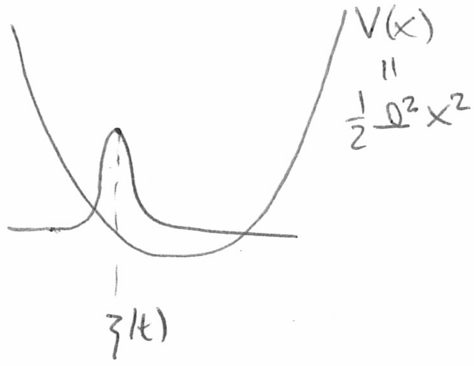
Describe next HW:

Ⓐ osillating bright soliton inside a parabolic trap.



- NLS: $i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = V(x)\psi$
- prove that Lagrangian is the same + $V(x)|\psi|^2$
- Ansatz: $\psi = \frac{\sqrt{N}}{\pi^{1/4}\sqrt{w}} e^{\left[-\frac{(x-\zeta)^2}{w^2} + i\beta(x-\zeta)^2 + i\phi\right]}$

Params: $w, \zeta, \beta, \cancel{\phi}$

- obtain $\ddot{\zeta} = -\frac{d}{d\zeta} V_{eff}(\zeta)$
- Compare PDE vs ODE.

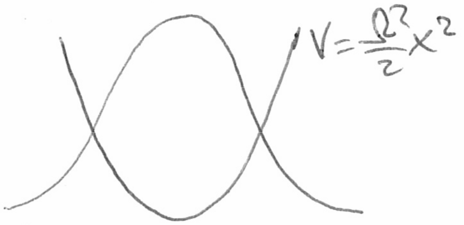
Ⓑ Nonlinear management

* optics index of refraction can be spatially modulated

* BEC nonlinearity (atom scattering) can be modulated:

NLS: $i\psi_t + \frac{1}{2}\psi_{xx} - U(t)|\psi|^2\psi = V(x)\psi$ (repulsive BEC)

start with steady state: and use again Gaussian ansatz.



Q2: what happens if $U(t) = U_0(1 + \varepsilon \sin(\alpha t))$

Q1: what happens if initial width/height are not @ steady state? $\longrightarrow \underbrace{\text{breathing}}_{\text{periodic}}$

Q3: Resonnances: what if α is tunned to ~~same~~ resonnate with periodic breathing?

Ⓒ a) Use a sech ansatz to recover NLS soliton

b) perturb with loss

c) — " — with loss/gain.

Perturbation theory within Variational Approach

[Scott: 7.4.2] p 316 – 320]

Take NLS:

$$iu_t + \frac{1}{2}u_{xx} \pm |u|^2 u = 0$$

+ : attractive
− : repulsive

Be careful, Scott's NLS is
$iu_t + u_{xx} + 2|u|^2 u = 0$
so everything has to be renormalized

Perturb the equation:

$$iu_t + \frac{1}{2}u_{xx} \pm |u|^2 u = \underbrace{\varepsilon}_{|\varepsilon|<<1} \overbrace{R(u,u_x,u_t,\dots\dots,x,t)}^{\text{perturbation}}$$

Lagrangian for unperturbed:

$$L = \frac{i}{2}[uu_t^* - u^*u_t] + \frac{|u_x|^2}{2} \mp \frac{1}{2}|u|^4$$

$$-L = i/2[u^*u_t - u_t^*u] - \frac{|u_x|^2}{2} \pm \frac{1}{2}|u|^4$$

soliton solution (bright ⊕)

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = 0$$

sol: $a \operatorname{sech} a(x-ct+x_0) \exp i[cx + \frac{a^2c^2}{2c}t + \phi_0]$

Ansatz: $\boxed{u_A = a\operatorname{sech}(a(x-\zeta))\exp i[c(x-\zeta)+ib]}$

Lag: $L_A = \frac{i}{2}[\quad \dots \quad] = L(u_A)$

E-L eq:

$$\begin{cases} \dot a = 0 \\ \dot c = 0 \\ \dot\zeta = c \\ \dot b = \cancel{a^2 - c^2}\,\overset{\frac{1}{2}}{\alpha} a^2 + \overset{\frac{1}{2}}{\beta}\zeta^2 \quad (\alpha,\beta)\in\mathbb{R} \end{cases}$$

Show VA_NLS.m

HW

$u = Ae^{i\phi}$

$* \boxed{\frac{i}{2}(uu_t^* - u^*u_t)}$

$= \frac{i}{2}(Ae^{+}(A_te^{-} - iA\phi_te^{-}) - \dots$

$= \boxed{A^2\phi_t}$

$* \boxed{\frac{1}{2}|u_x|^2 =}$

$\frac{1}{2}u_xu_x^* = $ —

$\dots = \frac{1}{2}$ —

$= \boxed{\frac{1}{2}(A_x^2 + A^2\phi_x^2)}$

So if ansatz = sech * modulation with chirp
⟹ one recovers the NLS soliton

Perturbed: Do Euler-Lag eq:

unperturbed: $\frac{d}{dt}\frac{\partial \bar L_A}{\partial \dot p} = \frac{\partial \bar L_A}{\partial p}$ ⟹ perturbed $\frac{\partial \bar L_A}{\partial p} - \frac{d}{dt}\frac{\partial \bar L_A}{\partial \dot p} = ?$

$$L \equiv L_A = L(u, u^*, u_t, u_t^*, u_x, u_x^*)$$

RHS: $\frac{\partial \bar L_A}{\partial p} = \frac{\partial}{\partial p}\int_{-\infty}^{\infty} L_A\, dx = \int_{-\infty}^{\infty}\left[\frac{\partial L}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial L}{\partial u_t}\frac{\partial u_t}{\partial p} + \underline{\frac{\partial L}{\partial u_x}\frac{\partial u_x}{\partial p}}\right] + \underset{u^*,\, u_t^*,\, u_x^*}{\text{c.c.}}\; dx$

but $\int_{-\infty}^{\infty}\frac{\partial L}{\partial u_x}\frac{\partial u_x}{\partial p}\,dx = \int_{-\infty}^{\infty} uv' = uv - \int u'v$

$= [\boxed{\ }]_{-\infty}^{\infty} \to 0 - \int \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x}\frac{\partial u}{\partial p}\,dx$

$u = \frac{\partial L}{\partial u_x}$ $\quad u' = \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x}$

$v = \frac{\partial u}{\partial p}$ $\quad v' = \frac{\partial u_x}{\partial p}$

$$\text{RHS} = \frac{\partial \bar L_A}{\partial p} = \int_{-\infty}^{\infty}\left[\frac{\partial L}{\partial u}\frac{\partial u}{\partial p} + \cancel{\frac{\partial L}{\partial u_t}\frac{\partial u_t}{\partial p}} - \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x}\frac{\partial u}{\partial p}\right] + \text{cc}$$

now LHS: $LHS = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot p}\right) = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial u_t}\frac{\partial u_t}{\partial \dot p}\right] + cc$

since the only term with $\dot p$ is u_t. $\Rightarrow \frac{\partial L}{\partial u}\cancel{\frac{\partial u}{\partial \dot p}}^{0} + \frac{\partial L}{\partial u_x}\cancel{\frac{\partial u_x}{\partial \dot p}}^{0}$

now, $\frac{\partial u_t}{\partial \dot p} = \frac{\partial \frac{\partial u}{\partial t}}{\partial \frac{\partial p}{\partial t}} = \frac{\partial u}{\partial p}$ if everything smooth.

$$\Rightarrow LHS = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial u_t}\frac{\partial u}{\partial p}\right] + cc.$$

$$= \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right)\frac{\partial u}{\partial p} + \frac{\partial L}{\partial u_t}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial p}\right)\right] + cc$$

$$= \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right)\frac{\partial u}{\partial p} + \cancel{\frac{\partial L}{\partial u_t}\frac{\partial u_t}{\partial p}}\right] + cc$$

cancel with previous page

$\therefore$ RHS−LHS. $\Rightarrow \int_{-\infty}^{\infty}\frac{\partial u}{\partial p}\Big[\underbrace{\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right)}_{①}\Big] + cc$

① → $L = \frac{i}{2}\left[u u_t^* - u^* u_t\right] + \frac{1}{2}|u_x|^2 \mp \frac{1}{2}|u|^4$ $\quad \mp\frac{1}{2}u^2u^{*2}$

$\Rightarrow \frac{\partial L}{\partial u} = \frac{i}{2}u_t^* \mp \frac{1}{2}2uu^{*2}$ ‖ $\frac{\partial L}{\partial u_x} = \frac{1}{2}u_x^*$ ‖ $\frac{\partial L}{\partial u_t} = -\frac{i}{2}u^*$

$\Rightarrow$ [illegible]

$① = \frac{i}{2}u_t^* \mp \frac{1}{2}2uu^{*2} - \frac{1}{2}u_{xx}^* + \frac{i}{2}u_t^*$
$= iu_t^* - \frac{1}{2}u_{xx}^* \mp |u|^2u^*$
$= -NLS^*$

~~Remember that $\delta\iint L\,dx\,dt = 0 \Leftrightarrow$ NLS~~

~~since NLS $\Leftrightarrow \frac{\delta L}{\delta u^*} = 0 \Leftrightarrow \frac{\partial}{\partial t}\frac{\partial L}{\partial u_t^*} + \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x^*} - \frac{\partial L}{\partial u^*} = 0$~~

~~$\therefore -NLS^* = 0 \Leftrightarrow \underbrace{-\frac{\partial}{\partial t}\frac{\partial L}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} + \frac{\partial L}{\partial u}}_{①} = 0$~~

[See Malomed Prog. Opt. 43 (2002) 71.]

Now, add perturb: $\overset{-NLS^*}{①} = 0 \Rightarrow \overset{-NLS^*}{①} = -\varepsilon R^*$

[Check SIGN] ✓

$\therefore$ RHS − LHS $= -\varepsilon\int_{-\infty}^{\infty}\frac{\partial u}{\partial p}R^* + cc$ $= -2\varepsilon\,\mathrm{Re}\left[\int_{-\infty}^{\infty} R\frac{\partial u^*}{\partial p}\right]$

$$\therefore \boxed{\frac{\partial L}{\partial p} - \frac{d}{dt}\frac{\partial L}{\partial \dot p} = -\varepsilon\int_{-\infty}^{\infty}\left(R^*\frac{\partial u}{\partial p} + R\frac{\partial u^*}{\partial p}\right)dx}$$

Perturbed E-L equns.

Ex 1: add loss to NLS: [Loss: suppose $|u|\ll 1$ & ~~u const~~ $u(x,t) = u(t)$ $\Rightarrow \cancel{i}u_t = -\cancel{i}\varepsilon u \Rightarrow u(t) = u_0 e^{-\cancel{i}\varepsilon t}$]

$$iu_t + \frac{1}{2}u_{xx} \pm |u|^2u \quad + \varepsilon i u = 0$$

, ie: $\varepsilon R = -i\varepsilon u$

Loss: dispersion relation: $u = Ae^{i(kx-\omega t)}$

$\Rightarrow iA(-i\omega)\cancel{e} + \frac{1}{2}A(ik)^2\cancel{e} \pm A^3\cancel{e} + \varepsilon i A\cancel{e} = 0$

$\rightarrow A\omega - \frac{1}{2}Ak^2 \pm A^3 + \varepsilon i A = 0$

$\Rightarrow \boxed{\omega = \underbrace{\frac{1}{2}k^2 \mp A^2}_{\text{dispersion}} \underbrace{- i\varepsilon}_{\text{dissipation}}} \Rightarrow u = Ae^{ik\left(x-\left(\frac{1}{2}k^2 \mp A^2\right)t\right)}\underbrace{e^{-\varepsilon t}}_{\text{dissipation}}$

Q: what happens to soliton when we introduce some small dissipation? This is what happens in an optical fiber.

Perturbation : $R=-iu$

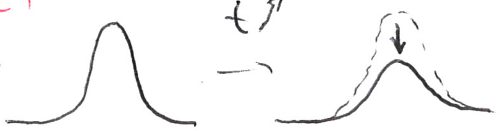
perturbed E-L eqs $\Longrightarrow$ - - - $\bigcirc\!\!\!\!\!\!\!\text{1+IX}$

(I think $\alpha = \frac{1}{2} = \beta$ (?)) $\Leftarrow$ See Maple VA_NLS.

$\boxed{\gamma=+2}$

$(\alpha, \beta \in \mathbb{R})$

after neglecting ε^2 terms $\Longrightarrow$
$$\begin{cases} \dot a = -\gamma\varepsilon a \\ \dot c = 0 \\ \dot\xi = c \\ \dot\sigma = a^2 \cancel{-c^2} = \alpha a^2 + \beta\xi^2 \end{cases}$$



Show NLS.m
NLS_spectral:
meq= αuu^* - $\boxed{0.01}\,i$ ε
meq= αuu^* - $0.01\,i$

$a(t) = a_0 e^{-\varepsilon\gamma t}$ $\therefore$ soliton keeps its shape but decays exponentially

$\underline{\gamma=+2}$ $\Longrightarrow$ soliton decays twice as fast as a low amplitude sinusoidal wave.

Matlab } try plane wave $\rightarrow e^{-\varepsilon t}$
try soliton $\rightarrow e^{-2\varepsilon t}$

$\underline{\text{Ex2}}$: add loss AND gain:

In an optical fiber one can add gain but the gain has to have a saturation (physical) :

$$\varepsilon R = -i\underbrace{\Gamma u}_{\substack{\text{damping}\\\text{loss}}} + i\underbrace{\frac{g_0}{1+P/P_s}\,u}_{\substack{\text{forcing}\\\text{gain}}} \qquad P_s, g_0, \Gamma > 0$$

the gain is saturated for large wave amplitudes.

$P = \int_{-\infty}^{\infty} |u|^2 dx$

and P_s: saturation power level. Ie. when $P \sim P_s$ then saturation kicks in.

Similar eq to, since thus, is for $iu_t + u_{xx} + 2|u|^2u = 0$ [Scott: 7.4.2]

$\underline{\text{Perturbed E-L}}$: - - -
$$\dot a = -2\Gamma a + \frac{2g_0}{1+\frac{2a^2}{P_s}}\, a \qquad \bigcirc\!\!\!\!\!\!\!\text{1+IX}$$

For $g_0 = 0$ we recover previous result, but for $g_0 > 0$ we indeed have a stationary solution when:

$$\Gamma = \frac{g_0}{1+2a^2/P_s} \Rightarrow \boxed{a = a_\infty \equiv \sqrt{\frac{(g_0-\Gamma)P_s}{2\Gamma}}}$$

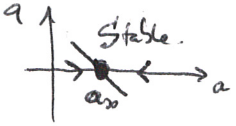
Ie: the soliton will adjust to $a \to a_\infty$

- $a(0) < a_\infty \Rightarrow$ gain > loss $\Rightarrow a \nearrow$
- $a(0) > a_\infty \Rightarrow$ gain < loss $\Rightarrow a \searrow$

$\therefore$ $a = a_\infty$ is a stable equilibrium.

