Chapter 3

Water Waves

Many of the general ideas about dispersive waves originated in the problems of water waves. This is a fascinating subject because the phenomena are familiar and the mathematical problems are various.

——— *G. B. Whitham*

3.1 Governing Equations for Water Waves

3.1.1 Euler equations

In this chapter, water is assumed to be an inviscid, incompressible fluid. Body force is the earth's gravity. From section 2.1.2, the conservation of mass and momentum yields

$$u_x + v_y + w_z = 0, \tag{3.1.1}$$

$$u_t + uu_x + vu_y + wu_z = -\frac{1}{\rho} p_x, \tag{3.1.2}$$

$$v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho} p_y, \tag{3.1.3}$$

$$w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho} p_z - g. \tag{3.1.4}$$

These equations are valid in the fluid domain and are called the *Euler equations.* Here $(u, v, w) = \mathbf{u}$ is the velocity field, ρ is the density, p denotes pressure, and g is the gravitational acceleration (see Fig. 3.1).

Boundary conditions vary from problem to problem. For surface waves in an open ocean (without shorelines), there are two boundaries. One is on the ocean bottom, which is assumed to be rigid. The other is on the free surface of the water, which is to be determined. On the free surface two conditions need to be specified. The first one is the dynamical condition, reflecting the external actions on the free surface. The other one is the geometrical condition, which states that the water particles on the free surface should always stay on the free surface. At the bottom, since there is no other fluid interacting with the water, the only condition which applies is the geometrical one showing that it is impossible for water to penetrate the rigid bottom.

Let $z = \eta(x, y, t)$ and $z = -h(x, y)$ be the free surface and the bottom topography respectively. Then, on the free surface $z = \eta(x, y, t)$, we have

$$p = \overline{p}(x, y, t) \quad \text{(dynamical)}, \tag{3.1.5}$$

$$\eta_t + u\eta_x + v\eta_y = w \quad \text{(geometrical)}. \tag{3.1.6}$$

And on the bottom $z = -h(x, y)$, we have

$$uh_x + vh_y + w = 0 \quad \text{(geometrical)}. \tag{3.1.7}$$

In (3.1.5), $\overline{p}$ is a given surface pressure disturbance and the geometrical boundary condition (3.1.6) was derived earlier in section 2.2. Therefore, the water wave problem in an open ocean is a free surface problem defined by (3.1.1) - (3.1.7). Solving such a wave motion problem is a challenging task.

3.1.2 Potential flow

The *potential flow* is a flow whose velocity field is irrotational, i.e.,

$$\omega = \nabla \times \mathbf{u} = 0$$

where ω is called the *vorticity*. From (3.1.1) - (3.1.4), we can derive an equation

$$\frac{D}{Dt}\left(\frac{\omega}{\rho}\right) = \left(\frac{\omega}{\rho} \cdot \nabla\right) \mathbf{u}. \tag{3.1.8}$$

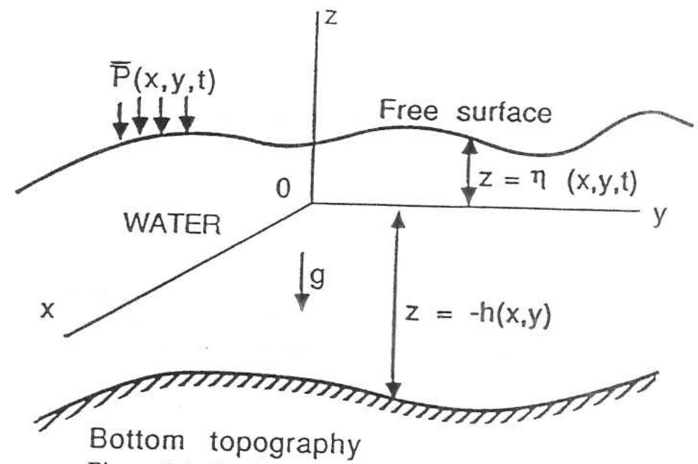

Figure 3.1: Surface water waves in an open ocean

This equation is called the *vorticity equation.*

Exercise: Derive (3.1.8) from (3.1.1) - (3.1.4).

Equation (3.1.8) has a solution $\omega = 0$. Hence, $\omega = 0$ is a solution to the Euler equations. Thus, irrotationality is a mathematically realistic assumption. Thus, it is claimed that an irrotational flow does exist in the Euler equation sense. Specifically, a uniform flow in a channel is irrotational. Of course, we always bear in mind that mathematical models (such as the Euler equations) and laboratory observed phenomena (such as the water waves) are only mutual approximations under certain conditions. For instance, any flow in the real world has a viscosity which is not zero and a density which is not uniform. The viscosity and the nonuniform density (the later is called the *stratification of a fluid*), always cause rotation of fluid elements. So, abosulte irrotational flows can never be found in nature. But in many cases, the vorticity is so weak that the flow can be considered to be approximately irrotational.

