$\boxed{\text{Envelope waves}}$ (for linear waves)

see "group velocity Wiki" on www.

- envelope vel = modulation vel = group vel $\equiv v_e$
- carrier vel = phase vel = v_c

- $v_e \neq v_c$ in general. (They can even be opposite ~~[illegible]~~ in sign) $\to$ "group velocity" in Wiki.

Dispersion relation:

$k(x - \overset{v_c}{\overbrace{\tfrac{\omega}{k}}}\, t)$

- $u = A e^{i(kx - \omega t)} \Rightarrow \boxed{v_c = \dfrac{\omega(k)}{k}}$
- what about group vel?

* Suppose a wave packet 

$t = 0 \quad u(x,0) = \sum a_n e^{ikx}$ [Fourier series]

$\Rightarrow t > 0 \quad u(x,t) = \sum a_n e^{i(kx - \omega(k) t)}$ [disp. rel.]

* Suppose wave has mostly one k (i.e. almost monocromatic) $\to k_0$

$\Rightarrow \quad \omega(k) \approx \underset{\leftarrow\, \omega(k_0)}{\omega_0} + (k - k_0)\, \omega'(k_0) \qquad \omega' = \dfrac{d\omega}{dk}$

$$\Rightarrow u(x,t) = \sum a_n e^{i(kx - \omega_0 t - (k - k_0)\omega_0' t)}$$
$$= e^{it(k_0 \omega_0' - \omega_0)} \sum a_k e^{ik(x - \omega_0' t)}$$

Look at the magnitude of this wave:

$$|u(x,t)| = \underbrace{|e^{it(k_0\omega_0' - \omega_0)}|}_{1}\ \underbrace{\left|\sum a_k e^{ik(x - \omega_0' t)}\right|}_{|u(x - \omega_0' t, 0)|}$$

$\therefore$ the magnitude travel at vel $= v_e = \omega_0' t = \left.\dfrac{d\omega}{dk}\right|_{k=k_0}$

$\therefore \boxed{v_e = \dfrac{\partial \omega}{\partial k}}$ [valid for linear waves with wave packets]

Nonlinear Schrödinger Eq (NLS)

- NLS is a generic eq. : lowest order nonlinear wave eq. for envelopes ~~of~~ carrier waves.

- As we saw before for linear eqs it is useful to express wave as combination of plane waves (ie. Fourier modes):

$$u(x,t) = \sum a_k e^{i(kx-\omega t)}$$

- Dispersion: $\omega = \omega(k)$, ie: $\neq k$ travel @ different speeds.

- + envelope (or modulation or group) velocity:

$$v_e = \frac{\partial \omega(k)}{\partial k}$$

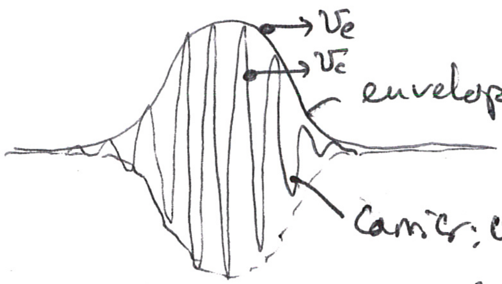
  + Carrier (or phase) velocity

$$v_c = \frac{\omega}{k} \qquad [(kx-\omega t) = k(x - v_c t)]$$

  + $v_e = v_c \iff \omega \propto k$

- We want an eq. that describes:

show Matlab wave-packet.m

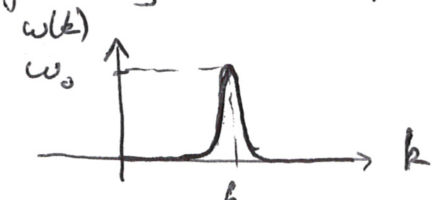


ie: $u \sim$ plane wave $*$ envelope

Maybe first show slide for this

- Such a wave has the following Fourier spectrum:

FT(u) :



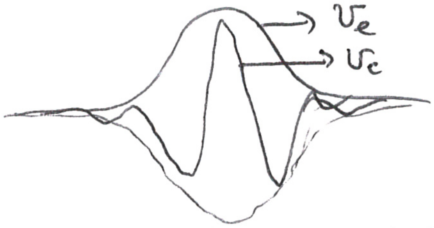
show Matlab Fourier-packet.m

- Let us choose a simple dispersion relation that behaves like this:
  expand about k_0: $\omega(k) = \omega_0 + b_1(k-k_0) + b_2(k-k_0)^2 + \cdots$

$$u(x,0) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, e^{ikx}$$

$$\Rightarrow \quad u(x,t) \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, e^{i[kx-\omega_0 t - b_1(k-k_0)t - b_2(k-k_0)^2 t]}\,dk$$

$$\simeq \underbrace{\frac{e^{i(k_0 x-\omega_0 t)}}{2\pi}}_{\substack{\text{plane wave} \\ @\ v_c = \omega_0/k_0}} \underbrace{\int_{-\infty}^{\infty} F(k)\, e^{i[(k-k_0)x - b_1(k-k_0)t - b_2(k-k_0)^2 t]}\,dk}_{\text{envelope: } \phi(x,t)}$$



Show matlab wave-packet

In general: $v_c \neq v_e$

- Let us follow envelope: $\phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(K+k_0)\, e^{i(Kx - b_1Kt - b_2K^2t)}\, dK$
  $K \equiv k - k_0$

$$\Rightarrow \frac{\partial \phi}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} -i(b_1K + b_2K^2)\, F(K+k_0)\, e^{i(\,)}\, dK$$

$$= -\frac{b_1}{2\pi}\int_{-\infty}^{\infty} iK\, F(K+k_0) e^{i(\,)}\, dK + \frac{ib_2}{2\pi}\int_{-\infty}^{\infty} \underbrace{-K^2}_{(iK)^2} F(K+k_0)\, e^{i(\,)}\, dK$$

$$\therefore \frac{\partial \phi}{\partial t} = -b_1 \frac{\partial \phi}{\partial x} + i b_2 \frac{\partial^2 \phi}{\partial x^2} \qquad \left[\mathcal{F}[f^{(n)}(x)] = (ik)^n \mathcal{F}[f(x)]\right]$$

$$\boxed{\phi_t = -b_1 \phi_x + i b_2' \phi_{xx}}$$ (1) Evolution eq. for pulse envelope.

- The above was linear, let us now add a nonlinear term in dispersion relationship:

+ ~~Without~~ Dispersion: $\phi = A e^{i(kx - \omega t)}$

$$\overset{(1)}{\Longrightarrow} -iA\omega \cancel{e^{()}} = -ib_1 k \cancel{A e^{()}} - i b_2' k \cancel{A e^{()}}$$

$$\Rightarrow \boxed{\omega = b_1 k + b_2' k^2}$$ what we started with.

+ let us add a dependence on envelope amplitude.

since  ϕ and  $-\phi$ should give rise to same wave packet, we need to add something that does not depend on sign. The lowest nonlinear term that achieves this is $|\phi|^2$.

β (+ or −) depending on response.

$$\therefore \boxed{\omega = b_1 k + b_2' k^2 + \beta|\phi|^2}$$ (2) lowest order nonlinearity added.

Here, dispersion also depends on amplitude of envelope.

- Find eq. whose dispersion rel. behaves like (2):

$$i(\phi_t + b_1 \phi_x) + b_2' \phi_{xx} - \beta|\phi|^2 \phi = 0$$

Check: $\phi = A e^{i(kx - \omega t)} \Rightarrow \left[i(-i\omega + ib_1 k) - b_2 k^2 - \beta|\phi|^2\right] A e^{i()} = 0$

$$\Rightarrow \omega - b_1 k - b_2 k^2 - \beta|\phi|^2 = 0 \quad ✓$$

- Now perform change of variables:

$$\psi(\xi,\tau) = \phi(x,t) \qquad \xi = x - b_1 t, \quad t = \tau \qquad \text{(same trick as the)}$$

$$\Rightarrow \quad \psi_t = \psi_\xi \frac{\partial \xi}{\partial t} + \psi_\tau \frac{\partial \tau}{\partial t} = -b_1 \psi_\xi + \psi_\tau$$

$$\psi_x = \psi_\xi \frac{\partial \xi}{\partial x} + \psi_\tau \cancel{\frac{\partial \tau}{\partial x}}^{\,0} = \psi_\xi$$

$$\therefore \quad i\left[-\cancel{b_1 \psi_\xi} + \psi_\tau + \cancel{b_1 \psi_\xi}\right] + b_2^2 \psi_{\xi\xi} - \beta |\psi|^2 \psi = 0$$

- Adimensionalize: $\psi = A u$; $\xi = \sqrt{2 b_2}\, x$: $\partial_\xi = \frac{1}{\sqrt{2b_2}} \partial_x$

$$\Rightarrow \quad i A u_\tau + \frac{A}{2} u_{xx} - \beta A^3 |u|^2 u = 0$$

$$\Rightarrow \quad i u_\tau + \frac{1}{2} u_{xx} - \beta A^2 |u|^2 u = 0$$

choose $A^2 = 1/|\beta| \Rightarrow -\beta A^2 = -\mathrm{sign}(\beta)$ — sign(β)

$$\therefore \quad \boxed{i u_t + \frac{1}{2} u_{xx} \pm |u|^2 u = 0}$$ Nonlinear Schrödinger Eq.

NLS

time dep., cubic, nonlinear, Schr. eq.

(+) Attractive/focusing nonlinearity $(\beta < 0)$

(−) Repulsive/defocusing —"— $(\beta > 0)$

Dispersion relation

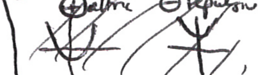
$u = e^{i(kx - \omega t)}$

$\Rightarrow i(-i\omega)\cancel{\phi} + \frac{1}{2}(ik)^2 \cancel{\phi} \pm \phi = 0$

$\Rightarrow \omega = \frac{1}{2}k^2 \mp 1$

$\therefore u_{xx} = $ dispersion

⊕ attractive ⊖ repulsive



- NLS describes propagation of an envelope wave riding over a carrier wave.
- u_{xx} = dispersion [not dissipation because $\underline{\underline{i}} u_t$]
- $|u|^2 u$: lowest order nonlinearity.

$\therefore$ NLS = generic eq. describing carrier wave with dispersive and nonlinear effects (No dissipation).

Applications:
- Hydrodynamic wave on deep water
- Heat pulses in solids
- "Langmuir" waves in plasmas
- Nonlinear waves in fluid-filled viscoelastic tube
- monochromatic waves is fibers and other optical nonlinear media {NLS follows from Maxwell's eq.
- Bose-Einstein condensation.
- Waves in piezoelectric semi-conductors

- $u \in \mathbb{C}$ so, for most depictions, we'll use $|u|^2$.
- $u = f(x - ct)$ cannot work because $\underline{\underline{i}} u_t$ [and thus speed ~~has to~~ would be complex]

$\rightarrow$ nonetheless if we decompose u into envelope and carrier waves the travelling ansatz works.