∘ $0<a<<1$
$\Rightarrow \dot a \simeq -2\Gamma a + 2g_0 a$
$\simeq +2a[g_0-\Gamma]$
If $g_0 > \Gamma \Rightarrow \dot a > 0$ for $a << 1$

• $a >> 1 \Rightarrow \dot a \simeq -2\Gamma a < 0$
$\therefore$ $a >> 1 \Rightarrow \dot a < 0$



Soliton - soliton interactions using perturbation theory and the VA. [Based on Gerdjikov et. al, PRE, 55 (1997) 6039]

- NLS : $i u_t + \frac{1}{2} u_{xx} + |u|^2 u = 0$

- suppose we have 2 solitons that are well separated

such that if

$$u_i = 2\nu_i \operatorname{sech} 2\nu_i[x - \xi_i(t)] e^{i\left[\frac{\mu_i}{\nu_i}[2\nu_i(x-\xi_i(t))] + \delta_i(t)\right]}$$

$$\xi_i(t) \equiv 2\mu_i t + \xi_{i0}$$
$$\delta_i(t) = 2(\mu_i^2 - \nu_i^2) t + \delta_{i0}$$

|| 4 params $\vec{p}(t) = \begin{pmatrix} \mu_i(t) \\ \nu_i(t) \\ \xi_i(t) \\ \delta_i(t) \end{pmatrix}$

- suppose that $u = u_1 + u_2$ is a sol. to NLS:

$$i(u_{1t} + u_{2t}) + \frac{1}{2}(u_{1xx} + u_{2xx}) + (u_1+u_2)^*(u_1+u_2)(u_1+u_2) = 0$$

$$\Rightarrow \quad \text{———} \quad \text{———} \quad + (u_1^* + u_2^*)(u_1^2 + 2u_1u_2 + u_2^2) = 0$$

$$\Rightarrow i(\underline{u_{1t}} + \underline{\underline{u_{2t}}}) + \frac{1}{2}(\underline{u_{1xx}} + \underline{\underline{u_{2xx}}}) + \left[\underline{|u_1|^2 u_1} + \underline{\underline{|u_2|^2 u_2}} + \underline{2|u_1|^2 u_2} + \underline{\underline{u_1^* u_2^2}} + \underline{\underline{2|u_2|^2 u_1}} + \underline{u_2^* u_1^2}\right.$$

$$\Leftarrow \quad i u_{it} + \frac{1}{2} u_{ixx} + |u_i|^2 u_i + 2|u_i|^2 u_j + u_i^2 u_j^* = 0 \qquad \{i,j\} = \{1,2\} \ \& \ \{2,1\}$$

$$\Rightarrow \quad i u_{it} + \frac{1}{2} u_{ixx} + |u_i^2| u_i = -\underbrace{(2|u_i|^2 u_j + u_i^2 u_j^*)}$$

perturbation induced on u_i by soliton j. $= iR$

$$\therefore \quad R = i(2|u_i|^2 u_j + u_i^2 u_j^*) \quad \leftarrow \text{small if solitons are separated}$$

- In Gerdjikov PRE 55 (1997) 6039 they consider 3 solitons and disregard higher order neighbors.

$$\begin{tikzcd} u_{i-1} \arrow[r, bend right, leftrightarrow] \arrow[rr, bend right=60, leftrightarrow, "\times"] & u_i \arrow[r, bend right, leftrightarrow] & u_{i+1} \end{tikzcd}$$

- Apply perturbation with VA and:

$$\frac{d}{dt}\vec{p}_k(t) = \frac{d}{dt}\begin{bmatrix} \nu_k \\ \mu_k \\ \xi_k \\ \delta_k \end{bmatrix} = f\left[\vec{p}_k, \vec{p}_{k-1}, \vec{p}_{k+1}\right]$$

see [Gerdjikov p6043 Eqs ~~(30)-(33)~~ (35)-(38)]

- More simplifications
  - $\nu_n \to \nu$, $\mu_n \to \mu$ in S & C's
  - $S, C \ll \mu$'s, ν's

$\Rightarrow$ Eqs (70)-(73)

- change of variables:

$$\boxed{\frac{d^2 q_k}{d\tau^2} = e^{q_{k+1}-q_k} - e^{q_k-q_{k-1}}}$$ ~~Eq~~ Toda Chain

with $q_k = q_k(\nu, \xi_k, \delta_k, \delta, \mu)$ see Eq (88)-(89)

- Very good agreement between numerics & approximation (even for large overlap). see Fig 3 & Fig 4

Ex: What is the dynamics of two nearby bright solitons? Dependence of phase?

Matlab

Perturbation theory for dark solitons

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Base on [Kivshar+..., PRE $\underline{49}$ (1994) 1657]

- NLS: $iU_t + \frac{1}{2}u_{xx} - |u|^2 u = 0$ (2) $z \to t$
- plane wave $u = u_0 e^{i(kx - \beta t)}$ (7)

$\Rightarrow$ dispersion relation: $\beta = \frac{1}{2}k^2 + u_0^2$ (7)

- Integrals of motion for plane wave sol.

~~Energy~~ $E = \dots = \frac{P^2}{2L} + \frac{I^2}{2P}$

Momentum $I = \dots = k u_0^2 L = kP$ $\quad \int_{-\infty}^{\infty} \to \int_{-L}^{L}$

mass (power) $P = \dots = u_0^2 L$

$\frac{1}{2}ku_0^2(2L)$

- Perturbation of plane wave:

$$u = \underbrace{(u_0+v)}_{\text{amplitude pert}} e^{ikx - i\beta t + i\overset{\text{phase pert.}}{\psi}}$$

NLS $\Rightarrow$ $i\,\cancel{(u_0+v)}(-i\beta + i\dot\psi)\cancel{e} + \frac{1}{2}\cancel{(u_0+v)}(ik + i\psi_{xx})^2\cancel{e} - (u_0+v)^{\cancel{3}\,2}\cancel{e} = 0$

$\Rightarrow \beta - \dot\psi + \frac{1}{2}(ik + i\psi_{xx})^2 - (u_0+v)^2 = 0$

$\Rightarrow \beta - \dot\psi + \frac{1}{2}(-k^2 - 2k\psi_{xx} - \psi_{xx}^2) - (u_0^2 + 2u_0 v + v^2) = 0$

$|v| \ll 1$ & $|\psi_{xx}| \ll 1$

$\Rightarrow \beta - \dot\psi + \frac{k^2}{2} - k\psi_{xx} - u_0^2 - 2u_0 v = 0$

$\Rightarrow \psi_t + k\psi_{xx} = \beta - \frac{k^2}{2} - u_0^2 - 2u_0 v$

similar to what we did before

NLS $\Rightarrow$ $iv_t\,~~(u_0+v)~~\,e + i(u_0+v)(-i\beta + i\psi_t)e +$

skip

separate the 2 eqns. and suppose

$v \propto e^{+i\Omega t - iqx} = e^{-i(qx - \Omega t)}$

$\psi \propto e^{+i\Omega t - iqx}$

We obtain the dispersion relation for plane wave solutions:

$$(\Omega - kq)^2 = q^2\left(u_0^2 + \frac{1}{4}q^2\right)$$

$\therefore$ ~~if $\Omega > kq$ then $\Omega \in \mathbb{R}$~~ $\Rightarrow$ no exponential growth! $\Rightarrow$ ~~stable~~.

$\Rightarrow \Omega = q\left(\sqrt{u_0^2 + q^2/4} + k\right) \in \mathbb{R}$ $\therefore$ all perturbations are stable

- Now consider DS with a ~~part of~~ plane carrier wave (7) ~~[illegible]~~

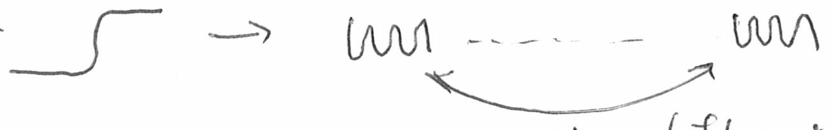
$$DS: \; u(x,t) = u_0 \{A \tanh[u_0 A(x-Vt)] + iB\} e^{-ikx - i\beta t + i\phi_0} \qquad (12)$$

and $\beta = \frac{1}{2}k^2 + u_0^2$ and $u_0 B = V - k$, $A^2 + B^2 = 1$

DS: 3 params A, B, V but 2 equs ie, only one indep. and the other 2 fixed by background plane wave.

- Note that Sol (12) has a plane wave asymptotics @ $x,t \to \pm\infty$

But since tanh



phase shift. $= \Delta\phi$

$$\rightarrow \Delta\phi = 2\left[\tan^{-1}\left(\frac{B}{A}\right) - \frac{\pi}{2}\right] = -2\tan^{-1}\left(\frac{A}{B}\right)$$

~~see: $x = -\infty$ $\quad u_\infty \sim (A + iB)e = (A^2+B^2)e^{+i\tan^{-1}\frac{B}{-A}}$~~

~~$u_{-\infty} \sim (A + iB)e = (A^2+B^2)e^{+i\tan^{-1}\frac{B}{A}}$ $\Big\} \Delta\phi =$~~

- let us now use integrals of motion
  - → problem: integrals are ∞
  - → let us remove the background since we are interested in DS only.
    - → commonly known as regularizing the integral (also renormalizing).

- let us do the regularization for $k=0$, ie: flat background:

$$\beta = u_0^2 \;, \quad u_0 B = V \;, \quad A^2 + B^2 = 1 \;, \quad \boxed{C = u_0}$$

also for $k = 0$ disp. relation for background wave.

skip

- **Power:** $\boxed{P_s = \int_{-\infty}^{\infty} (|u|^2 - u_0^2)\, dx}$ (since $P = u_0^2 L$)

$$P_s = 2u_0 A$$

- **Momentum:** one has to substract a term related with phase shift. because $k = k(x)$
  - → since $I = \frac{1}{2}k u_0^2 \cancel{L}$ ← contribution of background.
  - → @ $x = \pm\infty$ this is zero, but close to DS is not, so one has to remove $\frac{1}{2}\int u_0^2 k = \frac{1}{2}u_0^2 \int k(x)\,dx$

and $\underset{\substack{\uparrow \\ \text{wave \#}}}{\int_{-\infty}^{\infty} k(x)\,dx} = \Delta\phi = -2\tan^{-1}\left(\frac{A}{B}\right)$

skip

skip

$$\therefore \boxed{I_s = \frac{i}{2}\int_{-\infty}^{\infty} (u u_x^* - u^* u_x)dx - u_0^2 \Delta\phi} \underset{k=0}{=} I - u_0^2\Delta\phi$$

$$= -2V\sqrt{c^2-V^2} + 2c^2 \tanh^{-1}\left(\frac{\sqrt{c^2-V^2}}{V}\right)$$

where $\boxed{c \equiv u_0}$

- **Energy** : similarly :

$$\boxed{E_s = \int_{-\infty}^{\infty}\left(\frac{1}{2}|u_x|^2 + \frac{1}{2}(|u|^2 - u_0^2)^2\right)dx}$$

$$\underset{k=0}{=} \frac{4}{3}(c^2-V^2)^{3/2}$$

Now we are ready to use this conservation laws that remove background and follow the DS:

III.A Constant background

- perturbed NLS : $iu_t + \frac{1}{2}u_{xx} - |u|^2 u = \epsilon P(u)$
- assume perturbation does not change background @ $x = \pm\infty$
  ie: $P(u(x=\pm\infty)) = 0$
- constant background $(k=0) \Rightarrow$ plane wave $u = u_0 e^{-iu_0^2 t}$
  $(\beta = \frac{1}{2}k^2 + u_0^2)$

- Let us then factor out the constant background from the solution we are interest on:

$$u(x,t) = u_0 e^{-iu_0^2 t} v(x,t)$$

then follow $v(x,t)$:

$u(x,t)$ in NLS $\Rightarrow$ $$i v_\tau + \frac{1}{2}v_{\xi\xi} - (|v|^2-1)v = \varepsilon \tilde{P}(v) \quad (23)$$

where $\tau = u_0^2 t$ & $\xi = u_0 x$

- Analyze how DS params change due to presence of $\varepsilon\tilde{P}(v)$
- Ansatz for DS in eq for $\varepsilon = 0$

$$\boxed{\begin{gathered} v(\overset{\xi}{\cancel{x}},\tau) = \overset{\eta}{\cancel{\cos\phi}} \tanh\eta(\xi - \Omega\tau) - i\,\overset{\Omega}{\cancel{\sin\phi}} \\ \eta = \cos\phi,\ \Omega = \sin\phi = \text{vel.} \end{gathered}}$$

with ~~a single~~ a single parameter $\phi(\tau)$ ~~(since we already assume it is going @ speed c due to the background level)~~ (since we assume a const. background $\equiv 1$ because u_0 is factored out)

- Applying pert. machinery:

$$\boxed{\frac{d\phi}{d\tau} = \frac{\varepsilon}{2\cos^2\phi\sin\phi}\,\mathrm{Re}\left[\int_{-\infty}^{+\infty}\left(\tilde{P}(v)\frac{\partial v^*}{\partial \tau}\right)d\xi\right]} \quad (29)$$

Application : ~~loss~~/gain with saturation

NLS : $i u_t + \frac{1}{2} u_{xx} - |u|^2 u = \underbrace{i\alpha u}_{\text{~~laser~~ linear gain}} - \underbrace{iK|u|^2 u}_{\text{nonlinear absorption}}$

[α, K small] !

dispersion relation : $u = \boxed{u_0(t)}\, e^{i(kx-\beta t)}$ ← keep this to account for NONLINEARITY

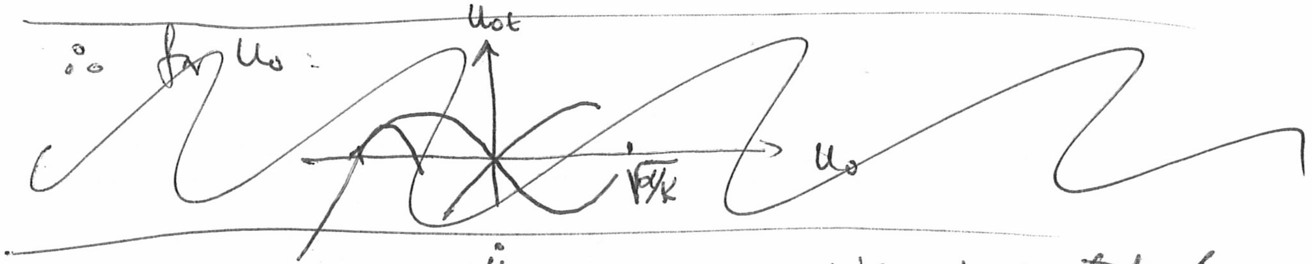
$$i(u_{0t}\cancel{e} + u_0(-i\beta)\cancel{e}) + \frac{1}{2}u_0(ik)^2\cancel{e} - u_0^3(t)\cancel{e} = i\alpha u_0\cancel{e} - iK u_0^3\cancel{e}$$

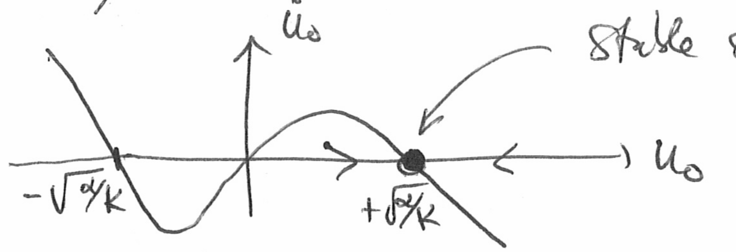
$$\rightarrow i\left(\frac{u_{0t}}{u_0} + (-i\beta)\right) + \frac{1}{2}u_0k^2 - u_0^2 = i\alpha - iKu_0^2$$

$$\rightarrow i\frac{u_{0t}}{u_0} + \beta - \frac{1}{2}u_0k^2 - u_0^2 = i\alpha - iKu_0^2$$

$$\Rightarrow \begin{cases} u_{0t} = \alpha u_0 - K u_0^3 = u_0(\alpha - Ku_0^2) \\ \beta = \frac{1}{2}u_0k^2 + u_0^2 \end{cases}$$

∴ for u_0 :





stable steady state for $u_0^2 = u_\infty^2 = \frac{\alpha}{K}$

If $u_0 < u_\infty \rightarrow$ gain
$u_0 > u_\infty \rightarrow$ loss
} interplay between linear and the nonlinear gain.

