \Rightarrow \lambda_1 = -1$
  $\text{vel}_2 = 16 \Rightarrow k_2 = 2 \Rightarrow \lambda_2 = -4$

Remember (6) $\Rightarrow w_{\frac{1}{2}x} = 2\lambda + \frac{1}{2} w_{\frac{1}{2}}^2$

$$\Longrightarrow \begin{cases} w_1 = -2k \tanh k(x-x_0-4k^2t) & \text{if } |w_1| < 2k \\ w_2 = -2k \coth k(x-x_0-4k^2t) & \text{if } |w_2| > 2k \end{cases}$$

$\therefore$ take $\begin{cases} w_1 = -2\tanh(x-4t) & k_1 = 1 \longleftarrow \text{non-singular} \\ w_2 = -4\coth(2x-32t) & k_2 = 2 \longleftarrow \text{singular} \end{cases}$



$$\Longrightarrow w_{12} = 0 - \frac{4(-1+4)}{-2t+4c} = \frac{6}{2\coth - \tanh}$$

and $u_{12} = (w_{12})_x$ = the 2-sol solution that we have written a few times. (p25 in my notes) $\longleftarrow$ non-singular.