Solitons in NLS

$$\boxed{iu_t + \tfrac{1}{2}u_{xx} \pm |u|^2 u = 0} \quad 1\times\mathbb{C}\ \text{PDE}$$

$$u = u(x,t) = f(x,t)\, e^{ig(x,t)} \qquad f, g \in \mathbb{R}, \quad f: \text{envelope}, \quad g: \text{carrier}.$$

$\Rightarrow$
- $u_x = f_x e^{ig} + i f g_x e^{ig}$
- $u_{xx} = f_{xx} e^{ig} + i f_x g_x e^{ig} + i(f_x g_x + f g_{xx}) e^{ig} - f g_x^2 e^{ig}$
- $u_t = f_t e^{ig} + i f g_t e^{ig}$
- $|u|^2 u = f^3 e^{ig}$

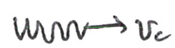
$\therefore$ NLS $\Rightarrow e^{ig}\left[i f_t - f g_t + \tfrac{1}{2}\left(f_{xx} + 2i f_x g_x + i f g_{xx} - f g_x^2\right) \pm f^3 \right] = 0$

$$\Rightarrow \boxed{\begin{array}{ll} \mathbb{R} & \left\{ f_t + f_x g_x + \tfrac{1}{2} f g_{xx} = 0 \right. \\ \mathbb{I} & \left\{ -f g_t + \tfrac{1}{2} f_{xx} - \tfrac{1}{2} f g_x^2 \pm f^3 = 0 \right. \end{array}} \quad 2\times\mathbb{R}\ \text{PDE's}$$

Now suppose traveling waves in BOTH carrier and envelope:

$$f(x,t) = \tilde{f}(x - v_e t) = \tilde{f}(\eta)$$

$$g(x,t) = \tilde{g}(x - v_c t) = \tilde{g}(\eta)$$



drop ~'s $\Rightarrow$

$$\begin{cases} -v_e f' + f' g' + \tfrac{1}{2} f g_{xx} = 0 & \text{(a)} \\ +v_c f g' + \tfrac{1}{2} f'' - \tfrac{1}{2} f (g')^2 \pm f^3 = 0 & \text{(b)} \end{cases}$$

We need to solve this system of coupled ODEs (with appropriate BCs) to find solitons:

- (a)$\cdot f$: $-v_e f f' + f f' g' + \tfrac{1}{2} f^2 g'' = 0$

$\Rightarrow -\frac{v_e}{2}(f^2)' + \tfrac{1}{2}(f^2)' g' + \tfrac{1}{2} f^2 g'' = 0$

$\times 2 \Rightarrow -v_e (f^2)' + [f^2]'[g'] + [f^2][g']' = 0$

$\Rightarrow -v_e (f^2)' + (f^2 g')' = 0 \overset{\int}{\Rightarrow} -v_e f^2 + f^2 g' = A$ $\qquad \eta = x - v_e t$

As before, we want decay in envelope $\Rightarrow f(\pm\infty) = 0 \Rightarrow A = 0$

$\therefore f^2(-v_e + g') = 0 \overset{f \neq 0}{\Rightarrow} \boxed{g' = v_e} \Rightarrow \boxed{g(\eta) = v_e \eta + \phi_0}$

- 2(b) : $2 v_c v_e f + f'' - f v_e^2 \pm 2 f^3 \Rightarrow \boxed{f'' = v_e f (v_e - 2v_c) \mp 2 f^3}$ (d)

$$= -\frac{d}{df}\left[-\frac{v_e}{2} f^2 (v_e - 2v_c) \pm \frac{1}{2} f^4 \right]$$

$\therefore m f'' = -\frac{dV}{df}$ with $m = 2$ & $V(f) = f^2\left[v_e(2v_c - v_e) \pm f^2\right]$

or $V(f) = \pm f^4 - v_e(v_e - 2v_c) f^2$ $\qquad$ (+) focusing/attra. (−) defocusing/repul.

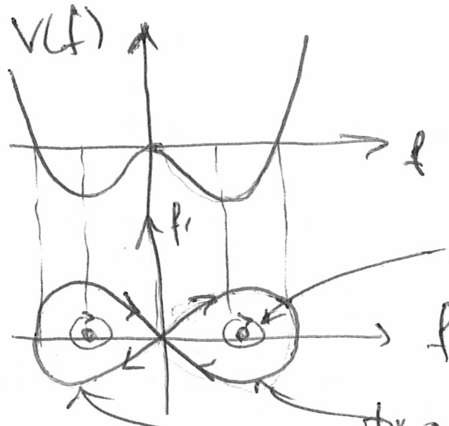
(+): Focusing nonlinearity:

$$V(f) = f^4 - v_e(v_e - 2v_c)\, f^2$$

- If $v_e - 2v_c < 0 \Rightarrow V(f) \sim f^4 + a f^2 = \sqcup + \cup = \cup$
  $\rightarrow$ No homoclinic orbits
- If $v_e - 2v_c > 0 \Rightarrow V(f) \sim f^4 - a f^2 = \sqcup + \cap = W$
  $\rightarrow$ yes!

Restriction : $\boxed{v_e > 2v_c}$ for focusing nonlinearity:

$\Rightarrow$

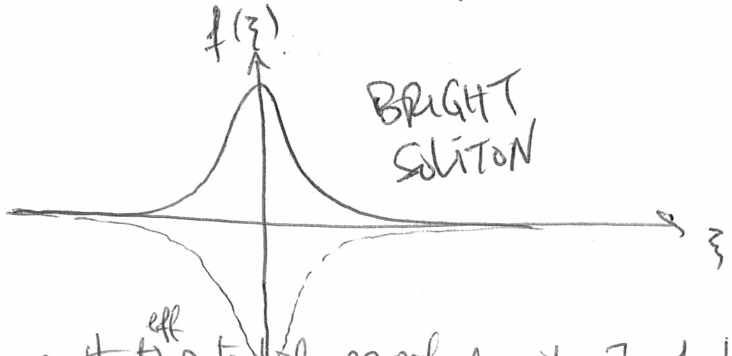


oscillatory profiles.

in between cn-oidal waves [KdV : $\operatorname{sech}^2, \operatorname{cn}^2, \cos^2$
NLS : $\operatorname{sech}, \operatorname{cn}, \cos$]

two homoclinic orbits

$\rightarrow$



BRIGHT SOLITON

Show Matlab
wave-packet.m
alpha = +1

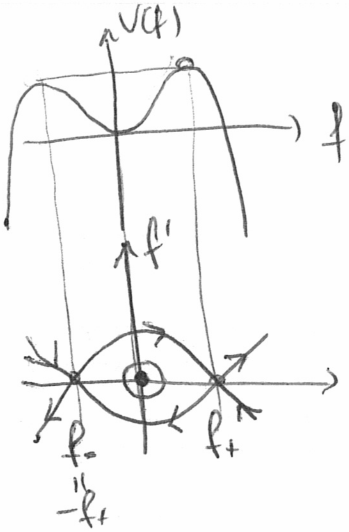
$\rightarrow$ discuss that eff potential approach provides $\exists$ of trav. sol. but it does not say anything about the stability !!!

(−): Defocusing nonlinearity

$$V(f) = -f^4 - v_e(v_e - 2v_c)\, f^2$$

- If $v_e - 2v_c > 0 \Rightarrow V(f) = -f^4 - af^2 = \pitchfork + \pitchfork$
  $\rightarrow$ No interesting orbits
- If $v_e - 2v_c < 0 \Rightarrow V(f) = -f^4 + af^2 = \pitchfork + \Psi = \pitchfork$
  $\rightarrow$ heteroclinic orbits.

Restriction: $\boxed{v_e < 2v_c}$



Show Matlab
Wave-packet.m , alpha = -1

Explicit solutions :

(*) $\displaystyle\int \frac{dx}{x\sqrt{b-ax^n}} = \frac{-2}{n\sqrt{b}} \tanh^{-1}\left[\frac{\sqrt{b-ax^n}}{\sqrt{b}}\right]$

(d) : $f'' - v_e f\,(v_e - 2v_c) \pm 2f^3 = 0$

$\times f'$: $f'f'' - v_e f f'(v_e - 2v_c) \pm 2f^3 f' = 0$

$\int$ $\quad \frac{1}{2}(f')^2 - \frac{v_e}{2} f^2 (v_e - 2v_c) \pm \frac{1}{2} f^4 = A/2$

(+) focusing $(v_e > 2v_c)$: $\quad f' = \pm f\sqrt{\underbrace{v_e(v_e-2v_c)}_{\equiv b > 0} - f^2}$ AND $A = 0$ BCs

$\Rightarrow \displaystyle\int \frac{df}{\pm f\sqrt{b - f^2}} = \xi - x_0$

$\overset{(*)}{\Rightarrow} \left[\dfrac{\mp \cancel{2}}{\cancel{2}\sqrt{\underbrace{v_e(v_e-2v_c)}_{b}}} \tanh^{-1}\left[\dfrac{\sqrt{\overbrace{v_e(v_e-2v_c)}^{b} - f^2}}{\sqrt{\underbrace{v_e(v_e-2v_c)}_{b}}}\right] = \xi - x_0\right]$

$b \equiv v_e(v_e - 2v_c) \Rightarrow \dfrac{\sqrt{b - f^2}}{\sqrt{b}} = \tanh(\mp\sqrt{b}\,(\xi - x_0))$

$\Rightarrow b - f^2 = b \tanh^2 \Rightarrow f^2 = b(1 - \tanh^2) = b\,\mathrm{sec}^2$

$\therefore \quad f = \overset{\text{since envelope}}{\oplus}\sqrt{b}\ \mathrm{sech}\sqrt{b}(\xi - x_0)$, $\xi = x - v_e t$

and $g = v_e \eta + \phi_0$; def : $a \equiv \sqrt{b} = \sqrt{v_e(v_e - 2v_c)} =$
$\eta = x - v_c t$
$\boxed{v_e \equiv c} \Rightarrow a^2 = c^2 - 2cv_c$
$\Rightarrow \boxed{v_c = -\dfrac{a^2 - c^2}{2c}}$

$u = fe^{ig}$

$\Rightarrow \boxed{u(x,t) = a\ \mathrm{sech}\, a(x - ct - x_0)\ e^{i[c(x + \frac{a^2 - c^2}{2c} t) + \phi_0]}}$

bright soliton solution of NLS.

Show Matlab
NLS.m
$u = u_1$
$\alpha = -1$

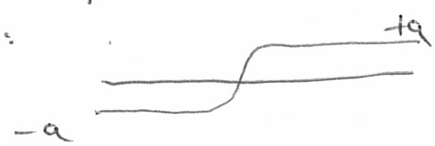
4 free params : a, c, x_0, ϕ_0

Envelope : $|u|^2 = a^2\,\mathrm{sech}^2 a(x - ct + x_0)$

(−) Defocusing ($v_e < 2v_c$) : $(f')^2 - v_e(v_e - 2v_c) f^2 - f^4 = A$

Here, since we are after heteroclinic solution we cannot make $A=0$

Let us try for :



f : $f(\pm\infty) = \pm a$, $f'(\pm\infty) = 0$

Thus : $\pm\infty$: $\underbrace{-v_e(v_e - 2v_c)}_{\equiv 2b > 0 \text{ (since } v_e < 2v_c)} a^2 - a^4 = A \Rightarrow a^4 - 2ba^2 + A = 0$

Also, let us choose v_e such that $\boxed{A = b^2}$ $\Rightarrow a^4 - 2ba^2 + b^2 = 0$

[a more general DS will be discussed later]

$\Rightarrow (a^2 - b)^2 = 0$

$\Rightarrow \boxed{b = a^2}$

$\therefore (f')^2 + 2\overset{2a^2}{\widehat{b}} f^2 - f^4 = a^4$

$\Rightarrow (f')^2 = (f^2 - a^2)^2 \Rightarrow f' = \pm(f^2 - a^2)$

$\Rightarrow \displaystyle\int \frac{df}{\pm(f^2 - a^2)} = \int_\xi d\xi = \xi - x_0 \Rightarrow \mp\frac{1}{a}\tanh^{-1}\frac{f}{a} = \xi - x_0$

$\Rightarrow f = \mp a \tanh a(\xi - x_0)$

and $g(\eta) = v_e \eta + \phi_0$

but : $b = a^2 \Rightarrow 2b = 2a^2 \Rightarrow -v_e(v_e - 2v_c) = 2a^2$

$\boxed{v_e \equiv c} \Rightarrow -c(c - 2v_c) = 2a^2$

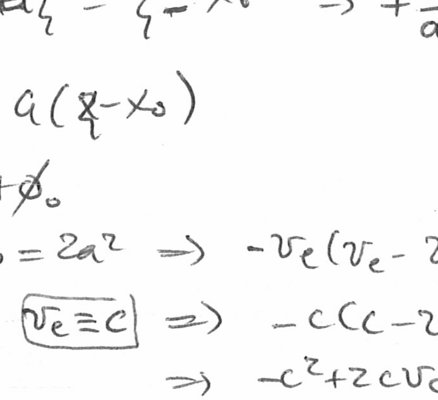
$\Rightarrow -c^2 + 2cv_c = 2a^2 \Rightarrow v_c = \dfrac{2a^2 + c^2}{2c}$

$\therefore \boxed{v_c = \dfrac{a^2}{c} + \dfrac{c}{2}} \Rightarrow g(\eta) = v_e(x - v_c t) + \phi_0$

$$\therefore \boxed{u(x,t) = \pm a \tanh a(x - ct \mp x_0)\, e^{i[cx - (a^2 + \frac{c^2}{2})t + \phi_0]}}$$

DARK Soliton Sol.

envelope : $|u|^2 = a^2 \tanh^2 a(x - ct \mp x_0)$

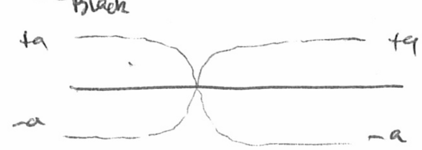


Show Matlab
NLS.m $u = u_3$
$\alpha = +1$

→ play with dynamics
- $phi_1 = 0$, $phi_2 = \pi$
- $phi_1 = 0$, $phi_2 = 0$
- $vel = 0$
- ...