- Let us study a DS on this stable background.

$\rightarrow$ ansatz $\quad u(x,t) = u_\infty\, v(x,t)\, e^{-iu_\infty^2 t}$ ⊛

$\xrightarrow{NLS}$ $\quad i v_y + \frac{1}{2} v_{zz} - (|v^2| - 1)v = -iK(|v|^2 - 1)v$] α no longer here because normalization $y = \frac{t\alpha}{K}$

$y \equiv \frac{t\alpha}{K}$, $z \equiv x\sqrt{\frac{\alpha}{K}}$

- Note that $v = 1$ we recover stable $u = u_\infty$ background, but as soon as $|v| \neq 1$ then we have a perturbation ~~constant~~ of size K

⊛ in (29) $\Rightarrow P(v) = -iK(|v|^2-1)v$, $v = \eta\tanh\eta(z - \Omega y) - i\Omega$, $\eta \equiv \cos\phi$, $\Omega \equiv \sin\phi$, $y \equiv \frac{t\alpha}{K}$

$\Rightarrow \frac{d\phi}{dy} = \frac{\epsilon^{=1}}{2\cos^2\phi\sin\phi} Re\left[\int_{-\infty}^{\infty} -iK(|v|^2-1)v\frac{\partial v^*}{\partial y}\, dz\right] = \frac{1}{2c^2s} Re\left[\int_{-\infty}^{\infty} -iK\left[\left(\eta^2 t^2 + \Omega^2\right) - 1\right](\eta t - i\Omega)\frac{\partial}{\partial y}[\eta t + i\Omega]\, dz\right] = \ldots$

Also on (29) : $\frac{d\phi}{dt} = \frac{\alpha}{3}\sin(2\phi)$ (back to t $[y = \frac{t\alpha}{K}]$)

$\Rightarrow \phi(t) = \tan^{-1}\left[\tan\phi_0\, e^{\frac{2}{3}\alpha t}\right]$ (independent on K)

$\therefore$

- $\phi(0) = 0 \rightarrow \phi(t) = 0 \Rightarrow$ stationary solution
  $v = \eta \tanh \eta(\xi - \Omega\tau) - i\Omega$
  $\eta = \cos\phi = 1$, $\Omega = \sin\phi = 0$
  $\uparrow$ vel.

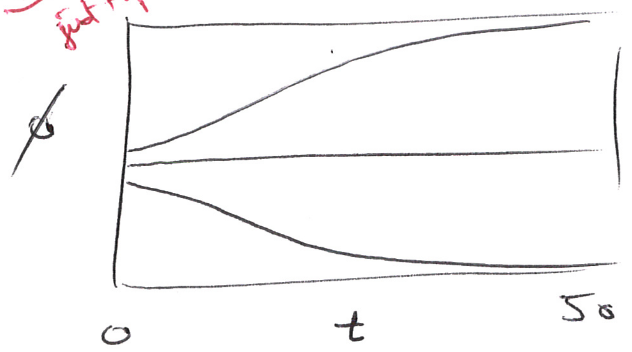
- $\phi(0) \neq 0 \Rightarrow \phi(\infty) = \tan^{-1}[\infty] = \frac{\pi}{2}\,\mathrm{sgn}(\phi_0)$
  $\Rightarrow \eta \to 0$, $\underset{\uparrow \text{vel}}{\Omega} = \pm 1$ $\therefore$ infinite width zero height.
  $\longrightarrow$ ie : decay of DS.

- Note that velocity of DS $= \Omega = \sin\phi$.
  So an initial steady DS $\Omega = 0$ $(\phi = 0)$ will be unstable since any change in ϕ will make $\phi \to \pm \pi/2$ and destroy solution.

Show Matlab spectral

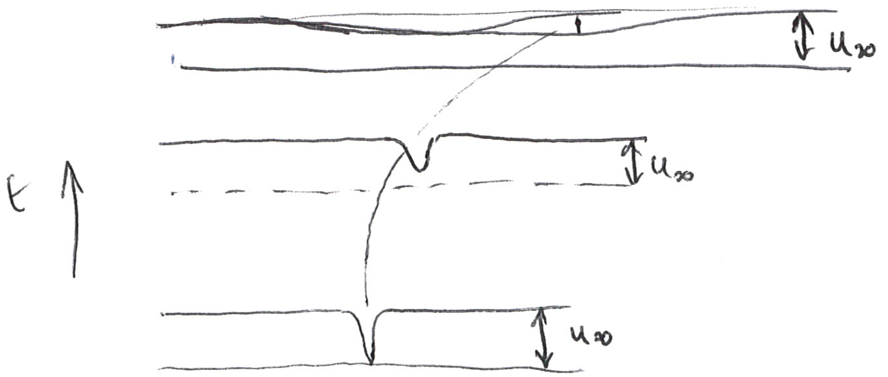
IC = (2 DS @ L/2 & -L/2).* exp(0.001*1i*x) -0.01*1i*x(1 - u.*conj(u))

just to pert.



$(\alpha = 0.1)$

- Dynamics $\longrightarrow$ if $\phi \neq 0 \Rightarrow$ soliton accelerates until it gets infinitely spread.



3D → 2D GPE Reduction

Let us start with fully dimensional 3D GPE eq:

$$i\hbar\psi_t = \Big[-\frac{\hbar^2}{2m}\nabla^2 + \underbrace{\frac{m}{2}(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2)}_{V(x,y,z)} + g|\psi|^2\Big]\psi \quad (1)$$

We suppose a "pancake" trap where $\omega_z \gg \omega_x \approx \omega_y \equiv \omega_r$

$$\Rightarrow \quad V(x,y,z) \longrightarrow \frac{m}{2}(\omega_r^2 r^2 + \omega_z^2 z^2)$$

In this quasi-2D scenario the wave function will not be able to be excited in the z-direction ⟹ in the z-direction the system will be in its "ground state".
[ground state of a harmonic trap → quantum harmonic oscillators
→ Gaussian]

So let us do:
a) separation of variables : $\psi_{3D} = \psi_{2D}\cdot\phi_0(z)\cdot f(t)$
b) method of averaging over z-direction

3D→2D : $\psi_{3D} = \psi_{2D}(x,y)\phi_0(z)e^{-i\frac{\gamma t}{\hbar}}$ (separation of variables)

$$\Rightarrow \phi_0\, i\hbar\frac{\partial}{\partial t}[\psi_2 e^{-i\gamma t}] = \Big\{-\Big(\frac{\hbar^2}{2m}\nabla_2^2\psi\Big)\phi_0 e + \Big\{-\frac{\hbar^2}{2m}\phi_0'' e\psi + \Big[V + g\phi_0^2|\psi_2|^2\Big]\phi_0\psi e$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}[\psi_2 e] = -\frac{\hbar^2}{2m}\nabla_2^2\psi e - \frac{\hbar^2}{2m}\frac{\phi_0''}{\phi_0}e\psi + [V + g\phi_0^2|\psi_2|^2]\psi e$$

$$\Rightarrow i\hbar\Big(\frac{\partial}{\partial t}\psi_2\Big)\cancel{e} + i\hbar\psi_2\Big(-i\frac{\gamma}{\hbar}\Big)\cancel{e} = -\frac{\hbar^2}{2m}\nabla_2^2\psi\cancel{e} - \frac{\hbar^2}{2m}\frac{\phi_0''}{\phi_0}\cancel{e}\psi + [\quad]\psi\cancel{e}$$

$$\div\psi_2 \Rightarrow i\hbar\frac{\partial_t\psi_2}{\psi_2} + \frac{\hbar\gamma}{\hbar} = -\frac{\hbar^2}{2m}\frac{\nabla_2^2\psi_2}{\psi_2} - \underline{\underline{\frac{\hbar^2}{2m}\frac{\phi_0''}{\phi_0}}} + [\underset{=}{V} + g\phi_0^2|\psi_2|^2]$$

$$i\hbar\frac{\partial_t\psi_2}{\psi_2} + \frac{\hbar^2}{2m}\frac{\nabla_2^2\psi_2}{\psi_2} - \frac{1}{2}m(\omega_x^2x^2+\omega_y^2y^2) - g\phi_0^2|\psi_2|^2 = -\frac{\hbar\gamma}{\hbar} - \frac{\hbar^2}{2m}\frac{\phi_0''}{\phi_0} + \frac{1}{2}m\omega_z^2z^2$$

$$\Rightarrow \begin{cases} \frac{\hbar^2}{2m}\phi_0'' - \frac{1}{2}m\omega_z^2 z^2\phi_0 + \frac{\hbar\gamma}{\hbar}\phi_0 = 0 \quad \Rightarrow -\frac{\hbar^2}{2m}\phi_0'' = \left[-\frac{1}{2}m\omega_z^2 z^2 + \gamma\right]\phi_0 \\ i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + \left[\frac{1}{2}V_2 + g_3\phi_0^2|\psi|^2\right]\psi \end{cases}$$

$\boxed{\phi_0}$

$$\phi_0(z) = A e^{-\frac{z^2}{2\beta^2}} \Rightarrow \cancel{\frac{\hbar^2}{2m}A\left(\frac{1}{4\beta^4}\right)} \quad \phi_0' = A\frac{(-\cancel{2}z)}{\cancel{2}\beta^2}e^{-\frac{z^2}{2\beta^2}}$$

$$\Rightarrow \phi_0'' = A\left[-\frac{1}{\beta^2}e - \frac{z}{\beta^2}\left(\frac{\cancel{2}z}{\cancel{2}\beta^2}\right)e\right]$$

$$\therefore \frac{\hbar^2}{2m}\left[-\frac{1}{\beta^2} + \frac{2z^2}{\beta^4}\right] - \frac{m}{2}\omega_z^2 z^2 + \frac{\hbar\gamma}{\cancel{\hbar}} = 0$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m\beta^2} + \frac{\cancel{\hbar}\gamma}{\cancel{\hbar}}\right] + \left[\frac{\hbar^2}{2m\beta^4} - \frac{m}{2}\omega_z^2\right]z^2 = 0$$

$$\Rightarrow \quad \gamma = \frac{\hbar^2}{2m\beta^2}, \quad m\omega_z^2 = \frac{\hbar^2}{m\beta^4}, \quad \gamma = \frac{\hbar^2}{2m}\frac{1}{\beta^2} = \frac{\hbar^2}{2m\,a_z^2} = \frac{\hbar^2}{2m}\frac{m\omega_z}{\cancel{\hbar}}$$

$$\boxed{\gamma = \frac{\hbar\omega_z}{2}}$$

$$\Rightarrow \boxed{\beta^2 = \frac{\hbar}{m\omega_z} = a_z^2} \qquad \left[a_z^2 \equiv \frac{\hbar}{m\omega_z}\right]$$

$$\therefore \phi_0(z) = A\,e^{-\frac{z^2}{2a_z^2}} \qquad \int_{-\infty}^{\infty} e^{-\frac{z^2}{2a^2}}dz = \sqrt{2\pi}\,a$$

$$\therefore \text{If } \underbrace{\int|\phi_0|^2 = 1}_{\text{Normalize}} \Rightarrow A^2\int e^{-\frac{z^2}{a_z^2}} = 1 \Rightarrow A^2 a_z\sqrt{\pi} = 1$$

$$\Rightarrow \boxed{A^{\cancel{2}} = \pi^{-1/4}a_z^{-1/2}}$$

$$\therefore \boxed{\phi_0(z) = \pi^{-1/4}a_z^{-1/2}e^{-\frac{z^2}{2a_z^2}}} \Rightarrow \phi_0'' = \pi^{-1/4}a_z^{-1/2}\left[-\frac{1}{a_z^2} + \frac{z^2}{a_z^4}\right]e$$

$$= \left[-\frac{1}{a_z^2} + \frac{z^2}{a_z^4}\right]\phi_0$$

$\boxed{\varphi_2}$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{1}{2}V_2 +$$

GPE : $i\hbar\psi_t = \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}V_3 + g_3|\psi|^2\right]\psi$

$$\psi_3 = \psi_2\phi_0 e^{-\frac{i\gamma t}{\hbar}}$$

Average over z $\boxed{\int GPE \cdot \phi_0^* \, dz}$ $\Rightarrow$ $i\hbar\frac{\partial\psi_2}{\partial t}\phi_0 + \gamma\psi_2\phi_0 = -\frac{\hbar^2}{2m}\nabla^2\psi_2\phi_0 - \frac{\hbar^2}{2m}\phi_0''\psi_2 + [V + g\phi_0^2|\psi_2|^2]\psi\phi_0$

~~$\Rightarrow$ $i\hbar\frac{\partial\psi_2}{\partial t} + \gamma\psi_2 = -\frac{\hbar^2}{2m}\nabla^2\psi_2 - \frac{\hbar^2}{2m}\left[-\frac{1}{a_z^2} + \frac{z^2}{a_z^4}\right]\psi_2 + [V + g$~~

$\times\phi_0^*$ $(= \times\phi_0)$ $\Rightarrow$ $i\hbar\frac{\partial\psi_2}{\partial t}\phi_0^2 + \gamma\psi_2\phi_0^2 = -\frac{\hbar^2}{2m}\nabla^2\psi_2\phi_0^2 - \frac{\hbar^2}{2m}\phi_0''\phi_0\psi_2 + [\quad]\psi_2\phi_0^2$

$\cancel{\int dz}$ $\Rightarrow$ $i\hbar\frac{\partial\psi_2}{\partial t}\phi_0^2 + \gamma\psi_2\phi_0^2 = -\frac{\hbar^2}{2m}\nabla^2\psi_2\phi_0^2 [-\frac{1}{2}m\omega_z^2 z^2 + \gamma]\phi_0\psi_2 + [\quad]\psi_2\phi_0^2$

$\int dz$ $\Rightarrow$ $i\hbar\frac{\partial\psi_2}{\partial t} + \cancel{\gamma\psi_2} = -\frac{\hbar^2}{2m}\nabla^2\psi_2 \cancel{\phi_0} - \cancel{\frac{1}{2}m\omega_z^2 z^2\psi_2} + \cancel{\gamma\psi_2} + [\cancel{V_1} + V_2]\psi_2 + g\int\phi_0^4\psi$