Stationary DS : $v_c = 0$, $v_e = 0 = c$

$\Rightarrow u_{Black} = \pm a \tanh a(x - x_0)\, e^{i[-a^2 t + \phi_0]}$



temp Freq : $\omega = -a^2$

Galilean boost for NLS solutions

Note that the general DS has a e^{icx} term $\longrightarrow$ this corresponds to a spatial oscillation that is induced by a constant velocity of the background where the DS is seeded.

To realise this let us see what are the effects of a "boost" $e^{i\alpha x}$ on a NLS solution:

Galilean Boost : suppose $u(x,t)$ solution to NLS: $iu_t + \frac{1}{2}u_{xx} \pm |u|^2 u = 0$

Q: what happens when we "kick" u ~~to~~ with $e^{i\alpha x}$?

let us define: $u = v\, e^{i\alpha(x-\beta t)}$

$$\overset{NLS}{\Longrightarrow} i(v_t e - i\alpha\beta v e) + \tfrac{1}{2}[v_x e + i\alpha v e]_x \pm |v|^2 v e = 0$$

$$\Rightarrow i(v_t \cancel{e} - i\alpha\beta v \cancel{e}) + \tfrac{1}{2}[v_{xx}\cancel{e} + i\alpha v_x \cancel{e} + i\alpha v_x \cancel{e} + (i\alpha)^2 v \cancel{e}] \pm |v|^2 v \cancel{e} = 0$$

$$\Rightarrow \underline{iv_t + \tfrac{1}{2}v_{xx} \pm |v|^2 v} + [\alpha\beta v - \tfrac{\alpha^2}{2} v + \alpha v_x] = 0$$

$$\Rightarrow i(v_t + \alpha v_x) + \tfrac{1}{2}v_{xx} \pm |v|^2 v + \alpha v(\beta - \alpha/2) = 0 \Rightarrow \boxed{\beta = \alpha/2}$$

$$\Rightarrow i(v_t + \alpha v_x) + \tfrac{1}{2}v_{xx} \pm |v|^2 v = 0$$

Now take a co-moving reference frame: $\left\{ \begin{array}{l} \xi = x - \alpha t \\ \tau = t \end{array} \right.$

$$\Rightarrow \left\{ \begin{array}{l} u_t = \frac{\partial u}{\partial \tau}\frac{d\tau}{dt} + \frac{\partial u}{\partial \xi}\frac{d\xi}{dt} = u_\tau + u_\xi(-\alpha) \\ u_x = \frac{\partial u}{\partial \tau}\frac{d\tau}{dx} + \frac{\partial u}{\partial \xi}\frac{d\xi}{dx} = 0 + u_\xi \Rightarrow u_{xx} = u_{\xi\xi} \end{array} \right.$$

$$\therefore\ i(v_\tau - \alpha v_\xi + \alpha v_\xi) + \tfrac{1}{2}v_{\xi\xi} \pm |v|^2 v = 0$$

$$\therefore\ iv_\tau + \tfrac{1}{2}v_{\xi\xi} \pm |v|^2 v = 0$$

$\therefore$ If $u(x,t)$ sol. to NLS $\Rightarrow$ $u e^{-i\frac{\alpha}{c}x}$ is also a sol. to NLS but in a co-moving ref. frame with velocity $\alpha = c$

$\therefore$

More general dark solitons → grey solitons

previous DS : $\underbrace{f(x)}_{\mathbb{R}}\, e^{i\underbrace{g(t)}_{\mathbb{R}}}$

→ let us generalize and find $f \in \mathbb{C}$:

Space-independent phase!
- this can be done always for a linear phase using a Galilean boost
- No back-flow!!!

$$u(x,t) = [f(\underbrace{x-ct}_{b\xi}) + iA]\, e^{i\boxed{\phi(t)}} \quad (1)$$

with f & $A \in \mathbb{R}$.

(1) in NLS : $iu_t + \frac{1}{2}u_{xx} - |u|^2 u = 0$

$$\Rightarrow \quad i[-cf'\cancel{e} + (f+iA)\, i\phi'\cancel{e}] + \frac{b^2}{2} f''\cancel{e} - [f^2+A^2][f+iA]\cancel{e} = 0$$

$$\Rightarrow \quad -icf' - B\phi' - iA\phi' + \frac{b^2}{2}f'' - (f^2+A^2)(f+iA) = 0$$

$$\begin{cases} \mathbb{R}=0 \Rightarrow -f\phi' + \frac{b^2}{2}f'' - [A^2+f^2]f = 0 \\ \mathbb{I}=0 \Rightarrow -A\phi' - cf' - [A^2+f^2]A = 0 \end{cases}$$

$$A\cdot\mathbb{R} - f\cdot\mathbb{I} \Rightarrow \frac{b^2}{2}Af'' + cff' = 0 \overset{\int}{\Rightarrow} \frac{b^2}{2}Af' + \frac{1}{2}cf^2 = d \Rightarrow \frac{df}{\alpha-\beta f^2} = d\xi$$

$$\therefore \quad f(b\xi) = a\tanh(b\xi)$$

HW

... choosing appropriate constants $(a, b, c, \dots)$ we can solve for f & ϕ to find:

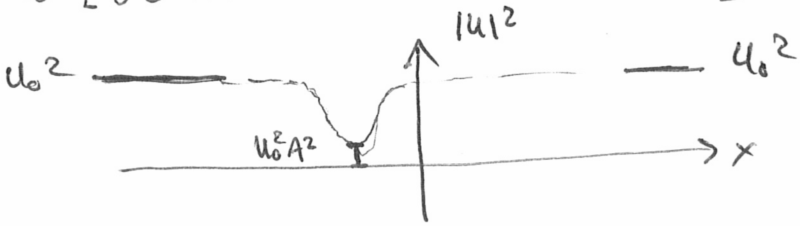
$$\boxed{u(x,t) = u_0[B\tanh(u_0 B(x-ct)) + iA]\, e^{-iu_0^2 t} \quad \text{with } c = Au_0 \text{ \& } A^2+B^2=1}$$

$$= [\tilde{B}\tanh(\tilde{B}(x-ct)) + iC]\, e^{-iu_0^2 t}, \quad u_0^2 = \tilde{B}^2 + C^2, \quad c \uparrow v_e$$

gral. form of dark (grey) soliton

Study this shape:

$$|u| = u_0^2[B^2t^2 + A^2] \ \cancel{= u_0^2[B^2t^2 + A^2 - B^2]} \ \begin{matrix} \xrightarrow{x\to\pm\infty} u_0^2[B^2+A^2] = u_0^2 \\ \xrightarrow{x\to 0} u_0^2A^2 \end{matrix}$$



MI for Dark Soliton Stripes

- Refs : [1] Kivshar + Luther Davis, Phys. Rep. 298(98) 81-197

$$i u_t + \tfrac{1}{2} \nabla^2 u - |u|^2 u = 0$$

$\boxed{\text{p92: Review on dark Solitons:}}$

$(\mu = +u_0^2)$

- $u(x,t) = u_0 [B \tanh\theta + iA] e^{-i u_0^2 t}$

$$\theta = u_0 B(x - A u_0 t) \quad \& \quad A^2 + B^2 = 1$$

now, if we do $A = \sin\phi$, $B = \cos\phi$
we have 2 params: u_0 & ϕ.

- $v_e = \text{vel} \overset{= c = u_0 A =}{=} u_0 \sin\phi \ (= A u_0) \Rightarrow [u_0 B \tanh\theta + i v_e] e^{-i u_0^2 t}$
$\Rightarrow [u_0 \cos\phi \tanh\theta + i u_0 \sin\phi] e^{-i u_0^2 t}$

* $\phi = 0 \Rightarrow v_e = 0 \Rightarrow A = \sin\phi = 0,\ B = \cos\phi = 1$

$\Rightarrow u(x,t) = u_0 \tanh u_0 (x)\, e^{-i u_0^2 t}$

= a stationary DS.

Δphase $= \pi$

* $\phi = \pi/2 \Rightarrow v_e = u_0 \Rightarrow A = 1,\ B = 0$

$\Rightarrow u(x,t) = u_0 e^{-i u_0^2 t}$

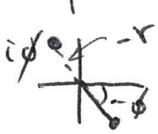
= a stationary steady state.

Δphase $= 0$

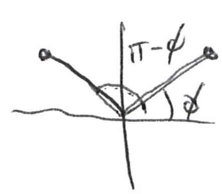
* ϕ : $x = +\infty \Rightarrow u_+ = u_0 [B + iA] e^{-i u_0^2 t}$
$x = -\infty \Rightarrow u_- = u_0 [-B + iA] e^{-i u_0^2 t}$

$\Rightarrow \tilde{u}_+ = u_+/u_0 e = B + iA = \cos\phi + i\sin\phi = e^{i\phi}$

$\tilde{u}_- = u_-/u_0 e = -B + iA = -\cos\phi + i\sin\phi$
$= -\cos(-\phi) - i\sin(-\phi) = -e^{-i\phi}$



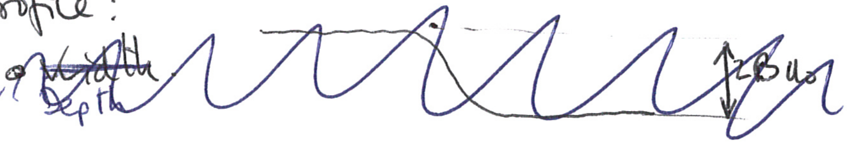
$\therefore$



$\therefore \Delta\text{phase} = (\pi - \phi) - \phi$

or $\boxed{\Delta\text{phase} = \pi - 2\phi}$, $\phi = 0 \Rightarrow \Delta\text{phase} = \pi$ ✓
$\phi = \pi/2 \Rightarrow \Delta\text{phase} = 0$ ✓

* profile:



- @ center : $x=0, t=0$ $\qquad u = u_0 [iA] e^{-iu_0^2 t}$

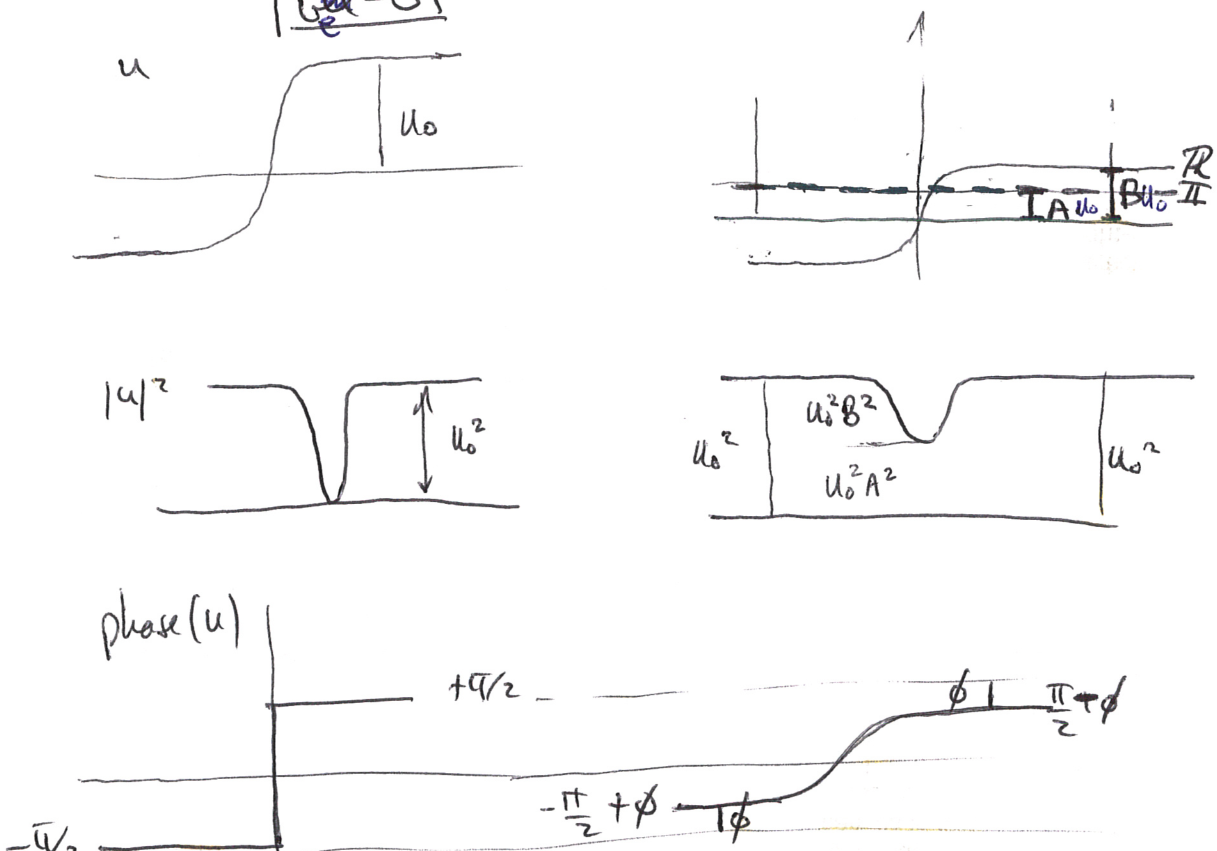
$\Rightarrow |u(0,0)| = u_0 A$

- $|u|^2$ @ $\pm\infty$: $|u_\pm|^2 = u_0^2 (\pm e^{\pm i\phi}) e^{-iu_0^2 t} = u_0^2$

- Depth $\} \; u_0^2 - u_0^2 A^2 = u_0^2 (1-A^2) = u_0^2 B^2$

- $\phi = 0$ $\qquad \phi, \boxed{v_e = u_0 \sin\phi}$

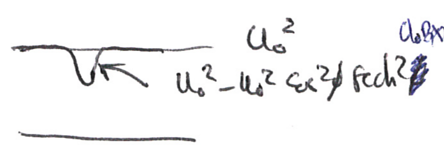
$\boxed{v_e = 0}$



Δphase $= \pi$

$$\Delta\text{phase} = \pi - 2\phi = \left(\tfrac{\pi}{2} - \phi\right) + \left(\tfrac{\pi}{2} - \phi\right) = \left(\tfrac{\pi}{2} - \phi\right) - \left(-\tfrac{\pi}{2} + \phi\right) \checkmark$$

- profile: $|u|^2 = u_0^2 \left(1 - \cos^2\phi \,\mathrm{sech}^2 (u_0 B x)\right)$



- DS travels @ speeds $\leq u_0$ = speed of sound.
  $\cdot\, v_e = u_0 \sin\phi$

Modulational Stab for NLS → I.e. How flat (homog.) steady state evolve → U/S?

$$i u_t + \frac{1}{2} u_{xx} \pm |u|^2 u = 0 \qquad \begin{matrix} + \text{ BS (Att)} \\ - \text{ DS (Rep)} \end{matrix}$$

Show numerics see p. 47

~~unif~~ Steady state: $u_s = u_0 e^{-i\mu t}$ · chemical potential in BECs · $-\mu$ = Freq. in optics.

$$\Rightarrow \quad \mu u_0 \pm u_0^3 = 0 \quad \Rightarrow \quad \mu \pm u_0^2 = 0$$

$$\Rightarrow \quad \boxed{\mu = \mp u_0^2}$$

$$\therefore \quad \boxed{u_s(x,t) = u_0 e^{\pm i u_0^2 t}} \qquad \begin{matrix} + \text{ Att} \\ - \text{ Rep.} \end{matrix}$$

~~mode~~

- Stab of unif. steady state:

suppose $\boxed{u = u_s(x,t) + \varepsilon v(x,t)}$ $\quad |\varepsilon| \ll 1$

$$|u|^2 u = u^2 u^* = (u_s^2 + 2\varepsilon u_s v + \varepsilon^2 v^2)(u_s^* + \varepsilon v^*)$$

$\overset{NLS}{\Longrightarrow}$ ~~$i u_{st} + \frac{1}{2} u_{sxx} \pm |u_s|^2 u_s = $~~

$$i u_{st} + \frac{1}{2} u_{xx} + i\varepsilon v_t + \frac{1}{2}\varepsilon v_{xx} \pm (u_s + \varepsilon v)^2 (u_s^* + \varepsilon v^*) = 0$$

$$\Rightarrow \cancel{i u_{st} + \tfrac{1}{2} u_{sxx}} + i\varepsilon v_t + \frac{1}{2}\varepsilon v_{xx} \pm \cancel{|u_s|^2 u_s} \pm \varepsilon [2 u_s v u_s^* + u_s^2 v^*] + \varepsilon^2 [\quad] = 0$$

$$O(\varepsilon): \quad \boxed{i v_t + \frac{1}{2} v_{xx} \pm 2|u_s|^2 v \pm u_s^2 v^* = 0}$$

$$\Rightarrow \boxed{i v_t + \frac{1}{2} v_{xx} \pm 2 u_0^2 v \pm u_0^2 e^{\pm 2 i u_0^2 t} v^* = 0} \quad (2) \text{ eq. for perturbation}$$

---

suppose $v(x,t) = e^{i(kx - \omega t)}$ → Q: dispersion relation = ?

$$\Rightarrow \quad \omega e + \frac{1}{2}(ik)^2 e \pm 2u_0^2 e \pm u_0^2 e^{\pm 2 i u_0^2 t} e^{-i(kx-\omega t)} = 0$$

$$\Rightarrow \quad \omega - \frac{1}{2}k^2 \pm 2u_0^2 \pm u_0^2 e^{\pm 2 i u_0^2 t - 2i(kx - \omega t)} = 0$$

does not work!
We need pert. to "rotate" in ℂ @ same rate as u_s

~~again:~~ $i v_t + \frac{1}{2} v_{xx} \pm 2 |u_s|^2 v \pm u_s^2 v^* = 0$ ✓ 1

$u_s = u_0 e^{\pm i u_0^2 t} \rightarrow i v_t + \frac{1}{2} v_{xx} \pm 2 u_0^2 v \pm u_0^2 e^{\pm 2 i u_0^2 t} v^* = 0$ 2

~~• perturb: $v(x,t) = e^{i(kx-\omega t)}$~~

~~$\Rightarrow$ - - - - does not work.~~

• perturb: $\boxed{v(x,t) = (a+ib) e^{\pm i u_0^2 t}}$ 3 $a = a(x,t)$, $b = b(x,t)$

we need perturb to be rotating @ same rate of u_s, then a & b will take care of pert.

2 & 3 $\Rightarrow$

$$i(a+ib)_t e^{\pm} + \overbrace{i(a+ib)(\pm i u_0^2)}^{(a+ib)(\mp u_0^2)} e^{\pm} + \frac{1}{2}(a+ib)_{xx} e^{\pm} \pm 2u_0^2 (a+ib) e^{\pm}$$
$$\pm u_0^2 e^{\pm 2} (a-ib) e^{\mp} = 0$$

$$\Rightarrow \begin{cases} \text{Im}: & a_t \mp \cancel{u_0^2 b} + \frac{1}{2} b_{xx} \pm \cancel{2u_0^2 b} \mp \cancel{u_0^2 b} = 0 \\ \text{Re}: & -b_t \mp u_0^2 a + \frac{1}{2} a_{xx} \pm 2u_0^2 a \pm u_0^2 a = 0 \end{cases}$$

$$\Rightarrow \boxed{\begin{cases} a_t + \frac{1}{2} b_{xx} = 0 \\ -b_t + \frac{1}{2} a_{xx} + \pm 2 u_0^2 a = 0 \end{cases}}$$ $\mp$ Att (BS) / Rep (DS)

Now let pert. be a plane wave:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} e^{+i(kx-\omega t)}$$

$$\Rightarrow \begin{cases} a_0(-i\omega) + \frac{1}{2}(ik)^2 b_0 = 0 \\ -b_0(-i\omega) + \frac{1}{2}(ik)^2 a_0 \pm 2 u_0^2 a_0 = 0 \end{cases}$$

$$\Rightarrow M\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = 0 \quad \text{with} \quad M = \begin{bmatrix} -i\omega & -\frac{k^2}{2} \\ -\frac{k^2}{2} \pm 2u_0^2 & i\omega \end{bmatrix}$$

non-trivial sols. to $M\begin{pmatrix} a \\ b \end{pmatrix} = 0$ only if $\det(M) = 0$

$$\Rightarrow \omega^2 = \left(-\frac{k^2}{2}\right)\left(-\frac{k^2}{2} \pm 2u_0^2\right)$$

Att: $\omega^2 = \left(-\frac{k^2}{2}\right)\left(-\frac{k^2}{2} + 2u_0^2\right) = \frac{k^2}{4}\left(k^2 - 4u_0^2\right)$

Rep. $\omega^2 = \left(-\frac{k^2}{2}\right)\left(-\frac{k^2}{2} - 2u_0^2\right) = \frac{k^2}{4}\left(k^2 + 4u_0^2\right) > 0$

$\Rightarrow \omega \in \mathbb{R} \Rightarrow$ Stable

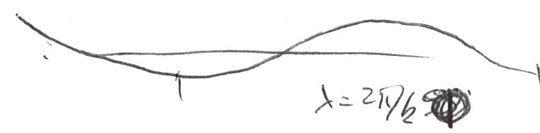
∴ Att * if $k^2 > 4u_0^2 \Rightarrow \omega \in \mathbb{R} \Rightarrow S$

$|k| > 2u_0 \Rightarrow S$

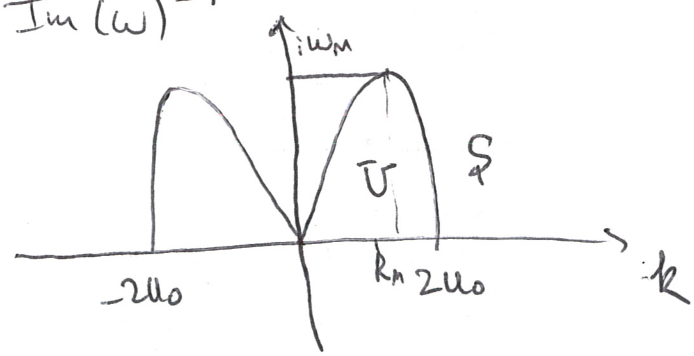
* if $|k| < 2u_0 \Rightarrow U$

∴ $\omega^2 < 0 \Rightarrow \omega = i\omega_i \Rightarrow U$

• $|k| < 2u_0 \Rightarrow \cos kx$



$$\mathrm{Im}(\omega) = \left|\frac{k}{2}\sqrt{4u_0^2 - k^2}\right|$$



$\omega_M = \frac{k}{2}\sqrt{4u_0^2 - 2u_0^2}$

$= \frac{\sqrt{2}k}{2}u_0$

$\omega_M = \frac{k u_0}{\sqrt{2}}$

maximum growth rate @ $k = k_M$ where $\left.\frac{\partial \omega}{\partial k}\right|_{k_M} = 0$

same location as $\left.\frac{\partial(\omega^2)}{\partial k}\right|_{k_M} = 0 \Rightarrow k_M^3 - 2k_M u_0^2 = 0$

$\Rightarrow k_M^2 = 2u_0^2 \Rightarrow \boxed{k_M = \sqrt{2}\,u_0}$

∴ If we randomly perturb NLS ~~for~~ a uniform steady state we should have a growth of an unstable mode @ $k_M = \sqrt{2}\, u_0$

Show NLS_instability.m

$\lambda \approx \frac{2\pi}{k_M} = \frac{2\pi}{\sqrt{2} u_0}$

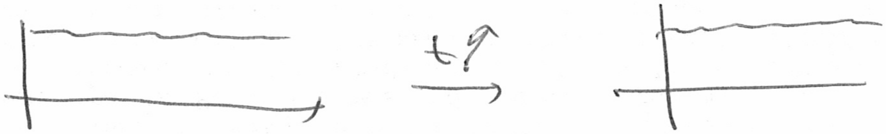
- $u_0 = 1 \Rightarrow \lambda \approx 4.44$
- $u_0 = 0.5$ ($H=20$, $L=50$) $\Rightarrow \lambda \simeq 8.88$

⇒ This MI creates solitons!