$[\int\phi_0^2 = 1]$

$$\Rightarrow i\hbar\frac{\partial\psi_2}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_2 + V_2\psi_2 + g_3\int\phi_0^4|\psi_2|^2\psi_2$$

$$\therefore g_2 \equiv g_3\int\phi_0^4 = g_3\frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{1}{a_z} = g_3\frac{1}{\sqrt{2\pi}a_z}$$

$$\therefore \boxed{g_2 = g_3\frac{1}{\sqrt{2\pi}a_z}}$$

$$\Rightarrow \boxed{i\hbar\frac{\partial\psi_2}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_2 + \left[\frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2) + g_2|\psi_2|^2\right]\psi_2} \quad (2)$$

$\psi_3 = \psi_2\phi_0 e^{-\frac{i\gamma t}{\hbar}}$, $\phi_0(z) = \pi^{-1/4}a_z^{-1/2}e^{-\frac{z^2}{2a_z^2}}$,

$a_z^2 \equiv \frac{\hbar}{m\omega_z}$, $\gamma \equiv \frac{\hbar\omega_z}{2}$ $\Rightarrow$ $\frac{\gamma}{\hbar} = \frac{\omega_z}{2} \Rightarrow [\;] = 1/t$ ✓

$g_2 \equiv g_3\frac{1}{\sqrt{2\pi}a_z}$, $g_3 = \frac{4\pi\hbar^2 a}{m}$ $\Rightarrow$ $\sqrt{2\pi}a_z g_2 = \frac{4\pi\hbar^2 a}{m}$

Adim : $\omega_r \equiv \omega_x = \omega_y$

$\tau = \omega_z t$, $R = r/\zeta$, $\zeta^2 = \frac{\hbar}{m\omega_z}$

$$\xrightarrow[t\to\tau,\ r\to R]{(2)} \qquad i\hbar\omega_z \frac{\partial\psi}{\partial\tau} = -\frac{\hbar^2}{2\cancel{m}}\frac{\cancel{m}\omega_z}{\hbar}\nabla_R^2\psi + \left[\frac{1}{2}\cancel{m}\omega_r^2\frac{\hbar}{\cancel{m}\omega_z}R^2 + g_2|\psi|^2\right]\psi$$

$$\div\hbar\omega_z \qquad i\psi_\tau = -\frac{1}{2}\nabla_R^2\psi + \left[\frac{1}{2}\Omega^2R^2 + \frac{g_2}{\hbar\omega_z}|\psi|^2\right]\psi$$

$$\boxed{\Omega \equiv \frac{\omega_r}{\omega_z}}$$

$$\bar\psi = (4\pi\zeta^2 a)^{1/2}\psi = \left(\cancel{4\pi}\frac{\cancel{\hbar}}{\cancel{m}\omega_z}\frac{g_3\cancel{m}}{\cancel{4\pi\hbar}}\right)^{1/2}\psi = \left(\frac{g_3}{\omega_z}\right)^{1/2}\psi$$

$$\Rightarrow \quad i\bar\psi_\tau = -\frac{1}{2}\nabla_R^2\bar\psi + \left[\frac{1}{2}\Omega^2R^2 + \frac{g_3}{\sqrt{2\pi}a_z}\frac{1}{\hbar\omega_z}\frac{\omega_z}{g_3}|\bar\psi|^2\right]\bar\psi$$

Adim ψ : $b\bar\psi = \psi \Rightarrow i\bar\psi_\tau = -\frac{1}{2}D^2\bar\psi + \left[\frac{1}{2}\Omega^2R^2 + \frac{g_2}{\hbar\omega_z}b^2|\bar\psi|^2\right]\bar\psi$

$\therefore \quad b^2 = \frac{\hbar\omega_z}{g_2}$

$$\left[\begin{array}{l} b^2 = \frac{\hbar\omega_z}{g_3} = \frac{\cancel{\hbar}\omega_z m}{4\pi\hbar^{\cancel{2}}a} = \frac{m\omega_z}{4\pi a\hbar} \\ 4\pi\zeta^2 a = 4\pi\frac{\hbar}{m\omega_z}a \end{array}\right]$$

$$\Rightarrow \boxed{\psi_2 = \sqrt{\frac{\hbar\omega_z}{g_2}}\,\bar\psi_2}$$

$$\Rightarrow \boxed{\begin{gathered} \boxed{i\frac{\partial\bar\psi}{\partial\tau} = -\frac{1}{2}D^2\bar\psi_2 + \left[\frac{1}{2}\Omega^2R^2 + |\bar\psi_2|^2\right]\bar\psi_2} \\ \psi_3 = \psi_2\phi_0\, e^{-\frac{i\gamma t}{\hbar}} , \quad \phi_0(z) = \pi^{-1/4}a_z^{-1/2}e^{-\frac{z^2}{2a_z^2}} \\ \zeta^2 = a_z^2 \equiv \frac{\hbar}{m\omega_z} , \quad \gamma \equiv \frac{\hbar\omega_z}{2} , \quad \Omega = \omega_r/\omega_z \\ \psi_2 = \sqrt{\frac{\hbar\omega_z}{g_2}}\,\bar\psi_2 , \quad g_2 \equiv g_3\frac{1}{\sqrt{2\pi}a_z} , \quad g_3 = \frac{4\pi\hbar^2 a}{m} \\ \tau = \omega_z t , \quad R = r/\zeta , \quad \zeta^2 = \frac{\hbar}{m\omega_z} > a_z^2 \end{gathered}}$$

Adimensionalized
&
reduced 3D-2D
GPE.

Ring dark solitons

[Theocharis + Frant. + PGK + BAM + Kivshar, $\underline{90}$ PRL (2003) 120403]

- Take a BEC in 3D:

$$iu_t + \frac{1}{2}\nabla^2 u \pm |u|^2 u = V(x,y,z)\, u$$

- use trapping potential:

$$V(x,y,z) = \frac{1}{2}\left(\Omega_x^2 x^2 + \Omega_y^2 y^2 + \Omega_z^2 z^2\right)$$

such that $\Omega_z \gg \Omega_x = \Omega_y \equiv \Omega$

$\Rightarrow$ 3D BEC $\rightarrow$ 2D "pancake" BEC. [See my chap contribution to BEC Book] Sec. 4.2

(Ω need to be rescaled when 3D$\to$2D)

$u = u(x,y)$, Repulsive:

$$\therefore \quad iu_t + \frac{1}{2}(u_{xx} + u_{yy}) - |u|^2 u = \frac{1}{2}\Omega^2 r^2 u \qquad (r^2 = x^2 + y^2)$$

- use polar coord. $\nabla^2 \rightarrow \partial_r^2 + \frac{1}{r}\partial_r$

~~and~~ consider $u(x,y,t) = u(r,\theta,t) = u(r,t)$ only:

$$iu_t + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - |u|^2 u = \frac{\Omega^2}{2} r^2 u$$

: 1D problem in radial direction.

- Thomas-Fermi approx [See my chap contribution to BEC book. Sec 5.1]

repulsive BEC: $iu_t + \frac{1}{2}\nabla^2 u + |u|^2 u = \frac{\Omega^2}{2} r^2 u$

$+ \iiint |u|^2\, dx\,dy\,dz \gg 1 \rightarrow$ slow spatial variation compared to potential

$\Rightarrow \nabla^2 u \ll V u$.

$\Rightarrow$ steady state: $u = u_{TF}\, e^{-i\mu t}$,

$$\Rightarrow \quad \mu \cancel{u_{TF}} - u_{TF}^{\cancel{3}\,2} \simeq V(r)\, \cancel{u_{TF}}$$

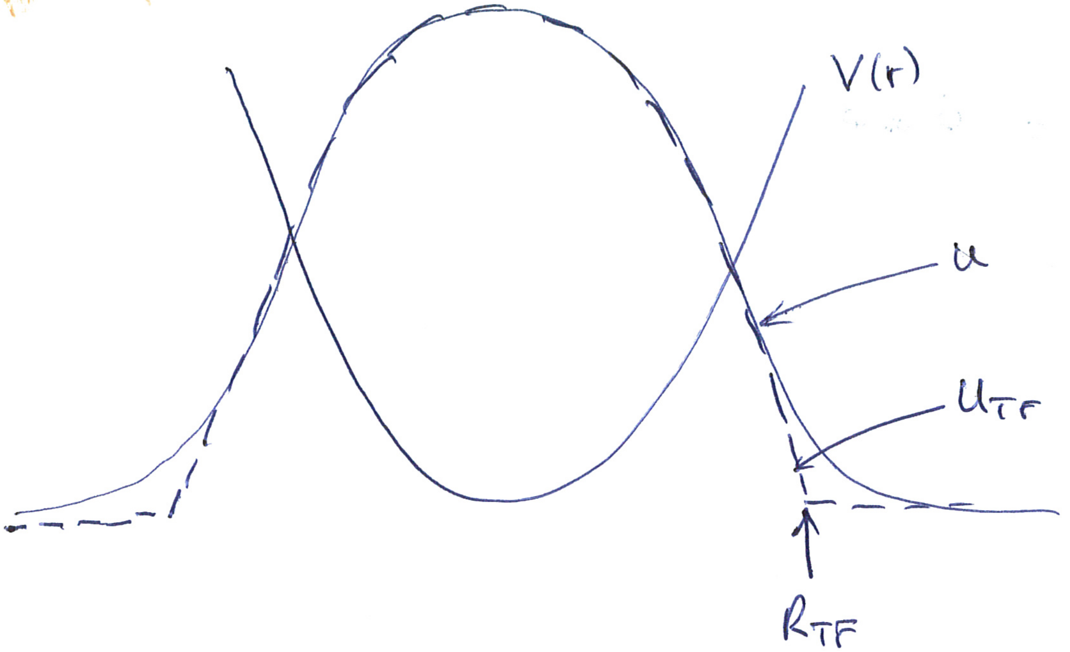
$$\Rightarrow \quad \boxed{u_{TF} \simeq \begin{cases} \sqrt{\mu - V(r)} \\ 0 \text{ if } r > R_{TF} \end{cases}}$$

$\leftarrow$ Thomas-Fermi approx

$\leftarrow$ Thomas-Fermi radius

- R_{TF}: $V(R_{TF}) = \mu \;\Rightarrow\; \frac{1}{2}\Omega^2 R_{TF}^2 = \mu$

$$\Rightarrow \quad \boxed{R_{TF} = \frac{\sqrt{2\mu}}{\Omega}}$$



- More approx : number of atoms N:

$$N = \iiint |u|^2 \, dx\,dy\,dz \simeq \iiint_{|r|<R_{TF}} \mu - V \, dr$$

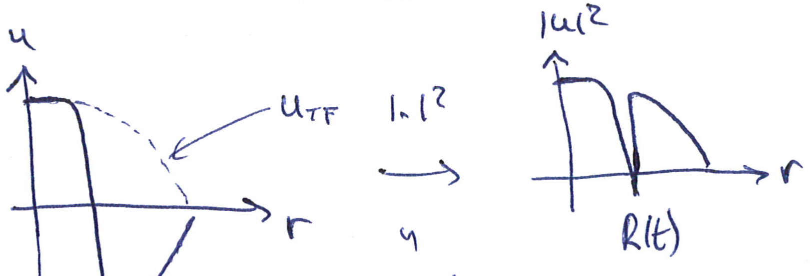
: $\sim$ [illegible] $N\uparrow$ if $\mu\uparrow$

- So if $N \gg 1 \Rightarrow \mu \gg 1$ and thus, for weak trap, $\Omega < 1$

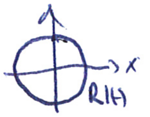
$\Rightarrow \mu \gg \Omega$

$$\Rightarrow u_{TF} \underset{|r|<R_{TF}}{\simeq} \sqrt{\mu - V} \simeq \sqrt{\mu} - \frac{1}{2\sqrt{\mu}}(V) = \sqrt{\mu} - \frac{\Omega^2}{4\sqrt{\mu}} r^2 \quad (1.5)$$

- Let us now try to describe a dark soliton about the r-direction:



ie: a RING-DARK-SOLITON . . . →



- [PRL] . . . . $t \to \mu t$, $r \to \sqrt{\mu}\, r$ and use

$$\boxed{u(r,t) = \overbrace{u_{TF}(r,t)}^{(1.5)} \cdot v(r,t)}$$

in NLS $\Longrightarrow$

$$i v_t + \frac{1}{2} v_{rr} - (|v|^2 - 1) v = \underbrace{P(v)}_{\text{pert.}}$$

where $P(v) = \frac{1}{\mu}\left[(1-|v|^2) v\, V(r) + \frac{1}{2} V'(r) v_r - \frac{\sqrt{\mu}}{2r} v_r\right]$ (2)

with $V(r) = \frac{\Omega^2}{2} r^2$

and all terms on ~~[illegible]~~ $P(v)$ are of same order (small because $\mu >> 1$)

∴ one can apply pert. theory for DS.

- DS (ring): $$\left.\begin{aligned} v(r,t) &= \cos\varphi(t)\tanh z + i\sin\varphi(t) \\ z &\equiv \cos(\varphi(t))\,[r - \underset{\uparrow \text{ position of DS.}}{R(t)}] \end{aligned}\right\}\ (2.5)$$

Perturbation $\xrightarrow{}$ . . . $\Longrightarrow$ $$\left\{\begin{aligned} \frac{d\varphi}{dt} &= -\frac{\cos\varphi}{2\mu}V'(R) + \frac{\cos\varphi}{3\sqrt{\mu}R} \\ \frac{dR}{dt} &= \sin\varphi \Rightarrow \dot{R} = \sin\varphi \Rightarrow 1 - \dot{R}^2 = \cos^2\varphi \end{aligned}\right.$$

$$\frac{d\varphi}{d\tau} = \frac{\varepsilon \,\mathrm{Re}\left[\int_{-\infty}^{\infty} p \frac{\partial v^*}{\partial \tau} d\xi\right]}{2\cos^2\varphi\sin\varphi}$$

$+ (2) + (2.5)$

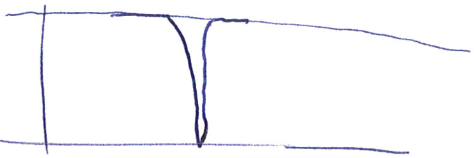
$\ddot{R} = \frac{d}{dt}(\sin\varphi) = \dot{\varphi}\cos\varphi$

∴ $$\boxed{\ddot{R} = \left[-\frac{1}{2}V'(R) + \frac{1}{3R}\right]\left[1 - \dot{R}^2\right]}$$ ($\mu = 1$ for convenience)

- Suppose very slow DS: ~~[illegible]~~, $|\dot{R}| = |\sin\varphi| \ll 1 \Rightarrow \underbrace{1 - \dot{R}^2}_{\cos^2\varphi} \simeq 1$

vel.