Repulsive: $\omega^2 = \frac{k^2}{2}\left(\frac{k^2}{2}+2u_0\right)^2 > 0$

∴ $\omega \in \mathbb{R}$ ∴ No MI for repulsive nonlinearity!

Show NLS_instability, alpha = +1, t = 150



∴ Stable.

⇒ DS cannot be created from MI.

→ this is natural because DS needs a topological charge, i.e. a defect on the phase. (this ~~is~~ would be a strange pert.).

Intro to BECs

- Rep/att
- V_{MT}
- Rep
  - * $M \ll 1$ limit → Gaussian
  - * $M \gg 1$ limit → TF.

$M \leftarrow \int |u|^2$ [$\int \rho = \int |u|^2$]

Repulsive

$\boxed{1D}$: $V_{MT} = \frac{1}{2}\Omega^2 x^2 \Rightarrow i u_t + \frac{1}{2} u_{xx} - |u|^2 u = V(x)\cdot u$

- $M \ll 1 \Rightarrow |u|^2 u \ll 1 \Rightarrow i u_t + \frac{1}{2} u_{xx} = V(x)\, u$

(linear limit) $u = v(x)\, e^{-i\mu t} \Rightarrow \mu v \cancel{\phi} + \frac{1}{2} v_{xx} \cancel{\phi} = V v \cancel{\phi}$

$\Rightarrow \mu v + \frac{1}{2} v'' = \frac{1}{2}\Omega^2 x^2 v$

$v \propto$ Gaussian: $v(x) = A e^{-\frac{x^2}{2\sigma^2}}$

$\Rightarrow v' = \left(-\frac{x}{\sigma^2}\right) v$

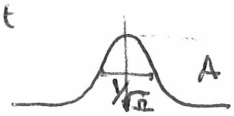
$\Rightarrow v'' = \left(-\frac{1}{\sigma^2}\right) v + \left(-\frac{x}{\sigma^2}\right) v' = -\frac{v}{\sigma^2} + \frac{x^2}{\sigma^4} v$

$\therefore \mu \cancel{v} + \frac{1}{2}\left(-\frac{\cancel{v}}{\sigma^2}\right) + \frac{1}{2}\frac{x^2}{\sigma^4}\cancel{v} = \frac{1}{2}\Omega^2 x^2 \cancel{v}$

$\Rightarrow \boxed{\mu = +\frac{1}{2\sigma^2}}$ & $\Omega^2 = \frac{1}{\sigma^4} \Rightarrow \boxed{\sigma^2 = 1/\Omega}$

$\Rightarrow \mu = +\frac{1}{2\sigma^2} = +\frac{1}{2}\Omega$

$\therefore u = A e^{-\mu x^2} e^{-i\mu t} = A e^{-\frac{1}{2}\Omega x^2} e^{-i\mu t}$

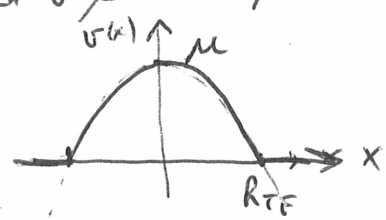


- $M \gg 1$ (Thomas-Fermi, TF, limit)

$Vu,\ u_t,\ |u|^2 u \gg u_{xx} \Rightarrow i u_t - |u|^2 u = V u$

$\therefore u = v(x)\, e^{-i\mu t} \Rightarrow \mu v \cancel{\phi} - v^3 \cancel{\phi} = V v \cancel{\phi} \Rightarrow v^2 = \mu - V$

$\therefore v(x) = \sqrt{\mu - V(x)}$



R_{TF}: $\mu = V(R_{TF}) \Rightarrow \mu = \frac{1}{2}\Omega^2 R_{TF}^2$

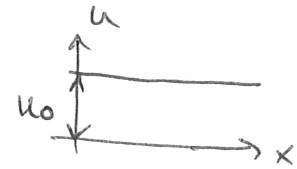
$\Rightarrow R_{TF} = \frac{\sqrt{2\mu}}{\Omega}$

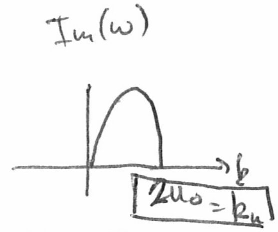
Avoiding MI by using MT

Kevrekidis et al PRA 70 (04) 023602
"Avoiding infrared catastrophes in trapped BECs"
IR UV

- NLS, attractive, $iu_t + \frac{1}{2}u_{xx} + |u|^2 u = 0$

- MI: $\omega^2 = \frac{k^2}{4}(k^2 - 4u_0^2)$





- BEC: NLS + V(x): $iu_t + \frac{1}{2}u_{xx} + |u|^2 u = V(x) u$

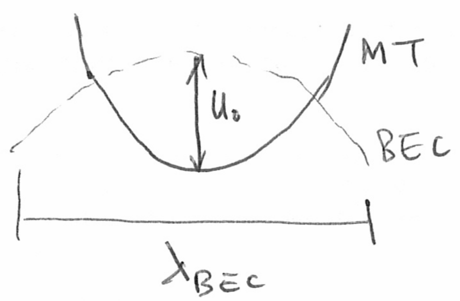
~~- In BEC if # atoms >> 1 (ie $\int |u|^2$ large)~~
~~- ⇒ slow spatial variations~~
~~⇒ approx $\nabla^2 = 0$ ($u_{xx} = 0$)~~
~~⇒ $iu_t + |u|^2 u \simeq V(x) u$~~
~~* steady state: $u = v\, e^{-i\mu t}$~~
~~⇒ $\mu \cancel{v} + v^{\cancel{3}\,2} \simeq V \cancel{v}$~~
~~⇒ $\boxed{v^2 \simeq V - \mu}$ Thomas-Fermi Approx.~~

~~only works for repulsive!~~

~~$\mu v + \frac{1}{2} v_{xx} + v^3 = V v$~~

~~$k = \frac{d}{2\pi}$~~

⇒



$k = 2\pi/\lambda$

∴ if $\lambda_{BEC} < \lambda_u = 2\pi / k_u = \pi / u_0$

⇒ one cannot support perturbations with $k < k_u$ and we suppress MI.

Conservation laws for NLS

As for KdV, it is possible to show that there is an ∞-# of conservation laws (indep). We can obtain them by appropriately massaging and $\int$ the NLS.

Ex: conservation of mass $M = \int_{-\infty}^{\infty} |u|^2 dx$

$\longrightarrow$ Mass in BECs total power in optical fibers

NLS: $i u_t + \frac{1}{2} u_{xx} \pm |u|^2 u = 0$ ①

$\overline{\text{NLS}}$: $-i \bar{u}_t + \frac{1}{2} \bar{u}_{xx} \pm |u|^2 \bar{u} = 0$ ②

①$\times \bar{u}$ − ②$\times u$: $i(u_t \bar{u} + \bar{u}_t u) + \frac{1}{2}(u_{xx}\bar{u} - \bar{u}_{xx} u) \pm |u|^2 (\cancel{u\bar{u}} - \cancel{\bar{u}u}) = 0$

$$i \frac{\partial}{\partial t}(u\bar{u}) + \frac{1}{2}\frac{\partial}{\partial x}(u_x \bar{u} - \bar{u}_x u) = 0$$

$\int \Rightarrow \quad i \frac{d}{dt}\int_{-\infty}^{\infty} |u|^2 dx + \frac{1}{2}\cancel{[u_x\bar{u} - \bar{u}_x u]_{-\infty}^{\infty}} = 0$

$\therefore$ as long as $u(\pm\infty) = 0 = u_x(\pm\infty)$
i.e. localized solution

$$\Rightarrow \boxed{M = \int_{-\infty}^{\infty} |u|^2 dx = \text{const}}$$

HW:

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2}|u_x|^2 \mp \frac{1}{2}|u|^4\right) dx = \text{const}$$

$$P = i\int_{-\infty}^{\infty} (u\bar{u}_x - \bar{u}u_x)\, dx = \text{const}.$$

$\vdots$

Applications:

- Use conservation laws to check numerics are ok.
- Use — " — " — with an ansatz (trial function) to obtain dynamics for this ansatz

ex:  (however since we introduce an external potential just M & E are now conserved).

$\longrightarrow$ we'll do this ~~[illegible]~~ in HW

Variational Approximation

Follow Anderson PRA 27 (1983) 3135 $\left[\begin{array}{l} t \to x \\ x \to t \end{array}\right]$

and also see Malomed Prog. in Optics 43 (2002) 71

NLS $\Leftrightarrow$ ψ is Extrema of Lagrangian L.

$$\text{NLS}: \quad i\psi_t = \alpha \psi_{xx} + k|\psi|^2\psi$$

$$L = \frac{i}{2}\left[\psi\psi_t^* - \psi^*\psi_t\right] - \alpha|\psi_x|^2 + \frac{k}{2}|\psi|^4$$

[$\alpha = -1/2$, $k = \pm 1 = \begin{cases} \text{repuls} = \text{defoc} \\ \text{attractive} = \text{foc} \end{cases}$
and $L \to -L$ is usual notation
~~or $\alpha = 1/2$, $t \to -t$, $k = \pm$ attractive / repulsive~~]

[Variational principle: $\delta \iint_{-\infty}^{\infty} L\, dx\, dt = 0$, $\left.\frac{\delta F}{\delta \psi}\right|_u \equiv \lim_{\lambda \to 0}\left[\frac{d}{d\lambda} F(u+\lambda v)\right]$

that is $\quad$ NLS $\Leftrightarrow$ $\frac{\delta L}{\delta \psi^*} = \frac{\partial}{\partial t}\frac{\partial L}{\partial \psi_t^*} + \frac{\partial}{\partial x}\frac{\partial L}{\partial \psi_x^*} - \frac{\partial L}{\partial \psi^*} = 0$

check: $\partial_t\left[\frac{i}{2}\psi\right] + \partial_x\left[-\alpha\psi_x\right] + \frac{i}{2}\psi_t + k\psi^2\psi^* = 0$

$\Rightarrow \frac{i}{2}\psi_t - \alpha\psi_{xx} + \frac{i}{2}\psi_t + k|\psi|^2\psi = 0 \Leftrightarrow$ NLS ✓]

Ansatz: Let us compute L but on a restricted space spanned by an ansatz.

Suppose $\psi(x,t)$ can be approximated by $\psi_A(x,t;\vec{p})$
↖ parameters.

and suppose $\vec{p} = \vec{p}(t)$ [but not $p(x)$, ie space indep].

$\Rightarrow \quad \delta \iint_{-\infty}^{+\infty} L(x,t)\,dx\,dt = 0 \quad \langle \simeq \rangle \quad \delta \iint_{-\infty}^{+\infty} L_A(x,t)\,dx\,dt = 0 \quad L_A = L(x,t)\big|_{\psi_A}$

Since $p \neq p(x)$ we can define an averaged Lagrangian:

$$\bar{L}_A = \int_{-\infty}^{\infty} L_A(x,t)\,dx$$

Now $\bar{L}_A = \bar{L}_A(t, \vec{p}(t))$

- Variational principle reads: $\delta \int_{-\infty}^{+\infty} \bar{L}_A(t,\vec{p}(t))\,dt = 0$

- Euler-Lagrange eqs:

suppose $f = f(\vec{p}, \dot{\vec{p}}, t)$, the extrema for $I = \int_{-\infty}^{+\infty} f(\vec{p}, \dot{\vec{p}}, t)\,dt$ must satisfy:

Euler-Lagrange: $\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_i}\right) = \frac{\partial f}{\partial p_i}$ $\quad$ where $\vec{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix}$

* Let us prove E-L for 1 param $\vec{p} = p$

$$I = \int f(p, \dot{p}, t)\,dt \quad \text{and we need } \delta I = 0$$

Let us, as in calculus, do a perturbation and do $\frac{\delta I}{\delta p(t)} = 0$

problem: how to perturb $p = p(t)$?

$$p(t) \to p(t) + \varepsilon q(t) \Rightarrow \dot{p} \to \dot{p} + \varepsilon \dot{q}$$

$$\therefore \delta I = \int f(p+\varepsilon q, \dot{p}+\varepsilon\dot{q}, t)\,dt - \int f(q, \dot{p}, t)\,dt$$

$$\overset{\text{Taylor}}{=} \int \cancel{f(q,\dot{p},t)} + \varepsilon \frac{\partial f}{\partial p} q + \varepsilon \frac{\partial f}{\partial \dot{p}} \dot{q} + O(\varepsilon^2) - \int \cancel{f(q,\dot{p},t)}\,dt$$

$$= \varepsilon \int \left(\frac{\partial f}{\partial p} q + \frac{\partial f}{\partial \dot{p}} \dot{q}\right) dt + O(\varepsilon^2)$$

Extremum $\Rightarrow \delta I = 0$ @ order $O(\varepsilon)$ $\Rightarrow \int \left(\frac{\partial f}{\partial p} q + \frac{\partial f}{\partial \dot{p}} \dot{q}\right) dt = 0$

parts: $u = \frac{\partial f}{\partial \dot{p}} \to \frac{d}{dt}\frac{\partial f}{\partial \dot{p}}\,dt$

$v = q(t) \leftarrow dv = \dot{q}\,dt$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{\partial f}{\partial p} q + \left[q(t) \frac{\partial f}{\partial \dot{p}}\right]_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} q \frac{d}{dt}\frac{\partial f}{\partial \dot{p}} = 0$$

$\left[\ \right] \to 0$ If $f(\pm\infty) \to 0$

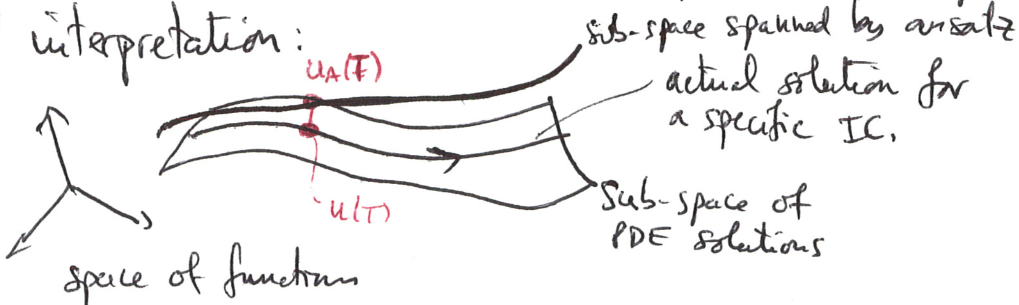
$$\Rightarrow \int_{-\infty}^{+\infty} \left[\frac{\partial f}{\partial p} - \frac{d}{dt}\frac{\partial f}{\partial \dot{p}}\right] q = 0 \Rightarrow \boxed{\frac{\partial f}{\partial p} - \frac{d}{dt}\frac{\partial f}{\partial \dot{p}} = 0} \quad \text{E-L}$$

* If one has an array of params: $\vec{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix}$, $\dot{\vec{p}} = \begin{pmatrix} \dot{p}_1 \\ \vdots \\ \dot{p}_N \end{pmatrix}$

$$\Rightarrow \frac{d}{dt}\frac{\partial f}{\partial \dot{p}_i} - \frac{\partial f}{\partial p_i} = 0 \quad \text{for } i = 1, \dots, N$$

↖ N eqs. (diff. eqs) for the parameters

∴ Geometrical interpretation:



If ansatz is good $\Rightarrow$ $u_A(T) \approx u(T)$

$\to$ N Euler-Lagrange Eqs.

For us: $\delta \int \bar{L}_A(t, \vec{p}(t))\,dt = 0 \iff \frac{d}{dt}\left(\frac{\partial \bar{L}_A}{\partial \dot{p}_i}\right) = \frac{\partial L_A}{\partial p_i}$

$\therefore$ NLS $\iff$ set of ODEs in ansatz parameters!

Ex: Let us try to approximate a NLS solution with a Gaussian profile.

Ansatz: $\Psi_A = \overbrace{A(t)}^{\mathbb{C}} e^{\left[-\frac{x^2}{2a^2(t)} + i b(t) x^2\right]}$ phase.

$A(t) = |A| e^{i\phi(t)}$

$a(t)$ = width

$|A(t)|$ = height

$2b(t)x$ = "chirp"

$e^{ix^2} \Rightarrow$ contractions.

compute averaged Lagrangian $\int e^{-\frac{x^2}{a^2}} = \sqrt{\pi}\, a$, $\int x^2 e^{-\frac{x^2}{a^2}} = \frac{\sqrt{\pi}}{2} a^3$

$$\bar{L}_A = \int_{-\infty}^{\infty} \frac{i}{2}\left[\Psi_A \Psi_{At}^* - \Psi_A^* \Psi_{At}\right] - \alpha |\Psi_{Ax}|^2 + \frac{k}{2}|\Psi_A|^4\, dx$$

$$= \int_{-\infty}^{\infty} \frac{i}{2}\left\{A^2 ee^* \frac{d}{dt}\left[-\frac{x^2}{2a^2} + ibx^2\right]^* - A^2 ee^* \frac{d}{dt}\left[-\frac{x^2}{2a^2} + ibx^2\right]\right\} - \alpha A^2\left(-\frac{2x}{2a^2} + 2ibx\right)\left(\frac{-2x}{2a^2} + 2ibx\right)^* ee^*$$

$$+ \frac{k}{2} A^4 (ee^*)^2\, dx$$

$$= \cdots = \frac{\sqrt{\pi}}{2}\left[|A|^2 a^3 \dot{b} - \alpha a^3 A^2\left(4b^2 + \frac{1}{a^4}\right) + \frac{ka}{\sqrt{2}} A^4\right]$$

Euler-Lag. Eqs:

$\boxed{A}$ $\underbrace{\frac{d}{dt}\left(\frac{\partial \bar{L}_A}{\partial \dot{A}}\right)}_{=0} = \frac{\partial \bar{L}_A}{\partial A} \Rightarrow \frac{\sqrt{\pi}}{2}\left[2Aa^3\dot{b} - 2\alpha a^3 A\left(4b^2 + \frac{1}{a^4}\right) + \frac{4ka}{\sqrt{2}} A^3\right] = 0$

$\boxed{a}$ $\underbrace{\frac{d}{dt}\left(\frac{\partial \bar{L}_A}{\partial \dot{a}}\right)}_{=0} = \frac{\partial \bar{L}_A}{\partial a} \to \frac{\sqrt{\pi}}{2}\left[3A^2 a^2 \dot{b} - 3\alpha a^2 A^2 4b^2 + \frac{\alpha A^2}{a^2} + \frac{k}{\sqrt{2}} A^4\right] = 0$

$\boxed{b}$ $\frac{d}{dt}\left(\frac{\partial \bar{L}_A}{\partial \dot{b}}\right) = \frac{\partial \bar{L}_A}{\partial b} \Rightarrow \frac{d}{dt}\left[\frac{\sqrt{\pi}}{2} A^2 a^3\right] = -\alpha a^3 A^2 8b$

Since our ansatz does not have a variation for the phase let us let $A(t)$ be complex ie: $A(t) = A_r(t) + i A_i(t)$

so we have now 4 parameters A_r, A_i, a, b

Instead of using A_i & A_r note that $A^* = (A_r - iA_i)/2$, $A = (A_r + iA)/2$

so let us use A & A^*.

thus $\vec{p} = \begin{pmatrix} A \\ A^* \\ a \\ b \end{pmatrix} \leftarrow$ parameters.

compute averaged Lagrangian, use $\int e^{\frac{-x^2}{a^2}} = \sqrt{\pi} a$

& $\int x^2 e^{-x^2/a^2} = \frac{\sqrt{\pi}}{2} a^3$

$$\longrightarrow \bar{L}_A(x,t) = \int_{-\infty}^{\infty} L(x,t)\Big|_{\psi_A} dx = [\text{Eq (16) of Anderson}]$$

$$\Rightarrow \bar{L}_A = \frac{\sqrt{\pi}}{2}\left[ia\left(A\dot{A}^* - A^*\dot{A}\right) + AA^* a^3 \dot{b} - \alpha a^3 AA^*\left(4b^2 + \frac{1}{a^4}\right) + \frac{1}{\sqrt{2}} k a AA^*\right]$$

Euler-Lag. eqns

$\boxed{A}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{A}}\right) = \frac{\partial L}{\partial A} \Rightarrow$ ODE (17) $\Rightarrow \frac{d}{dt}\left(-iaA^*\right) = ia\dot{A}^* + A^*_{a^3\dot{b}}$
$- \alpha a^3 A^*(\,) + \frac{1}{\sqrt{2}} k a A^*$

$\boxed{A^*}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{A}^*}\right) = \frac{\partial L}{\partial A^*} \Rightarrow$ ODE (18) $\rightarrow \frac{d}{dt}\left(iaA\right) = ia\dot{A} + Aa^3\dot{b}$
$- \alpha a^3 A(\,) + \frac{1}{2} k a A$

$\boxed{a}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right) = \frac{\partial L}{\partial a} \Rightarrow$ ODE (19)

$\boxed{b}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{b}}\right) = \frac{\partial L}{\partial b} \Rightarrow$ ODE (20)

* combining (17) & (18) $\Longrightarrow \frac{d}{dt}\left(a|A|^2\right) = 0 \qquad |A|^2 = AA^*$

$$\Rightarrow \left.\begin{array}{l} \left[-i\dot{a}A^* + ia\dot{A}^* = \ldots\right]\cdot A \\ \left[i\dot{a}A + ia\dot{A} = \ldots\right]\cdot A^* \end{array}\right\} \ominus \begin{array}{l} -i\dot{a}|A|^2 - ia\dot{A}^*A \\ +i\dot{a}|A|^2 - ia\dot{A}A^* = 0 \end{array}$$

$$- \cancel{2} i\left[\dot{a}|A|^2 + \frac{a}{2}\left(\dot{A}^*A + \dot{A}A^*\right)\right] = 0$$

$\therefore \boxed{a|A|^2 = \text{const}}$ (23)

this recovers conservation of mass:

$$\int |\psi_A|^2 dx = \text{const}$$

$$\Rightarrow M = \int_{-\infty}^{\infty} |A|^2 e\, dx = |A|^2 \int e^{-\frac{x^2}{2a^2}} = |A|^2 \sqrt{2\pi}\, a = \text{const}.$$

$$|A| = \sqrt{\frac{M}{\sqrt{2\pi}\, a}}$$

* (20) : $\frac{d}{dt}\left(a^3 |A|^2\right) = -4\alpha b a^3 |A|^2$

$$\frac{d}{dt}\left(\underbrace{a|A|^2}_{\text{const}} \cdot a^2\right) = a\cancel{|A|^2}\, 2\dot{a}a = -4\alpha b a^3 \cancel{|A|^2}$$

$$\Rightarrow \boxed{\dot{a} = -2\alpha b a} \quad (25) \quad \triangle \text{ factor of 2}$$

\+ combining (17)&(18) in (19):

$$a\dot{b} - 4\alpha a b^2 + \frac{\alpha}{a^3} - \frac{k}{2\sqrt{2}}\frac{|A|^2}{a} = 0 \quad (26)$$

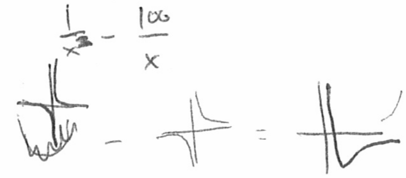
now $\frac{d}{dt}(25) \Rightarrow \ddot{a} = -2\alpha\left(\dot{b}a + b\dot{a}\right) = -2\alpha\left(\dot{b}a + 2\alpha b^2 a\right)$

$$\overset{(26)}{\Longrightarrow} \ldots \quad \boxed{\ddot{a} = \frac{4\alpha^2}{a^3} - \alpha k \sqrt{2}\frac{|A|^2}{a}} \quad (27)$$

It is convenient to eliminate $a|A|^2$

Since $\frac{M}{\sqrt{2\pi}} = |A|^2 a$

$$\therefore \quad \ddot{a} = \frac{4\alpha^2}{a^3} - \frac{\alpha k \sqrt{2}}{a^2} \frac{M}{\sqrt{2\pi}}$$



Show NLS.m with IC $= 1.1 A \operatorname{sech}[0.9A(\ldots)]$, $c=0$, $x_0=0$

$$\ddot{a} = \frac{4\alpha^2}{a^3} - \frac{\alpha k M}{\sqrt{\pi}\, a^2} = -\frac{d}{da}\left[+\frac{4\alpha^2}{2a^2} - \frac{\alpha k M}{\sqrt{\pi}\, a}\right]$$

$$\Rightarrow \boxed{\ddot{a} = -\frac{d}{da}\left[V_{eff}(a)\right]} \quad (28) \qquad \boxed{V_{eff}(a) = \frac{2\alpha^2}{a^2} - \frac{\alpha k M}{\sqrt{\pi}\, a}} \quad (29)$$

This ODE is solved for a and then b is obtained through (25) and $|A|^2 a$ is const.