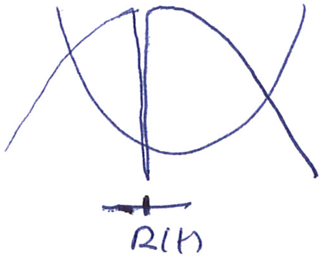
$$\Rightarrow \ddot{R} = -\frac{1}{2}V'(R) + \frac{1}{3R}$$



- now consider a very large trap, $R >> 1$

then the dynamics looks like: $$\boxed{\ddot{R} = -\frac{1}{2}V'(R)}$$

∴ know result for dynamics of DS stripe inside parabolic potential:



typo

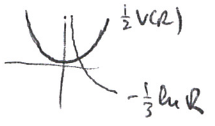
$$\Rightarrow \boxed{\ddot{R} = -\frac{\Omega^2}{2}R} = -\frac{d}{dR}\left[\frac{1}{2}\,\frac{\Omega^2}{2}R^2\right] = -\frac{d}{dr}\left[\frac{(R/\sqrt{2})^2}{2R}\right.$$

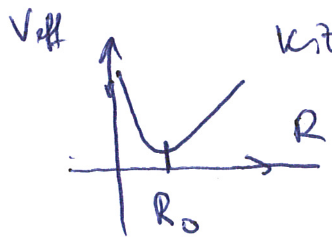
$\Rightarrow$ harmonic oscillation with freq ~~[illegible]~~ $\Omega/\sqrt{2}$

- Suppose slow DS but keep $1/3R$ ← this is "pressure" because of curvature effects of the ring.

$$\ddot{R} = \cancel{\varepsilon\frac{1}{2}\Omega^2 R} - \frac{1}{2}V'(R) + \frac{1}{3R}$$

Newton's law: $\ddot{R} = -\frac{1}{2}V'(R) + \frac{1}{3}[\ln R]' = -\frac{d}{dr}V_{\text{eff}}$





with $V_{\text{eff}} = \frac{1}{2}V(R) - \frac{1}{3}\ln R$

∴ ~~R_0: $\frac{1}{2}V(R_0) = \frac{1}{3}\ln R_0$~~

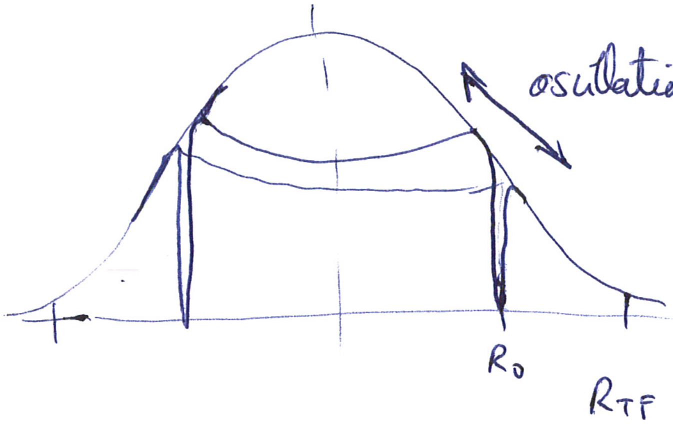
~~$\frac{1}{2}\Omega$~~

R_0: $V'_{eff}(R_0)=0 \Rightarrow \frac{1}{2}\Omega^2 R_0 - \frac{1}{3R_0} = 0 \Rightarrow R_0^2 = \frac{2}{3}/\Omega^2$

$$\therefore \boxed{R_0 = \frac{\sqrt{2/3}}{\Omega}}$$

and $R_{TF} = \frac{\sqrt{2\mu}}{\Omega} \overset{\mu=1}{=} \frac{\sqrt{2}}{\Omega} = \frac{\sqrt{2}}{\sqrt{3}\,\Omega}\sqrt{3} = R_0\sqrt{3}$

$$\therefore \boxed{R_0 = R_{TF}/\sqrt{3}}$$



oscillations inside V_{eff}

- ⚠ We know that DS stripes are MI. (see next)

∴ If R becomes too large → MI

From exp:

[See pics in PRL paper]

→ generation of vortices.

Transverse stability of $\left\{\begin{matrix}\text{Dark soliton stripes}\\ \text{plane solitons}\end{matrix}\right\}$

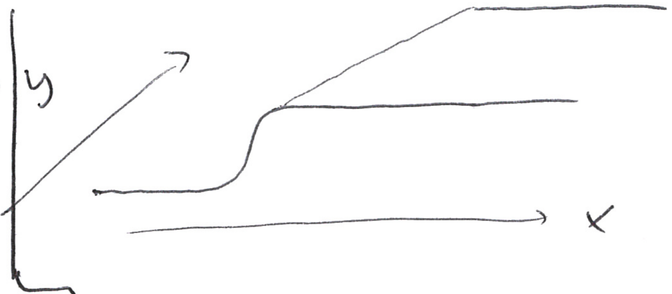
[Kivshar + Luther-Davis = [1]
Phys. Rep 298(98) 81-197]

follow a dark soliton stripe.

1D: $iU_t + \frac{1}{2}U_{xx} - |U|^2 U = 0$, $U_{1D}(x,t) = U_0[B\tanh\theta + iA]e^{-i\mu t}$

2D: $iU_t + \frac{1}{2}\nabla^2 U - |U|^2 U = 0$, $U_s(x,y,t) = U_{1D}$

i.e.: $U_s(x,y,t)$ is const w.r.t. y:



dark soliton stripe

<u>Transf:</u>

$U = v e^{-i\mu t}$

NLS $\Rightarrow i(v_t \cancel{e} + (-i\mu) v \cancel{e}) + \frac{1}{2}\nabla^2 v \cancel{e} - |v|^2 v \cancel{e} = 0$

$\Rightarrow i v_t + \frac{1}{2}\nabla^2 v + (\mu - |v|^2) v = 0$

so $\mu = 1 \Rightarrow (1-|v|^2)v$

This is done so that eq. with $(1-|v|^2)$ has a stationary steady state with $\mu = 0$
→ i.e. No rotation in complex plane

Q: What is the stability of such object?

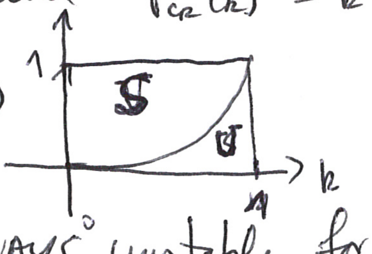
Rewrite DS stripe: $U_s = k\tanh[k(x - vt)] + iv$

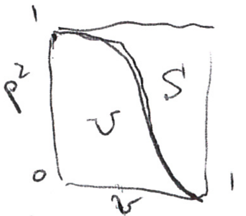
$v = AU_0$
$k = BU_0$
for $U_0 = 1$
$v^2 + k^2 = 1$
and $0 \leq k \leq 1$

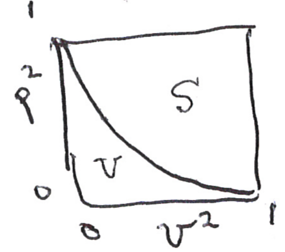
→ It turns out that DS stripes are unstable to transversal modes of the form $\sim \cos(py)$ for $p < p_{cr}(k)$ ~~and~~ $p^2_{cr}(k) = k^2 - 2 + 2\sqrt{k^4 - k^2 + 1}$ ($v = \sqrt{1-k^2}$)

$p_{cr}(k=0) = 0$
$p_{cr}(k=1) = 1 - 2 + 2 = 1$

So that $p^2_{cr}(k)$ ←





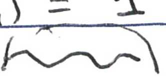


Ie: it is ALWAYS unstable for very long wavelengths

→ However, in practice, ~~as we saw for~~ the experiment will always have finite length, so if we choose k small enough. ($k = BU_0 = U_0\cos\phi$), namely $\phi \simeq \pi/2$, ie: very gray solitons then will be stable. [$k = \frac{2\pi}{\lambda}$

→ For dark sol: $k = U_0$ (= 1 in the above)

$\Rightarrow p^2_{cr}(k=1) = 1 \therefore p_{cr} = 1 \Rightarrow \lambda < 2\pi$ will be stable

→ See p. 180 in [1]: $\ominus$ ⊖  ← snaking instability

Vortices in NLS (defocusing & 2D)

- Consider a BEC with MT: $V_{MT} = \frac{1}{2}m(\omega_x^2+\omega_y^2+\omega_z^2)$
  such that $\omega_z \gg \omega_x = \omega_y = \omega$

$$\Rightarrow \quad iu_t + \frac{1}{2}\nabla^2 u - |u|^2 u = V_{MT}\,u$$

- Suppose that $\omega_x = \omega_y \ll 1$ so that

$$\Rightarrow \boxed{iu_t + \frac{1}{2}\nabla^2 u - |u|^2 u = 0}$$

- Sometimes you'll see eq. written as

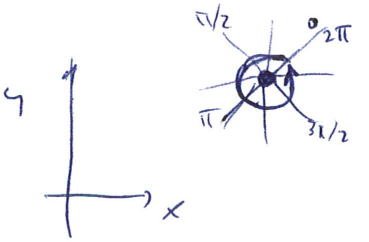
$$\rightarrow \quad iv_t + \frac{1}{2}\nabla^2 u + (d - |v|^2)v = 0$$

change of var: $u = v(x,t)\,e^{-i\lambda t}$

$$\xRightarrow{NLS} \quad i(v_t\cancel{e} - i\lambda v\cancel{e}) + \frac{1}{2}\nabla^2 v\cancel{e} - |v|^2 v\cancel{e} = 0$$

$$\Rightarrow \quad iv_t + \frac{1}{2}\nabla^2 v + (\lambda - |v|^2)v = 0$$

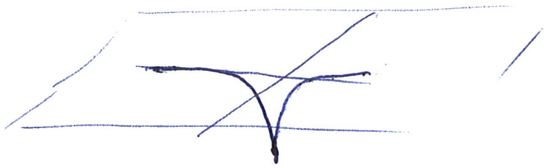
- **Vortex**: Solution that about the center of the vortex one has a phase jump of $S2\pi$



$S \equiv$ vortex charge (topological charge)

$S \in \mathbb{Z}$ — $S>0$ ↺, $S<0$ ↻

- **Profile**:

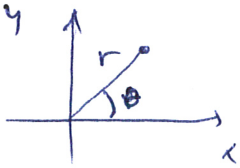


→ along any radial cut, the vortex looks like a DS:

} phase jump of π.

let us find this profile:

$$u(x,t) = \underbrace{R(r)}_{r}\,\underbrace{e^{iS\theta}}_{\theta}\,\underbrace{e^{-i\mu t}}_{t}$$



$$\xRightarrow[\text{steady state}]{NLS} \quad iR\cancel{e}\cancel{e}(-i\mu) + \frac{1}{2}\frac{1}{r}\partial_r(r\partial_r R\cancel{e}\cancel{e}) + \frac{1}{2}\frac{1}{r^2}\overset{(iS)^2}{\partial_\theta^2}(R\cancel{e}\cancel{e}) - R^3\cancel{e}\cancel{e} = 0$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} \qquad \Rightarrow \quad +\mu R + \frac{1}{2r}\partial_r(rR') + \frac{R}{2r^2}(-S^2) - R^3 = 0$$

$$\Rightarrow \mu R + \frac{1}{2r}(R' + rR'') - \frac{Rs^2}{2r^2} - R^3 = 0$$

$$\Rightarrow \boxed{\frac{1}{2}R'' + \frac{1}{2r}R' + \left(\mu - \frac{s^2}{2r^2}\right)R - R^3 = 0}$$

Need to solve this ODE with BCs: $R(0) = 0$ & $R(\infty) = \sqrt{\mu}$

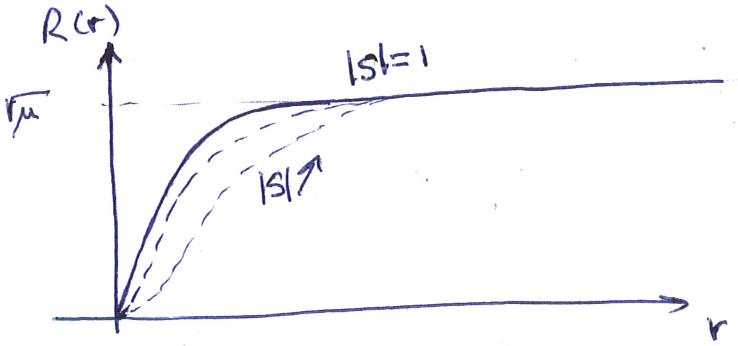
At $x = \infty$: $u = u_0 e^{is\theta} e^{-i\mu t} \Rightarrow \mu u_0 - u_0^3 = 0$

$\Rightarrow u_0(\mu - u_0^2) = 0 \Rightarrow \mu = u_0^2$

$\Rightarrow \boxed{u_0 = \sqrt{\mu}}$

⚠ the ODE for R does not have analytic solutions.
Not even for $S = \pm 1$

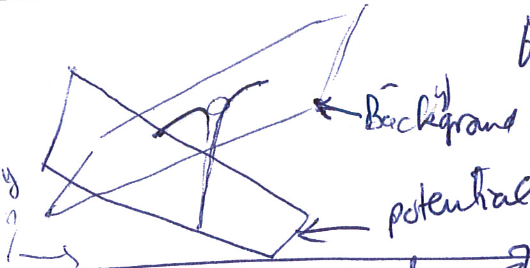
Numerical:



Show Matlab driver-sample-bvp

- Dynamics: Vortex in a slow-varying inhomogeneous background:

~~$\vec{v} \propto S\hat{z}\times\nabla B$~~



leave it with + since all my gradients are pointing upstream

$$\boxed{\vec{v} \propto \cancel{+S\hat{z}}\times\nabla V}\ \boxed{\cancel{\propto +S\nabla V\times\hat{z}}}$$

$$\vec{v} \propto -S\hat{z}\times\nabla V \propto S\hat{z}\times\nabla I_b$$

~~where is $B(x,y)$ coming from?~~ let us analyse NLS

~~With linear potential:~~

~~1D steady state~~ ~~$\mu u(x) + \frac{1}{2}u_{xx} - u^3 = Vu$~~

~~$V(x) = \alpha X$~~

~~$u(x) = \beta X + \gamma$~~

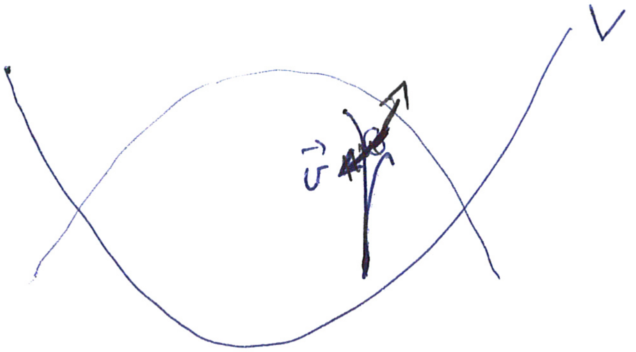
~~$\Rightarrow \mu\beta x + \frac{1}{2}\cdot 0 - \beta^2 x^2 \pm \alpha\beta x^2$~~

~~$\Rightarrow \mu - \beta^2 x^2 = \alpha X$~~

~~$\Rightarrow \mu\cancel{(\beta x+\gamma)} - (\beta x+\gamma)^{\cancel{3}\,2} = \alpha X\cancel{(\beta x+\gamma)}$~~

$= \mu - (\beta x+\gamma)^2 = \alpha X$

thus, in a TF cloud:



In fact, more specific [Kivshar et al., opt. comm. 152 (98) 198–206]

$$\vec{v} = 2\nabla\Theta_b + k\underbrace{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}_{R(\pi/2)}\nabla \ln I_b$$

$$k \equiv \ln\left(\frac{c\,\nabla \ln I_b(\vec{r}_0)\,e^{\gamma}}{4}\right)$$

$c \equiv 1.126$ ← found numerically

$\gamma \equiv$ Euler const $= \gamma = 0.577$

where background $= \sqrt{I_b}\, e^{i\Theta_b}$

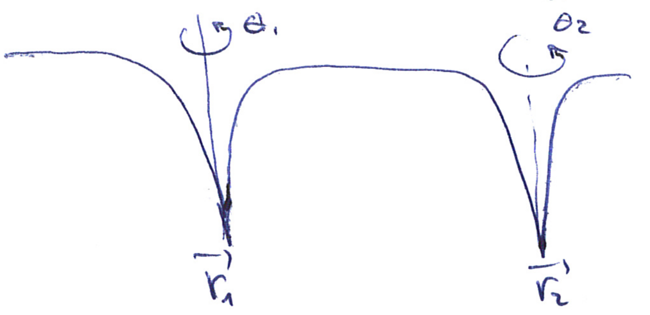
∴ { phase gradients ⟶ $\vec{v}$ // phase gradient
Intensity gradients ⟶ $\vec{v} \perp$ Intensity gradient.