- b eq: (24) $\frac{\dot{a}}{a} = -2\alpha b \Rightarrow \frac{d}{dt}\ln a = -2\alpha b \Rightarrow \boxed{b(t) = -\frac{1}{2\alpha}\frac{d}{dt}(\ln a)}$ (30)

- phase of A: $A(t) = \underbrace{B(t)\, e^{i\phi(t)}}$

this could have been another choice for ansatz $\Rightarrow \psi_A = B(t)\, e^{\left(-\frac{x^2}{2a^2} + ibx^2 + i\phi(t)\right)}$

this is more standard nowadays.

combining (17) & (18) $\Rightarrow$ eq (22) and use eq (26)

$$\Rightarrow \boxed{\dot{\phi}(t) = \frac{\alpha}{a^2} - \frac{5\sqrt{2}}{8} k |A|^2}$$

Steady state: minimum of $V_{eff} \Rightarrow \frac{4\alpha^2}{a^3} = \frac{\alpha k M}{\sqrt{\pi}\, a^2} \Rightarrow \frac{4\alpha}{a} = \frac{kM}{\sqrt{\pi}}$

$$\therefore \boxed{a = \frac{4\alpha\sqrt{\pi}}{kM}}$$

compare with $\psi = E \operatorname{sech} Ex$ of a standing soliton

with $\psi_A = A e^{-\frac{x^2}{2a}}$

where $M = \sqrt{2\pi}\, a|A|^2$ and $M = \int E^2 \operatorname{sech}^2 Ex = 2E$

and $a = \frac{4\alpha\sqrt{\pi}}{kM}$ $\Longrightarrow$ see Fig 5 of Anderson

Variational Approach in general

- nonlinear PDE : $PDE(\psi) = 0$
- Cast PDE as a variational Lagrangian:

$$\delta \iint L(\psi(x,t))\, dx\, dt = 0 \iff PDE(\psi) = 0$$

- Restrict sols to ansatz : $L_A = L(\psi_A(x, t, \vec{p}))$
- Average over x : $\overline{L_A}(t, \vec{p}) = \int L_A \, dx$
- Euler-Lagrange $\frac{d}{dt} \frac{\partial \overline{L_A}}{\partial \dot{p}_i} = \frac{\partial \overline{L_A}}{\partial p_i}$
- Find a system of ODE's for params:

most of the time E-L $\Longrightarrow$ $\boxed{M(\vec{p}) \frac{d\vec{p}}{dt} = f(\vec{p})}$

$M(\vec{p})$: mass matrix $f(\vec{p})$: forcing term.

⚠ the system of ODE's can be degenerate (non-invertible M)
$\Longrightarrow$ one has to try a $\neq$ ansatz.
<u>Ex</u>: Adding the chirp removes degeneracy!

non-chirped ansatz: $\boxed{V=0}$

$$u(x,t) = a \operatorname{sech} w(x-d)\, e^{i(b+cx)} \qquad p = \begin{bmatrix} a \\ b \\ c \\ d \\ w \end{bmatrix}$$

EL: ---

$$M(p)\,\dot{p} = f(p)$$

$$M = \begin{bmatrix} 0 & & \\ & \ddots & \\ & & \end{bmatrix}$$

~~Stax p20~~

$$= \begin{array}{c} \\ a \\ b \\ c \\ d \\ w \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & w \end{array} \\ \begin{bmatrix} 0 & 4a/w & 4ad/w & 0 & 0 \\ -4a/w & 0 & 0 & 0 & 2a^2/w^2 \\ 0 & 0 & 0 & -\frac{2a^2}{w} & 0 \\ 0 & 0 & 2a^2/w & 0 & 0 \\ 0 & -\frac{2a^2}{w^2} & -\frac{2a^2 d}{w^2} & 0 & 0 \end{bmatrix} \end{array}$$

$$f = \begin{bmatrix} f_1 \\ 0 \\ f_2 \\ 0 \\ f_3 \end{bmatrix}$$

① :
$$\begin{cases} \dot{b} + \dot{c} = \\ \dot{a} + \dot{w} = \\ \dot{d} = \\ \dot{c} = \\ \dot{b} + \dot{c} = \end{cases}$$

1 eqn for $\dot{a}, \dot{w}$

4 eqns for $\dot{b}, \dot{c}, \dot{d}$

$\Longrightarrow$ non invertible.

Chirped ansatz : $\boxed{V \neq 0}$

$$u(x,t) = a \,\mathrm{sech}\, w(x-d)\, e^{i(cx + b + ex^2)}$$

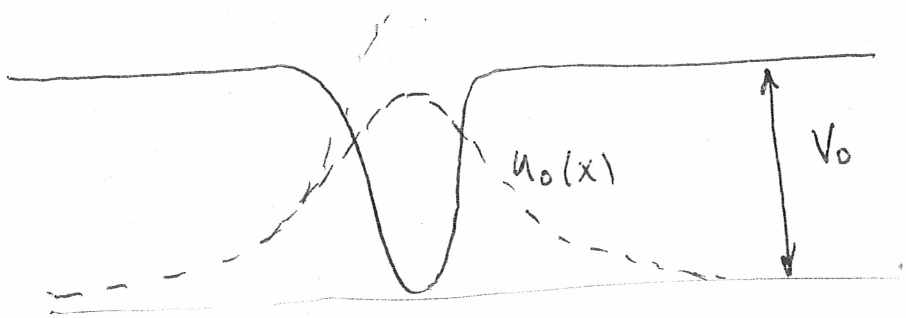
1	a
2	b
3	c
4	d
5	e
6	w

Variational Method

$$M(p) \frac{dp}{dt} = f(p)$$

With a well behaved M (i.e. invertible)

Ex : $V(x) = V_0 \tanh^2 x$



$$u_0(x,t) = \overbrace{\sqrt{1-V_0}}^{a_0} \,\mathrm{sech}\, x \; e^{i\omega_0 t} \qquad \omega_0 = \frac{1}{2} - V_0$$

ansatz : $u(x,t) = a \,\mathrm{sech}\, w(x-d)\, e^{i(cx + b + ex^2)}$

Show ODE system.

$M\dot{p} = f(p)$

M invertible ✓

Potential term

$\mathcal{L}_A = \int_{-\infty}^{\infty} L_A$

$= \ldots\ldots \underbrace{\int_{-\infty}^{\infty} V(x)|u_a|^2\, dx}_{\text{overlapping } \int}$

(6.41), be zero. This results in a system of 6 equations for the time derivatives a', b', c', d', e', and w',

$$M \begin{bmatrix} a' \\ b' \\ c' \\ d' \\ e' \\ w' \end{bmatrix} = f.$$

a : amplitude
b : phase (ω)
c : velocity
d : position (ξ)
e : chirp
w : width (ν)

The mass matrix is

$$M = \begin{bmatrix} 0 & \frac{4a}{w} & \frac{4ad}{w} & 0 & \frac{\pi^2 a}{3w^3} + \frac{4ad^2}{w} & 0 \\ -\frac{4a}{w} & 0 & 0 & 0 & 0 & \frac{2a^2}{w^2} \\ 0 & 0 & 0 & -\frac{2a^2}{w} & 0 & 0 \\ 0 & 0 & \frac{2a^2}{w} & 0 & \frac{4a^2 d}{w} & 0 \\ -2a\left(\frac{\pi^2}{6w^3} + \frac{2d^2}{w}\right) & 0 & 0 & -\frac{4a^2 d}{w} & 0 & \frac{a^2\pi^2}{2w^4} + \frac{2a^2 d^2}{w^2} \\ 0 & -\frac{2a^2}{w^2} & -\frac{2a^2 d}{w^2} & 0 & -a^2\left(\frac{\pi^2}{2w^4} + \frac{2d^2}{w^2}\right) & 0 \end{bmatrix}$$

and the forcing terms are

$$f = \begin{bmatrix} f_1 \\ 0 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}$$

where the entries are

$$\begin{aligned} f_1 &= \frac{4}{w}\alpha ac^2 + \frac{16}{w}\alpha acde + \frac{4\pi^2}{3w^3}\alpha ae^2 + \frac{16}{w}\alpha ad^2e^2 + \frac{4}{3}\alpha aw - \frac{8}{3w}\kappa a^3 - \\ &\quad 2a\int_{-\infty}^{\infty} V(x)\operatorname{sech}^2 w(x-d)\,dx \end{aligned}$$

$$f_3 = \frac{4}{w}\alpha a^2 c + \frac{8}{w}\alpha a^2 de$$

$$f_4 = \frac{8}{w}\alpha a^2 ce + \frac{16}{w}\alpha a^2 de^2 - 2a^2 w \int_{-\infty}^{\infty} V(x)\operatorname{sech}^2 w(x-d)\tanh w(x-d)dx$$

$$f_5 = \frac{8}{w}\alpha a^2 cd + 8\alpha a^2 e\left(\frac{\pi^2}{6w^3} + \frac{2d^2}{w}\right)$$

$$\begin{aligned} f_6 \quad = \quad & -\frac{2}{w^2}\alpha a^2 c^2 - \frac{8}{w^2}\alpha a^2 cde - 4\alpha a^2 e^2\left(\frac{\pi^2}{2w^4} + \frac{2d^2}{w^2}\right) + \frac{2}{3}\alpha a^2 + \frac{2}{3w^2}\kappa a^4 + \\ & 2a^2 \int_{-\infty}^{\infty} V(x)(x-d)\operatorname{sech}^2 w(x-d)\tanh w(x-d)dx. \end{aligned}$$