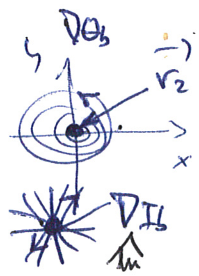
⚠ The above analysis is valid for weakly varying backgrounds.

Vortex-vortex interactions:

take one vortex and view the other as a perturbation to the background:

$R(\vec{r}_2)\, e^{+is\Theta_2}$





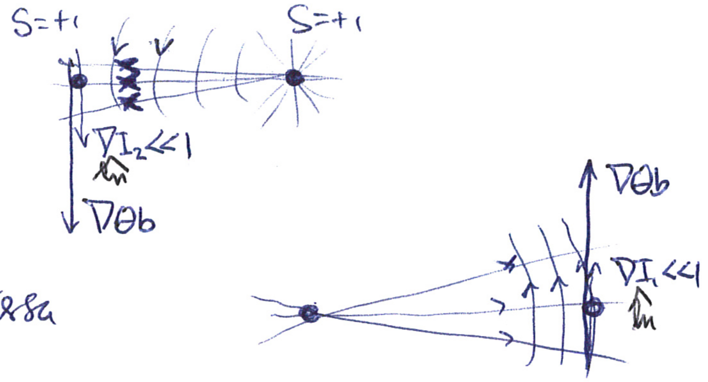
- the second vortex produces a phase gradient
- the second vortex produces an intensity gradient:

∴ since $\vec{v} \parallel \nabla\Theta_b$ & $\vec{v} \perp \nabla \ln I_b$

⟹ all velocities are ~~[illegible]~~ Azimuthal!!!

Same charge

∴

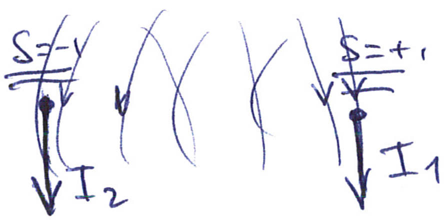


and vice-versa

∴ 2 vortices with same charge: will rotate with const. ang. velocity.

Opposite charge

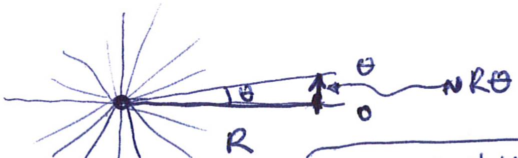
($\nabla \hat{I}$'s $\ll 1$ so neglected)



∴ 2 vortices with same charge will ~~rotate~~ move parallel to each other with constant speed

Rates for velocities:

$\theta_b \doteq$



$\therefore \|\nabla\theta_b\| \approx \frac{1}{R}$

$\theta_b = \tan^{-1} y/x$

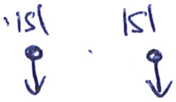
$\Rightarrow \nabla\theta_b = \left(\frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}\right) = \left(-\frac{y}{x^2}\cdot\frac{1}{1+\frac{y^2}{x^2}}, \frac{1}{x}\cdot\frac{1}{1+\frac{y^2}{x^2}}\right)$

$= \left(-\frac{y}{R^2}, \frac{x}{R^2}\right) \approx \frac{R^1}{R^2} \propto 1/R$

$\therefore \|\nabla\theta_b\|^2 = \frac{y^2}{R^4} + \frac{x^2}{R^4} = \frac{R^2}{R^4} = \frac{1}{R^2}$

$\therefore \boxed{\|\nabla\theta_b\| = \frac{1}{R}}$

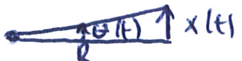
∴ • opposite charge



$\boxed{v_{lin} \propto \frac{|S|}{R}}$

• same signed charge

$\boxed{v_{ang} \propto \frac{|S|}{R^2}}$



$x(t) = R\theta(t) \Rightarrow \theta(t) = \frac{x(t)}{R}$

$\Rightarrow \dot{\theta} = \frac{\dot{x}}{R} = \frac{v}{R}$

but $v \propto \frac{1}{R} \Rightarrow \dot{\theta} \propto \frac{1}{R^2}$

Show all sorts of dynamics for vortices from Research ~~work~~.

Numerical techniques:

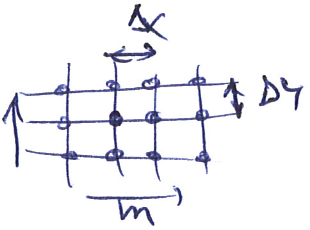
- Steady States $\Longrightarrow$ Newton method in high D
- Stability $\longrightarrow$ eigenvalue problem.

Steady States:

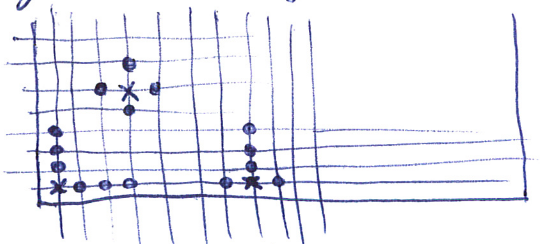
* Suppose $u_t = F(u, u_x, u_{xx}, u_y, u_{yy}, u_{xy}, \ldots)$
is a time dependent PDE.
We want to find u^* such that $u_t = 0$, ie $F(\cdots) = 0$
$\longrightarrow$ Finding a zero of an ∞D system $\longrightarrow$ Newton method.

* First, let us turn our ∞D system into a finite one by discretizing derivatives:

$$u(x,y,t) \longrightarrow U_{m,n}(t)$$



- $u_x = \dfrac{U_{m+1,n} - U_{m-1,n}}{2\Delta x} + O(\Delta x^2)$
- $u_{xx} = \dfrac{U_{m+1,n} - 2U_{m,n} + U_{m-1,n}}{\Delta x^2} + O(\Delta x^2)$
- $\nabla^2 u = u_{xx} + u_{yy} = \frac{1}{\Delta^2}\underbrace{(U_{m+1,n} + U_{m-1,n} + U_{m,n+1} + U_{m,n-1} - 4U_{m,n})}_{\text{discrete Laplacian}} + O(\Delta^2)$
- At boundaries you can use forward and backward differences.



* Thus, after discretization we have:

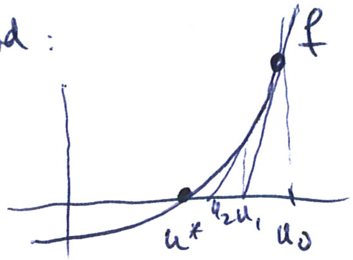
$$\frac{d}{dt}(U_{m,n}) = F(U\text{'s})$$

$\longrightarrow$ steady state becomes $F(U\text{'s}) = 0$
where $\vec{u} = (U_{1,1}, U_{2,1}, \ldots, U_{N,1}, U_{1,2}, \ldots, U_{N,2}, \ldots, U_{NN})$
ie all values of U unwrapped in a vector:

$$F(\vec{u}) = 0$$

* Newton method:

1D:



$$u^{n+1} = u^n - \frac{f(u^n)}{f'(u^n)}$$

ND: Use taylor to find formula:
Find $F(u^*)=0$ from initial guess u^0
If we are close: $\Delta u = u^* - u^0$ is small
and thus:

$$F(u^*)=0 \Rightarrow F(u^0+\Delta u)=0$$

$$\Rightarrow F(u^0) + J(u^0)\cdot\Delta u + \cdots = 0$$

$$\Rightarrow F(u^0) + J(u^0)(u^* - u^0) \simeq 0$$

$$\Rightarrow u^* \simeq u^0 - J^{-1}(u^0)F(u^0)$$

So, we do this in iterates:

$$\boxed{u^{n+1} = u^n - J^{-1}(u^n)F(u^n)}$$ Newton in higher D.

⚠ In practice → do NOT invert Jacobian [it is expensive $O(N^3)$]
* Just solve linear problem $J\,\Delta u = F$ for $\Delta u = u^{n+1} - u^n$

* **For NLS:**

$$i u_t + \nabla^2 u \pm |u|^2 u = V\cdot u.$$

$u_t = v e^{-i\mu t}$ st.-st. $\Rightarrow$ $\mu v + \nabla^2 v \pm |v|^2 v = V\cdot v$

$$\Rightarrow F(v) = [(\mu - V) + D^2 \pm |v|^2]v = 0$$

D^2 ← discrete Laplacian

$\Rightarrow \longrightarrow$ Newton method.

* Show sample on Matlab "bec2d" with driver.

Jacobian: If $v_n \in \mathbb{C}$ then one has to split real an imaginary part of Jacobian
→ this is what needs to be done for vortices for example

in 1D: $$D^2 u_n = \frac{u_{n-1} - 2u_n + u_{n+1}}{\Delta x^2}$$

$$\therefore\quad J = (\mu - V \pm 3 v_n) v_n \overset{N\times N}{\mathbb{1}} + \frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & & \phi \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ \phi & & 1 & -2 \end{bmatrix}$$

↑ if $|v_n|^2 = v_n^2$ (i.e. real solution)

~~but if $v_n \in \mathbb{C}$: $v = v^0 + \varepsilon w \Rightarrow |v|^2 v = v^2 \bar{v} = (v^0 + \varepsilon w)^2 (\bar{v}^0 + \varepsilon \bar{w}) = (v_0^2 + 2\varepsilon v_0 w + \varepsilon^2 w^2)(\bar{v}_0 + \varepsilon \bar{w})$~~
~~$= |v^0|^2 v^0 + \varepsilon \underbrace{[2|v^0|^2 w + v^{02}\bar{w}]}_{\text{linearization}} + O(\varepsilon^2)$.~~

~~$\therefore\; J = (\mu - V)v_n \pm 2$~~

Stability:

- In many ocassions, it is not possible to do a linear stability analysis analytically. For example, for ~~[illegible]~~ vortex sols the sol itself cannot be found analytically so one cannot ~~[illegible]~~ do the stab. analysis
- Let us concentrate in NLS, for other PDEs it is similar.

* Eq: $i u_t + \frac{1}{2}\nabla^2 u \pm |u|^2 u = V u$

* steady state $u = u^0 e^{-i\mu t} \Rightarrow \mu u^0 + \frac{1}{2}\nabla^2 u^0 + \alpha |u^0|^2 u^0 - V u^0 = 0$

* Newton → find steady state u^0.

* Stability take $u = u^0 e^{-i\mu t}$ and perturb:

~~$v = v^* + \varepsilon(a e^{\lambda t} + b e^{\bar\lambda t})$ perturbation~~

~~* Discretize and vectorize:~~

~~$v_n(t) = v_n^* + \varepsilon(a_n e^{\lambda t} + b_n e^{\bar\lambda t}) = v_n^* + \varepsilon w_n(t)$~~

~~NLS $\Rightarrow$ $i[\varepsilon \dot w_n$~~

$$u = [u^0 + \varepsilon\underbrace{(a e^{\lambda t} + b e^{\bar\lambda t})}_{\text{pert.} = v(x,t)}] e^{-i\mu t}$$

* Discretize and vectorize:

$$u_n(t) = [u_n^0 + \varepsilon\underbrace{(a_n e^{\lambda t} + b_n e^{\bar\lambda t})}_{v_n(t)}] e^{-i\mu t} = (u_n^0 + \varepsilon v_n) e^{-i\mu t}$$

$\xrightarrow{\text{NLS}}$ $i[\varepsilon \dot v_n \cancel{e} + (\underset{\sim\sim}{u_n^0} + \varepsilon v_n)(-i\mu)\cancel{e}] + \frac{\cancel{e}}{2}(\nabla^2 \underset{\sim\sim}{u_n^0} + \varepsilon \nabla^2 v_n) + \alpha |u_n^0 + \varepsilon v_n|^2 (u_n^0 + \varepsilon v_n)\cancel{e}$
$- V_n(\underset{\sim\sim}{u_n^0} + \varepsilon v_n)\cancel{e} = 0$

now expand $|u_n^0 + \varepsilon v_n|^2 (u_n^0 + \varepsilon v_n) = (u_n^0 + \varepsilon v_n)^2 (\bar u_n^0 + \varepsilon \bar v_n) = (u_n^{0\,2} + 2\varepsilon u_n^0 v_n + \varepsilon^2 v_n^2)(\bar u_n^0 + \varepsilon \bar v_n)$
$= \underset{\sim\sim}{|u_n^0|^2 u_n^0} + 2\varepsilon |u_n^0|^2 v_n + \varepsilon u_n^{0\,2} \bar v_n + O(\varepsilon^2)$

$\sim\sim$ = steady state

$\Rightarrow i \cancel{\varepsilon} \dot v_n + \mu \cancel{\varepsilon} v_n + \frac{\cancel{\varepsilon}}{2}\nabla^2 v_n + \alpha 2 \cancel{\varepsilon} |u^0|^2 v_n + \alpha \cancel{\varepsilon} u^{0\,2} \bar v - \cancel{\varepsilon} V_n v_n = 0$

$\Rightarrow i \dot v + \mu v + \frac{1}{2}\nabla^2 v + 2\alpha |u^0|^2 v + \alpha u^{0\,2} \bar v - V v = 0$

Remember $v_n = a_n e^{\lambda t} + b_n e^{\bar\lambda t}$

$\Rightarrow i(\lambda a_n e^{\lambda t} + b_n \bar\lambda e^{\bar\lambda t}) + \mu(a_n e^{\lambda t} + b_n e^{\bar\lambda t}) + \frac{1}{2}\nabla^2(ae + b\bar e) + 2\alpha |u^0|^2 (a_n e + b_n \bar e)$
$+ \alpha u^{0\,2}\, \overline{(ae + b\bar e)} - V(ae + b\bar e) = 0$

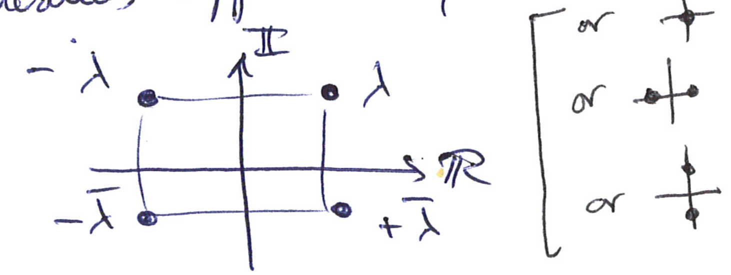
$$\rightarrow \quad e^{\lambda t}\left[i\lambda a+\mu a+\tfrac{1}{2}\nabla^2 a+2\alpha|u^0|^2 a+\alpha u^{0\,2}\bar{b}-Va\right]+ \qquad (1)$$
$$e^{\bar\lambda t}\left[i\bar\lambda b+\mu b+\tfrac{1}{2}\nabla^2 b+2\alpha|u^0|^2 b+\alpha u^{0\,2}\bar{a}-Vb\right]=0$$

but $e^{\lambda t}$ and $e^{\bar\lambda t}$ are linearly independent $\Rightarrow \begin{cases}[\ ]=0\\ [\ ]=0\end{cases} \Leftrightarrow \underbrace{M\vec{w}=\lambda\vec{w}}_{\text{eigenval. pb.}}$

Invariances: ① if $\lambda \leftrightarrow \bar\lambda$ and $a \leftrightarrow \bar{b}$ $\Rightarrow$ equs unchanged, so if λ is sol. so is $\bar\lambda$

② if $\lambda \leftrightarrow -\lambda$ and $a \leftrightarrow \bar{a}$ & $b \leftrightarrow \bar{b}$ $\Rightarrow$ equs unchanged, so if λ is sol. so is $-\lambda$

$\therefore$ eigenvalues appear in quartets:



Eigenvalue problem: $M\vec{w}=\lambda\vec{w}$

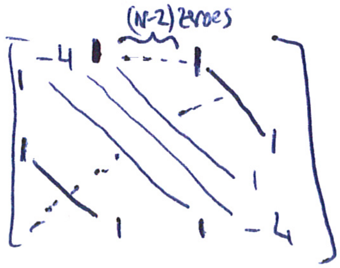
$\underline{(1)}\,a \times i \Rightarrow$
~~(1)~~ $(1)\,b \times i \Rightarrow$
$$\begin{cases}\lambda a = i\left[\mu a+\tfrac{1}{2}\nabla^2 a+2\alpha|u^0|^2 a+\alpha u^{0\,2}\bar{b}-Va\right]\\ \lambda\bar{b} = -i\left[\mu\bar{b}+\tfrac{1}{2}\nabla^2\bar{b}+2\alpha|u^0|^2\bar{b}+\alpha\bar{u}^{0\,2}a-V\bar{b}\right]\end{cases}$$

$\therefore$ if $\vec{w}=\begin{pmatrix}a\\ \bar{b}\end{pmatrix}$ & M:

$$M = i\;\begin{array}{c|c|c} & a & \bar{b}\\ \hline a & (\mu-V+2\alpha|u^0|^2)+\tfrac{1}{2}\nabla^2 & \alpha(u^0)^2\\ \hline \bar{b} & -\alpha\overline{(u^0)}^2 & -(\mu-V+2\alpha|u^0|^2)-\tfrac{1}{2}\nabla^2\end{array} = i\begin{bmatrix} M_{11} & M_{12}\\ -\bar{M}_{12} & -M_{11}\end{bmatrix} \qquad \begin{array}{l} M_{11}\in\mathbb{R}^{N^d\times N^d}\\ M_{12}\in\mathbb{C}^{N^d\times N^d}\end{array}$$

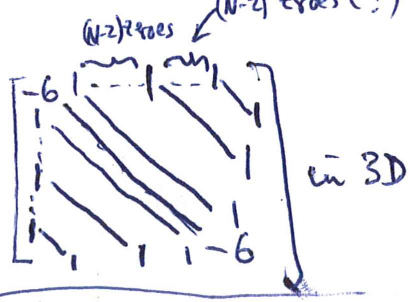
and $\nabla^2 = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & & \\ & & \ddots & 1\\ & & 1 & -2\end{bmatrix}$ in 1D

and $\nabla^2 =$



in 2D

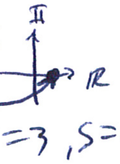
$\nabla^2 =$



in 3D

Show matlab

N = 80x50 $\quad \Omega = 0.05$
eigs = 100, $\sigma = 0.5$



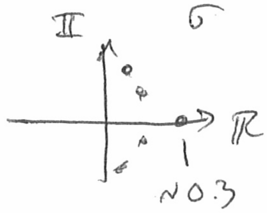
vortex = S=1, S=2, S=3, S=4

Instabilities for 2D

- A $S=\pm1$ vortex will always stable @ $(0,0)$ of V_{MT}
- Higher order vortices are unstable in some parameter windows → [H. Pu et al. PRA 59 (1999) ~~[illegible]~~ 1533]

- Numerics [See-newton-stability.zip]

$S=+3$, $N=100$, $\Omega=0.025$ → ~~λr~~ max



Dynamical reduction for vortices in BECs

when vortices are trapped there will be 2 contributions to eq.-of motion:

a) Vortex - trap interaction
b) Vortex - vortex interactions

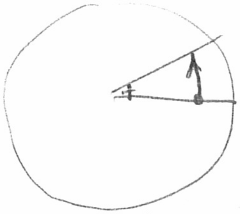
If we assume a) & b) are independent then we can just add these two contributions.

a) Vortex - trap interactions

Many ways to find this (usually in the TF limit): variational, perturbation, energetic, ...

[Refs: see nonlinear/w .../notebook/references/vortex-precession/notes.txt]

* Trap → approx. constant precession/ang. vel.



$\omega_{pr} = \omega_{pr}^{0}$ (precession @ center)

$\omega_{pr}^{0} = \omega_{pr}^{0}(\mu, \Omega)$

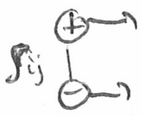
chem. potential

trap strength

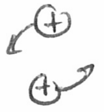
* correction (Fetter): $\omega_{pr} = \dfrac{\omega_{pr}^{0}}{\left(1-\frac{r^2}{R_{TF}^2}\right)}$

* even better correction: [Fetter PRA 89 (14) 023629
→ paper is in 2C but precession for 1C can be extracted from it]

b) Vortex - vortex:



$v_{lin} \propto \dfrac{1}{\rho_{mn}}$



$\omega_{mn} \propto \dfrac{1}{\rho_{mn}^2}$

Combining the 2:

a) $\dot{x}_m = -S_m \omega_{pr} y_m$
$\dot{y}_m = -S_m \omega_{pr} x_m$

$\Rightarrow \ddot{x}_m = -S_m \omega_{pr} \dot{y}_m = S_m^2 \omega_{pr}^2 x_m$

$\Rightarrow$ ang. vel $= S_m \omega_{pr}$

$$\boxed{\begin{aligned} \dot{x}_m &\propto - S_n \frac{y_m - y_n}{\rho_{mn}^2} \\ \dot{y}_m &\propto + S_n \frac{x_m - x_n}{\rho_{m,n}^2} \end{aligned}}$$
b)

Considering N vortices with position (x_i, y_i) & charge S_i we get

$$① \begin{cases} \dot{x}_{m} = -S_m\omega_{pr}\, y_m + \frac{\omega_{vort}}{2}\sum_{m\neq n} S_n \frac{y_m - y_n}{\rho_{mn}^2} \\ \dot{y}_{m} = -S_m\omega_{pr}\, x_m + \frac{\omega_{vort}}{2}\sum_{m\neq n} S_n \frac{x_m - x_n}{\rho_{mn}} \end{cases}$$

$S_j \frac{\omega_{vort}}{\rho_{mn}^2}$ = ang. vel. of 2 vortices separated by ρ_{mn}

Conserved quantities:

1) Total energy = Hamiltonian:

$$H = -\frac{\omega_{pr}}{2}\sum_{m=1}^{N} \ln(1 - r_m^2) + \frac{\omega_{vort}}{4}\sum_{m\neq n} S_m S_n \ln(\rho_{mn}^2)$$

2) Ang. momentum = L

$$L = \sum_{m=1}^{N} S_m r_m^2$$

Complex version of eq. of motion:

$z_m = x_m + i\, y_m$

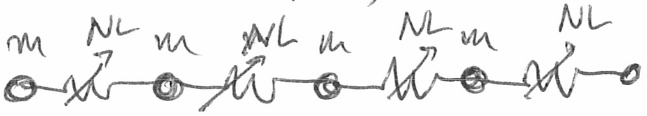
$$\boxed{\dot{z}_m = -i S_m \omega_{pr} z_m + i\frac{\omega_{vort}}{2}\sum_{m\neq n}\frac{z_m - z_n}{|z_m - z_n|^2}}$$

→ give the talk about vortices ~~[illegible]~~ in V_{MT}
(++, +-, pitchfork, etc, ...)

Nonlinear lattices

FPU Fermi-Pasta-Ulam [Los Alamos report 1955] [FPU in Scholarpedia]

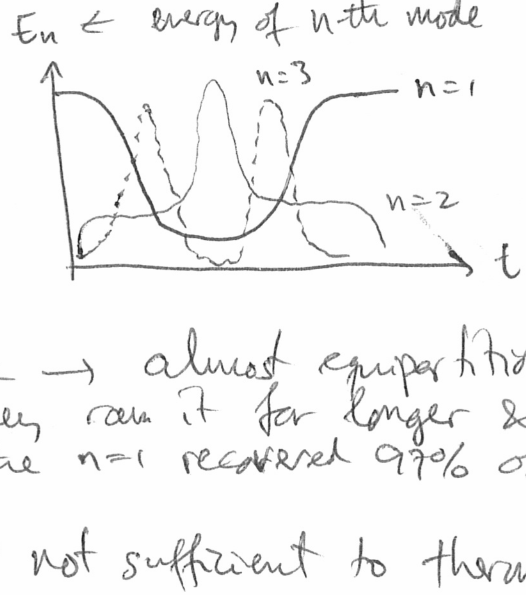
- Nonlinear lattice to model the thermalization of a solid → i.e. equipartition of energy in all modes
- A linear lattice has dynamics that is the linear combo of the normal modes of vibration
- nonlinearity ⟹ ~~connection~~ coupling between ≠ modes
  ⟹ energy transfer between ≠ mods
  $\xRightarrow{t\gg 0}$ thermalization
- Numerical experiments using one of the first computers (Maniac)
  model = linear spring-masses + small nonlinearity:



m=1 ⟹ k=1 linear

$$\ddot{u}_n = \underbrace{u_{n+1} - 2u_n + u_{n+1}}_{\substack{+k(u_{n+1}-u_n)\\ -k(u_n-u_{n-1})}} + \alpha\left[(u_{n+1}-u_n)^2 - (u_n-u_{n-1})^2\right]$$

- Surprise when they ran model:



→ Recurrence

Show Matlab FPU.m

* For short time → almost equipartition
* by mistake they ran it for longer & after ~157 periods of mode 1 the n=1 recovered 97% of its energy !!!

→ nonlinearity is not sufficient to thermalize why? Not equipartition?

$\longrightarrow$ solitons!

show kink-antikink solution

Matlab

where are these solitons coming from?

A $\rightarrow$ continuum limit approx. of FPU $\rightarrow$ KdV !!!

Toda lattice

- let us study in more detail a nonlinear lattice that can be solved analytically $\rightarrow$ Toda lattice (TL)

  [Non-linear wave propagation in the 1D TL, Itai Afek]

  + [M. Toda J. Phys. Soc. Japan $\underline{22}$ (1967) 431]

- Toda introduced this model so that it could be solved analytically.

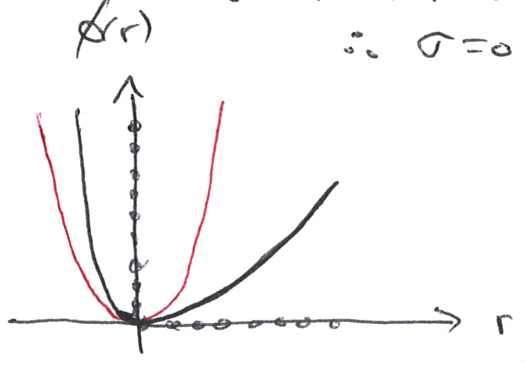
$$m_n \ddot{y}_n = -\phi'(y_n - y_{n+1}) + \phi'(y_{n+1} - y_n)$$

ϕ: potential interaction $\rightarrow$ $F = -\dfrac{\partial \phi}{\partial y}$

potential: $$\phi(r) = \frac{a}{b} e^{-b(r-\sigma)} + ar + \overset{\rightarrow 0}{\cancel{\text{const.}}}$$

- $ab > 0$
- $r \rightarrow r - \sigma$ same eq. with $\neq$ const

  $\therefore$ $\sigma = 0$ without loss of generality.



limits: ⓐ $b \rightarrow 0$ & $ab = $ const

$\longrightarrow$ ~~linear~~ $\phi \propto r^2$ $\cup$

$\longrightarrow$ linear force

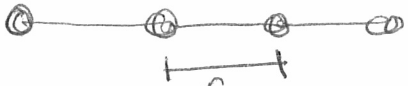
ⓑ $b \rightarrow \infty$ & $ab = $ const

$\longrightarrow$ ϕ is hard sphere

Dual lattice representation

Newton → $\boxed{m\ddot{y}_n = ae^{-b(y_n-y_{n-1}-r_0)} - ae^{-b(y_{n+1}-y_n-r_0)}}$ ①

$m_n = m$



r_0 ← equilibrium distance.