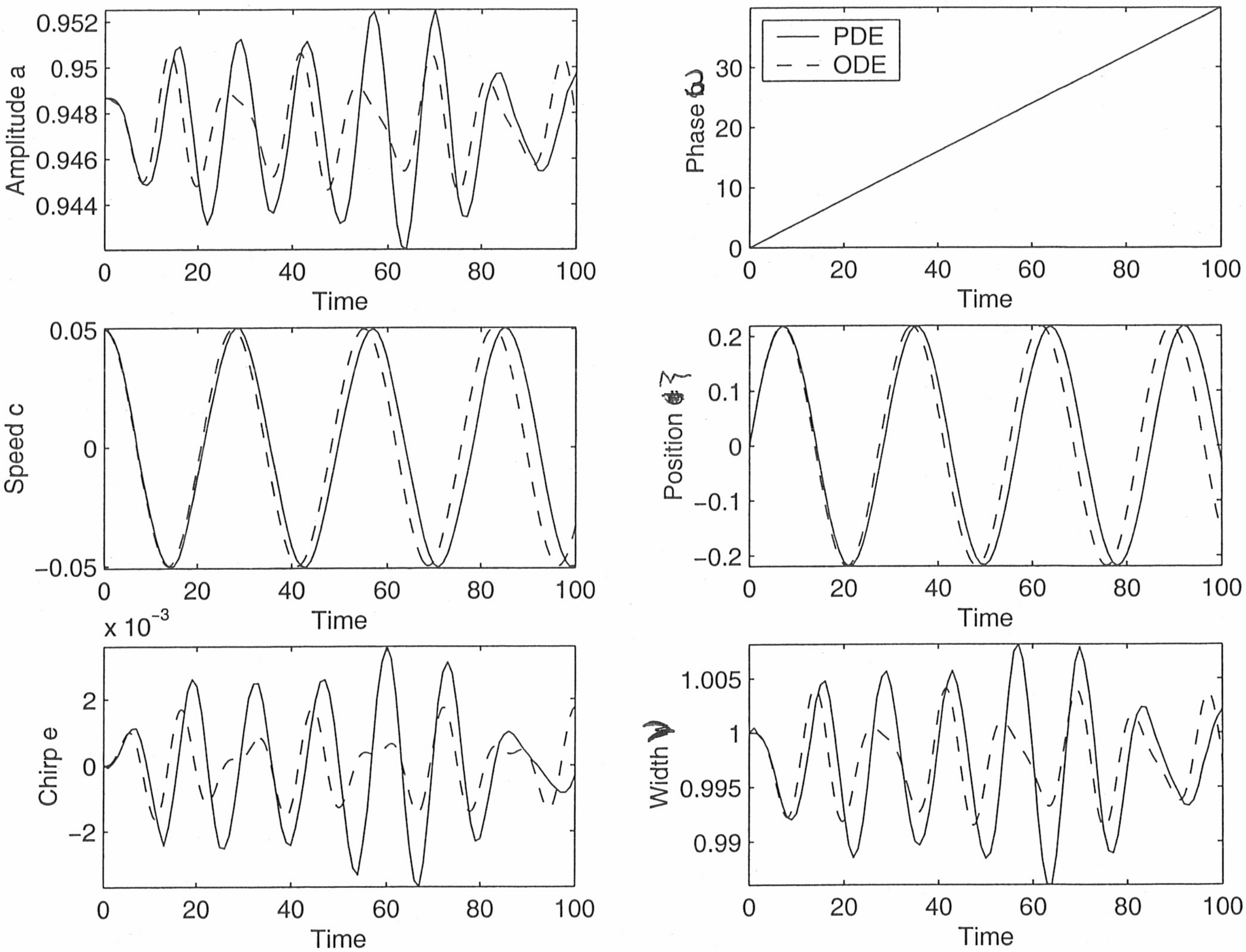
We note that M is now invertible. As we stated at the end of chapter 5, this is necessary for our ODE system to be well-posed and to yield an explicit solution of parameters. Thus our "chirped" ansatz produces an invertible mass matrix for the ODE parameter system. We now have the following inverse of the mass matrix M

$$M^{-1} = \begin{bmatrix} 0 & -\frac{3}{8}\frac{\left(\pi^2+4d^2w^2\right)w}{a\pi^2} & -3\frac{dw^3}{a\pi^2} & 0 & \frac{3}{2}\frac{w^3}{a\pi^2} & 0 \\ -\frac{3}{8}\frac{\left(-\pi^2+4d^2w^2\right)w}{a\pi^2} & 0 & 0 & -\frac{1}{2}\frac{dw}{a^2} & 0 & -\frac{1}{4}\frac{\left(12d^2w^2-\pi^2\right)w^2}{a^2\pi^2} \\ 3\frac{dw^3}{a\pi^2} & 0 & 0 & \frac{1}{2}\frac{w}{a^2} & 0 & 6\frac{w^4 d}{a^2\pi^2} \\ 0 & 0 & -\frac{1}{2}\frac{w}{a^2} & 0 & 0 & 0 \\ -\frac{3}{2}\frac{w^3}{a\pi^2} & 0 & 0 & 0 & 0 & -3\frac{w^4}{a^2\pi^2} \\ 0 & -\frac{1}{4}\frac{\left(\pi^2+12d^2w^2\right)w^2}{a^2\pi^2} & -6\frac{w^4 d}{a^2\pi^2} & 0 & 3\frac{w^4}{a^2\pi^2} & 0 \end{bmatrix}$$

and we may now write explicitly

$$P' = M^{-1} f \tag{6.42}$$

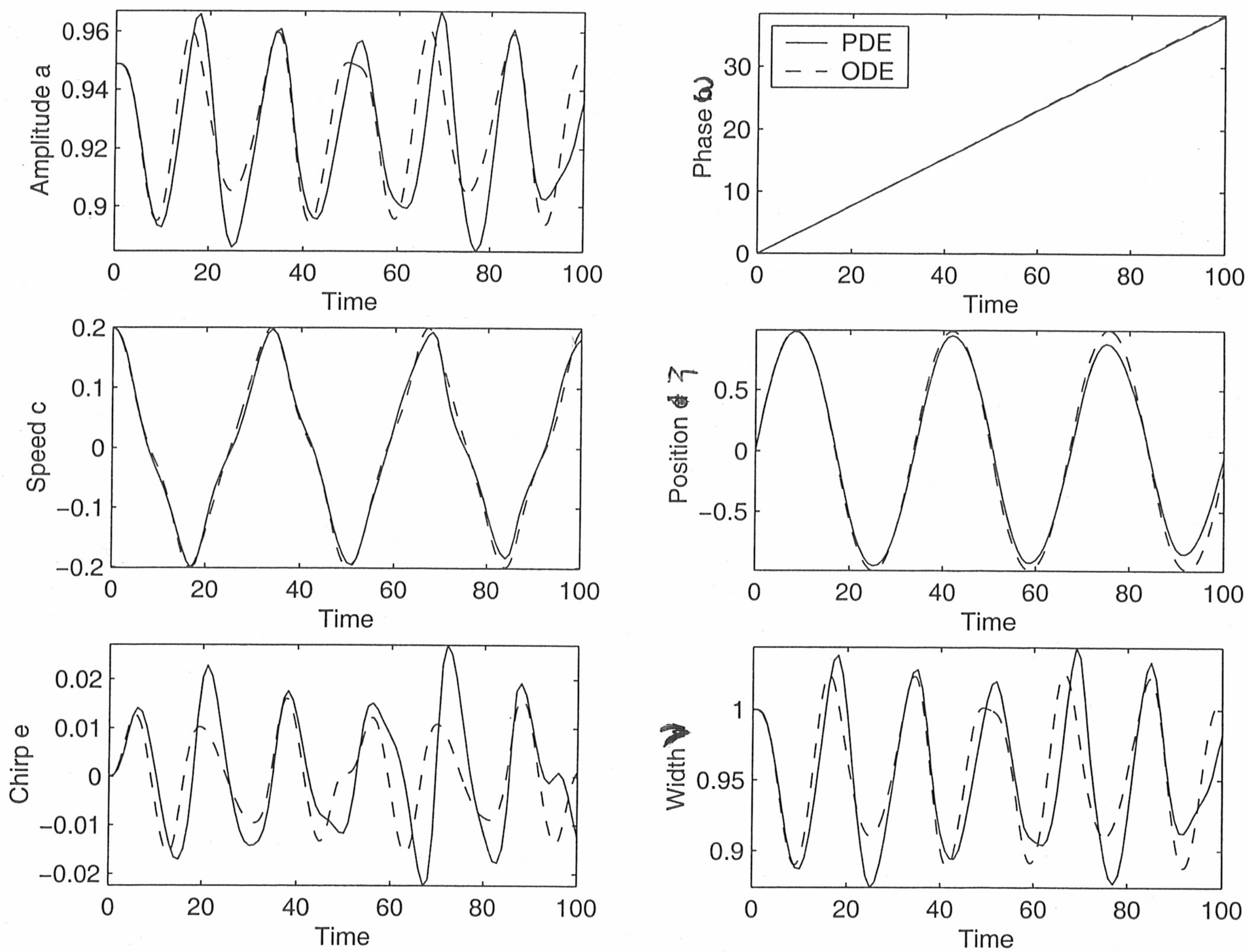
c = 0.05



IC:

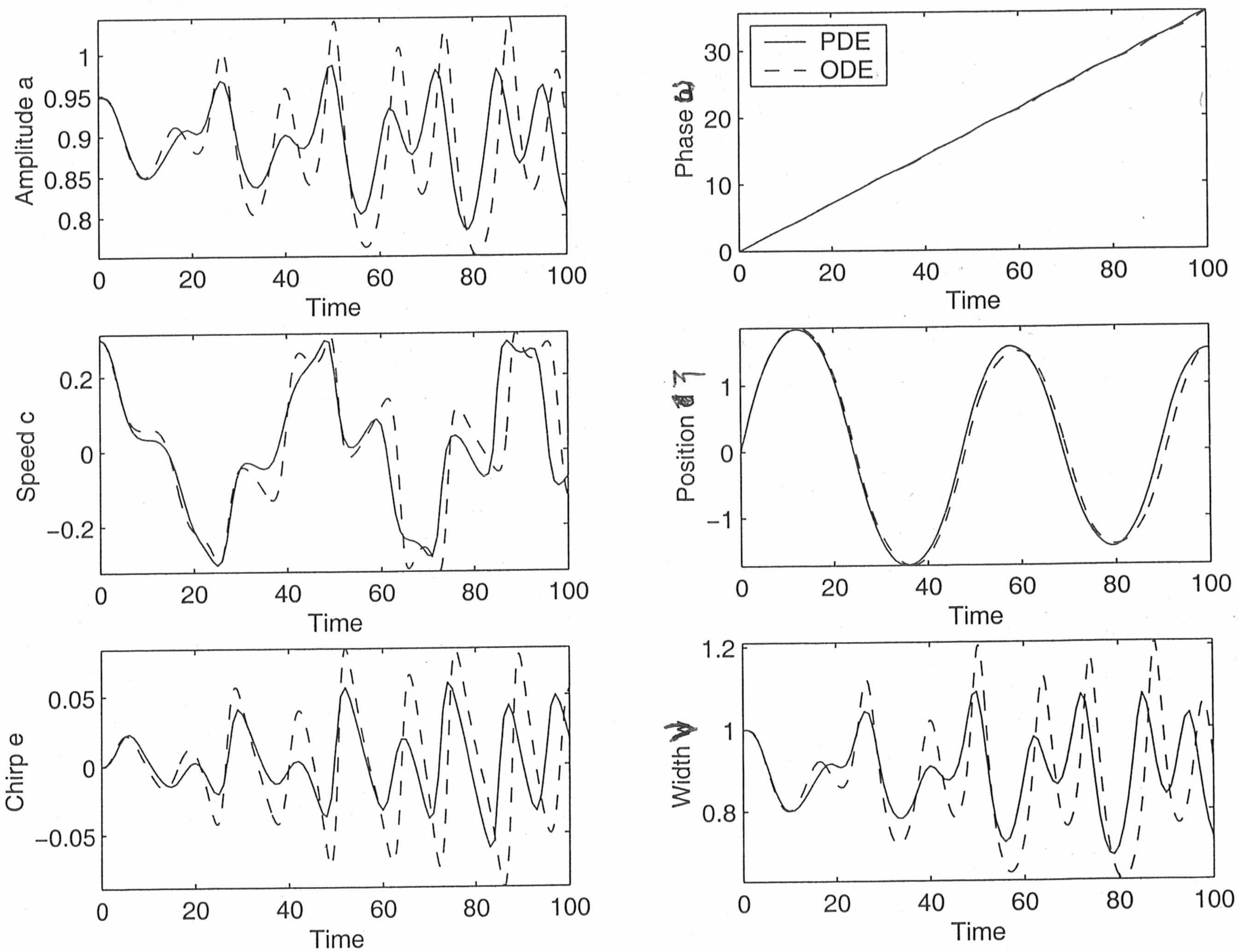
$$\begin{cases} V_0 = 0.1 \\ a = a_0 = \sqrt{1-V_0} \\ \omega = 0 \\ c = 0.05 \\ \zeta = 0 \\ \nu = 1 \end{cases}$$

c = 0.2 68



IC: $\begin{cases} V_0 = 0.1 \\ a = a_0 = \sqrt{1-V_0} \\ \omega = 0 \\ c = 0.2 \\ \zeta = 0 \\ \nu = 1 \end{cases}$

c = 0.3



ICs: $\begin{cases} V_0 = 0.1 \\ a = a_0 = \sqrt{1-V_0} \\ \omega = 0 \\ c = 0.3 \\ \zeta = 0 \\ \gamma = 1 \end{cases}$