Def: perturbation away from equilibrium:

$$\boxed{r_n \equiv y_n - y_{n-1} - r_0} \quad ②$$

$$\Rightarrow \ddot{r}_n = \ddot{y}_n - \ddot{y}_{n-1} = \frac{a}{m}e^{-b(r_n} - \cancel{\frac{a}{m}e^{-br_{n+1}}} - \frac{a}{m}e^{-br_{n-1}} + \frac{a}{m}e^{-br_n}$$

$$\Rightarrow m\ddot{r}_n = a\left[2e^{-br_n} \underbrace{- e^{-br_{n+1}} - e^{-br_{n-1}}}_{\text{this looks like a discrete Laplacian.}}\right]$$

Def: $\boxed{\dot{s}_n \equiv a[e^{-br_n} - 1]}$ ③ $\Rightarrow ae^{-br_n} = \dot{s}_n + a$

~~$\Rightarrow m\ddot{r}_n = \cancel{a}[2(\dot{s}_n + a) - (\dot{s}_{n+1} + a) - (\dot{s}_{n-1} + a)]$~~

~~$= \cancel{a}[2\dot{s}_n - \dot{s}_{n+1} - \dot{s}_{n-1}] \Rightarrow \boxed{m\ddot{r}_n = \cancel{a}[2s_n - s_{n+1} - s_{n-1}]}$~~

~~But $\ddot{r}_n = y_n - y_{n-1}$~~

~~① $\Rightarrow m\ddot{y}_n = \dot{s}_n - \dot{s}_{n-1} \Rightarrow m\ddot{y}_n = \dot{s}_n - \dot{s}_{n+1} \Rightarrow m\dot{r}_n =$~~

and $\ddot{s}_n = -ab\dot{r}_n e^{-br_n}$

$$= -\cancel{a}b\dot{r}_n a[\cancel{a}\underbrace{e^{-br_n} - 1}_{\dot{s}_n} + 1] = -b\dot{r}_n[\cancel{a}\dot{s}_n + a]$$

$$\Rightarrow \boxed{\frac{\ddot{s}_n}{\dot{s}_n + a} = -\frac{b}{m}(2s_n - s_{n+1} - s_{n-1})} \quad ④$$

$\int dt$: $\frac{d}{dt}\left[\ln\left(1 + \frac{\dot{s}_n}{a}\right)\right] = \frac{\ddot{s}_n}{a} \cdot \frac{1}{1 + \frac{\dot{s}_n}{a}} = \frac{\ddot{s}_n}{a + \dot{s}_n}$

∴ Def $P_n = \int s_n\,dt$ [or $s_n = \dot{P}_n$]

$$\Rightarrow \frac{d}{dt}\left[\ln\left(1 + \frac{\ddot{P}_n}{a}\right)\right] = -\frac{b}{m}[2\dot{P}_n - \dot{P}_{n+1} - \dot{P}_{n-1}]$$

$$\int dt \Rightarrow \quad \boxed{\ln\left(1+\frac{\ddot{P}_n}{a}\right) = -\frac{b}{m}\left(2P_n - P_{n+1} - P_{n-1}\right)} \quad ⑤$$

using this expression and the fact that

$$[\tanh]' = 1 - \tanh^2$$
$$\int \tanh = \ln(\cosh)$$
$$= \ln\left(\frac{1}{\operatorname{sech}}\right)$$

one can prove that the following is a solution for ⑤ by using $S_n = \pm \frac{\beta m}{b} \tanh(nk \pm \beta t) + \text{const.}$

$$\boxed{\ln\left[1 + \frac{\beta^2 m}{ab}\operatorname{sech}^2(nk \pm \beta t)\right] = -b\, r_n} \quad ⑥$$

$v = \beta/k$

sech-soliton solution !!!

Show Matlab

$A=1$, $A=0.1$

Discrete NLS (DNLS)

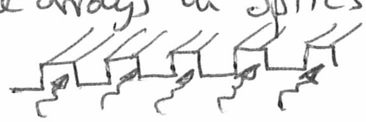
[Carretero's paper #59]

Intro Discrete versions of the NLS, i.e. DNLS, emerge from several applications:

- periodic potentials in BECs:

– using suitable basis (Wannier functions) ⟶ coupled ODE's on params ⟹ DNLS

- Waveguide arrays in optics



- photorefractive media
- DNLS is a prototypical model for envelope waves ~~[illegible]~~ in discrete media

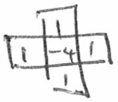
⋮

DNLS discrete Laplacian

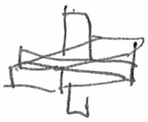
$$\boxed{i\dot{u}_n = -\varepsilon\Delta u_n + \beta|u_n|^2 u_n} \quad (1) \quad \{u_n\}_{n=-\infty}^{\infty} \quad \varepsilon, \beta \in \mathbb{R}$$

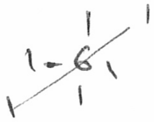
1D $\Delta u_n = u_{n-1} - 2u_n + u_{n+1}$

2D



3D





Steady states ~~Breathers~~

Sep. var. $u_n = v_n e^{i\lambda t}$ (2)

$$(2) \overset{(1)}{\Longrightarrow} -\lambda v_n \cancel{e} = -\varepsilon\Delta u_n \cancel{e} + \beta|v_n|^2 v_n \cancel{e}$$

- in 1D one can prove that $v_n \in \mathbb{R}$

$$\Rightarrow \lambda v_n = \varepsilon\Delta v_n - \beta v_n^3$$

$$\boxed{\lambda v_n = \varepsilon(v_{n-1} - 2v_n + v_{n+1}) - \beta v_n^3} \quad (3) \text{ Recurrence relationship}$$

- This second order recurrence: $G(v_n, v_{n-1}, v_{n+1}) = 0$ can be cast as $2 \times 1^{\text{rst}}$ order recurrences:

[same way as transf. a 1x 2nd order ODE → 2x 1st order ODE : i.e. intermediate variable.

$$w_n \equiv v_{n-1}$$

(3) ⇒ $\varepsilon v_{n+1} = -\varepsilon w_n + (2\varepsilon + \lambda) v_n + \beta v_n^3$

⇒ $$\boxed{\begin{cases} v_{n+1} = \frac{1}{\varepsilon}\left[(\lambda+2\varepsilon) v_n - \varepsilon w_n + \beta v_n^3\right] \\ w_{n+1} = v_n \end{cases}}$$ 2D recurrence.

⇔ 2D Map: $\begin{pmatrix} v_{n+1} \\ w_{n+1} \end{pmatrix} = M \begin{pmatrix} v_n \\ w_n \end{pmatrix}$

$$M : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{\varepsilon}\left[(\lambda+2\varepsilon)x - \varepsilon y + \beta x^3\right] \\ x \end{pmatrix}$$

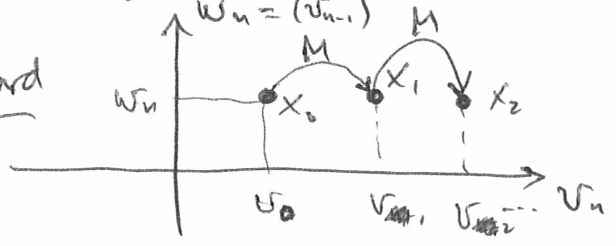
∴ Every stationary state $u_n = v_n e^{i\lambda t}$ must come from an orbit of the 2D Map M

- We need 2 ICs :

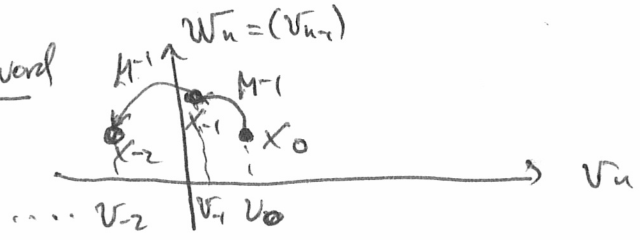
  IC : $X_0 = \begin{pmatrix} v_0 \\ w_0 \end{pmatrix}$ ~~and to get~~

- We then need to iterate forward & backward in n to get the shape of the whole lattice:

Forward



Backward



this way we get : $\{\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\}$

Properties of 2D M

- M & M^{-1} are identical after $v_n \leftrightarrow w_n$

  $\Longrightarrow$ Forward & backward orbits are symmetric w.r.t. $y=x$ line!

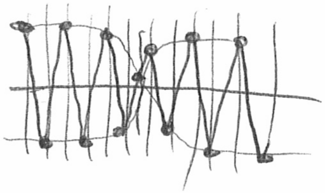
- Exchanging $v_n \to (-1)^n v_n$ (and thus $w_n \to (-1)^n w_n$), i.e. using the so-called STAGGERING TRANSF, the map transf. $M \to (-1)^n M$ with $\begin{cases} \lambda \to -\lambda - 4\varepsilon \\ \beta \to -\beta \end{cases}$

  ∴ one can go from attractive/focusing ($\beta<0$) to repulsive/defocusing ($\beta>0$) nonlinearities by just alternating signs of the u's!

Ex: + $\beta<0 \to$ attractive $\to$ bright soliton

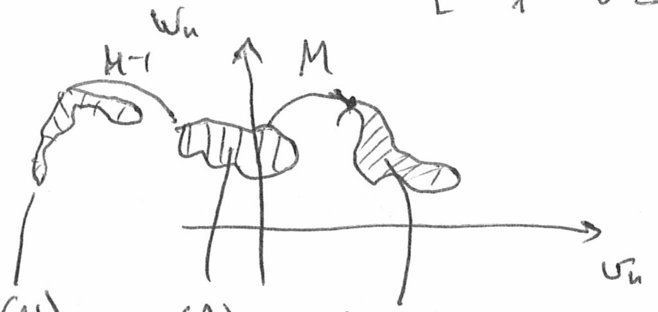
$\to$ 

* staggering with $\beta<0$ is equivalent to repulsive $\to$ dark soliton $\to$



- M is area preserving: $\det(J(M)) = 1$

Proof: $J(M) = \begin{bmatrix} \frac{1}{\varepsilon}(\lambda+2\varepsilon)+3\beta v_n & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow |J(M)| = 1 \to$ area preserv.



$\text{area}(M^{-1}(A)) = \text{area}(A) = \text{area}(M(A))$

Consequences:

a) linear centers are nonlinear centers i.e. linear stab. is enough

b) W^s & W^u for saddles must have must have same rates of attraction/repulsion.

Diffrent steady states

Homogeneous : $\begin{cases} v_{n+1} = v_n \\ w_{n+1} = w_n \end{cases}$ i.e. fixed pts.

but $\begin{matrix} w_{n+1} \equiv v_n \\ w_n \equiv v_{n-1} \end{matrix} \Rightarrow \begin{cases} v^*_{n+1} = v^*_n \\ v^*_n = v^*_{n-1} \end{cases}$

$\Rightarrow$ $\Rightarrow$ $v_n = \{\ldots, v^*, \ldots\}$
and $w_n = \{\ldots, v^*, \ldots\}$

∴ fixed pts need to be on y=x line.

$$(3) \Rightarrow \lambda v^* = \varepsilon(\underbrace{v^* - 2v^* + v^*}_{=0}) - \beta(v^*)^3$$

[Δ=0 because homogeneous!]

$$\Rightarrow v^*(\lambda + \beta v^{*2}) = 0$$

$$\Rightarrow v^* = \begin{cases} 0 \\ \pm\sqrt{-\lambda/\beta} \end{cases}$$

∴ Homogeneous steady states: $\begin{cases} u_n = 0\, e^{i\lambda t} = 0 \\ u_n = \pm\sqrt{-\lambda/\beta}\, e^{i\lambda t} \end{cases}$

* condition for nontrivial f.pts: $\lambda/\beta < 0$

Phase portrait: We need stability for f.pts:

$v^* = 0$ $\quad J = \begin{bmatrix} \frac{1}{\varepsilon}(\lambda+2\varepsilon) + 3\beta \cancelto{0}{v_n^2} & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\varepsilon}(\lambda+2\varepsilon) & -1 \\ 1 & 0 \end{bmatrix}$

eig(J): $|J - \mu \mathbb{1}| = 0 \Rightarrow \left(\frac{1}{\varepsilon}(\lambda+2\varepsilon) - \mu\right)(-\mu) + 1 = 0$

$$\Rightarrow \mu^2 + \frac{1}{\varepsilon}(\lambda+2\varepsilon)\mu + 1 = 0$$

$$\Rightarrow \mu_\pm = \frac{1/\varepsilon(\lambda+2\varepsilon) \pm \sqrt{\frac{(\lambda+2\varepsilon)^2}{\varepsilon^2} - 4}}{2}$$

∴ If $D = \frac{(\lambda+2\varepsilon)^2}{\varepsilon^2} - 4 \Rightarrow$ ~~D<0~~

$$= \frac{1}{\varepsilon^2}[\lambda^2 + 4\varepsilon\lambda + \cancel{4\varepsilon^2} - \cancel{4\varepsilon^2}]$$

$$= \frac{1}{\varepsilon^2}\lambda(\lambda+4\varepsilon)$$

$$\therefore \quad \mu_\pm = \frac{\lambda+2\varepsilon \pm \sqrt{\lambda(\lambda+4\varepsilon)}}{2\varepsilon}$$

D<0 : complex evals : $\mu_\pm = \frac{1}{2\varepsilon}\left[\lambda+2\varepsilon \pm \sqrt{\lambda(\lambda+4\varepsilon)}\right]$

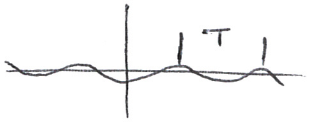
$$|\mu|^2 = \frac{1}{(2\varepsilon)^2}\left[(\lambda+2\varepsilon)^2 + \left(-\lambda(\lambda+4\varepsilon)\right)\right]$$

$$= \frac{1}{(2\varepsilon)^2}\left[\cancel{\lambda^2} + \cancel{4\varepsilon\lambda} + 4\varepsilon^2 - \cancel{\lambda^2} - \cancel{4\varepsilon\lambda}\right] = \frac{4\varepsilon^2}{4\varepsilon^2} = 1$$

$\therefore \quad |\mu|^2 = 1 \Longrightarrow$ center

Freq ~~≈~~ $\mathrm{Im}(\mu) = \frac{1}{2\varepsilon}\sqrt{-\lambda(\lambda+4\varepsilon)}$

$\therefore$ period T ~~≈~~ $\frac{2\pi}{\text{freq.}} =$



provided perturbation from $v^*=0$ is small.

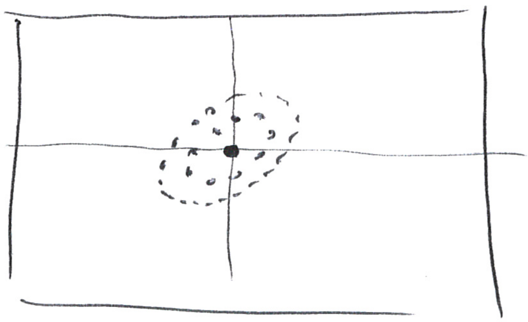
D>0 : Real evals : $\mu_\pm = \frac{1}{2\varepsilon}\left[\lambda+2\varepsilon \pm \sqrt{\lambda(\lambda+4\varepsilon)}\right]$

You can check that $\mu_+ = 1/\mu_-$ so that attraction ~~convergence~~ over W^s : $(\mu_-)^n$ is balance with repulsion over W^u : $(\mu_+)^n$.

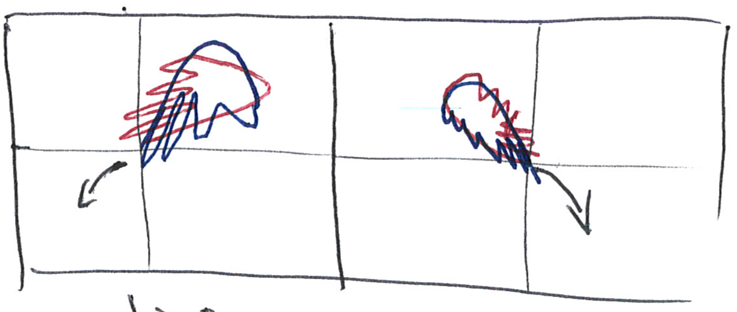
$\therefore$ SADDLE.

2 qualitative ≠ case for phase portrait

D<0



D>0



$\lambda > 0$

* SHOW PICS FROM MY PAPER/#59

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