If $\nabla \times \mathbf{u} = 0$, then there exists a potential ϕ such that

$$\mathbf{u} = \nabla\phi.$$

In this situation, the equations (3.1.1) and (3.1.8) may be considered as the governing equations since they are dependent on (3.1.1) and (3.1.2) - (3.1.4). Equation (3.1.8) is satisfied if $\omega = 0$. Hence the only equation which needs to be satisfied is (3.1.1). This is the case when ϕ is a harmonic function, i.e.

$$\nabla^2\phi = 0. \tag{3.1.9}$$

Using

$$\nabla(\mathbf{u} \cdot \mathbf{u}) = 2(\mathbf{u} \cdot \nabla)\mathbf{u} + 2\mathbf{u} \times (\nabla \times \mathbf{u}),$$

equation

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p - g\mathbf{k}$$

(i.e. equations (3.1.2) - (3.1.4)) can be written as

$$\mathbf{u}_t + \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) + \boldsymbol{\omega} \times \mathbf{u} = -\nabla\left(\int^p \frac{dp}{\rho} + gz\right). \tag{3.1.10}$$

Let

$$H = \int^p \frac{dp}{\rho} + \frac{1}{2}|\mathbf{u}|^2 + gz. \tag{3.1.11}$$

This H is called the *head of the flow*. If $\boldsymbol{\omega} = \nabla \times \mathbf{u} = 0$, there exists a scalar potential function ϕ such that $\mathbf{u} = \nabla\phi$. Consequently, equation (3.1.10) gives

$$\nabla(\phi_t + H) = 0.$$

Hence,

$$\phi_t + \int^p \frac{dp}{\rho} + \frac{1}{2}|\nabla\phi|^2 + gz = C(t) \tag{3.1.12}$$

where $C(t)$ is the arbitrary constant from spatial integration and is a function of t. Equation (3.1.12) is called the *Bernoulli equation*, which is very useful in hydrodynamics.

Now we are ready to formulate the boundary conditions for the Laplace equation (3.1.9) for an open ocean. On the free surface $z = \eta(x,\, y,\, t)$, we have

$$\phi_t + \int^p \frac{dp}{\rho} + \frac{1}{2}|\nabla\phi|^2 + g\eta = C(t) \quad \text{(dynamical condition)}, \tag{3.1.13}$$

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z \quad \text{(kinematic condition)}. \tag{3.1.14}$$

On the bottom $z = -h(x, y)$, we have

$$\phi_x h_x + \phi_y h_y + \phi_z = 0 \quad \text{(kinematic condition)}. \tag{3.1.15}$$

The function $C(t)$ in (3.1.13) can be determined by a known condition at a point on the free surface, such as the point at infinity.

Example 1: If $h(x,\, y) = -1$, and $\rho = 1$, then $\phi = V_0\, y,\ \eta = 0,\ C(t) = V_0^2/2$ is a solution of the problem (3.1.9), and (3.1.13) - (3.1.15). Physically, this solution represents a uniform flow with velocity V_0 along the y-axis.

Example 2: There is right cylindrical water container which holds water of depth H. At the bottom of the container there is a small hole through which the water flows out. The area of the small hole is S which is much smaller than the base area A of the right cylindrical container. We may use the Bernoulli theorem to compute the total time T needed for all the water to drain out (see Fig. 3.2).

By the Bernoulli theorem, we can first evaluate the flow speed at the draining hole. Since the depth decrease is very slow, the flow may be considered

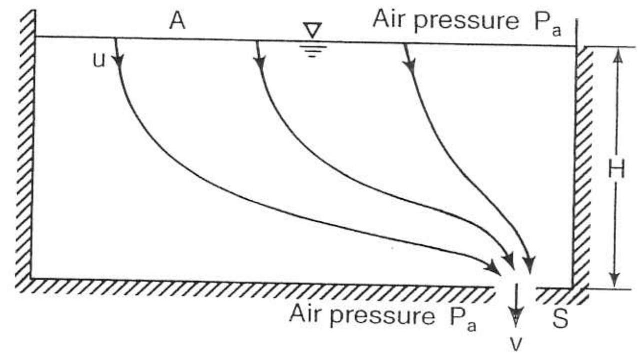

Figure 3.2: Application of the Bernoulli theorem to the draining water problem.

approximately stationary. We also regard the water density as a constant. Then the head on the free surface of the water is equal to the head at the hole:

$$\frac{p_a}{\rho} + \frac{1}{2}u^2 + gH = \frac{p_a}{\rho} + \frac{1}{2}v^2.$$

In the above, we have chosen the bottom of the container as the zero point for the vertical coordinate. On both the free surface and the hole, the water pressure should be equal to the air pressure p_a. The water flowing speed at the hole is v and the free surface is moving down at a speed equal to u. Both u and v are unknowns. We need an additional equation to solve the above problem. This is the conservation of flux:

$$uA = vS.$$

Solving the above coupled equations, we obtain

$$v = \sqrt{\frac{2gH}{1 - S^2/A^2}}, \quad \text{and} \quad u = \frac{S}{A}\sqrt{\frac{2gH}{1 - S^2/A^2}}.$$

Since $S \ll A$, we have an approximation

$$v \approx \sqrt{2gH}.$$

The total time needed to drain the water is denoted by T. The conservation of mass leads to

$$-A\, dh = Sv\, dt.$$

Integrating the above with respect to h from H to zero and with respect to t from zero to T, we can obtain that

$$T = \sqrt{\frac{2}{m_e\, g}H} \tag{3.1.16}$$

where $m_e = (S^2/A^2)(1 - S^2/A^2)^{-1}$ is considered as an effective gravitational coefficient. If we consider $m_e g$ as the effective gravitational acceleration, then equation (3.1.16) gives the time needed for a free falling particle to reach the ground from the height H in this effective gravitational field. This is a familiar result from general physics or elementary calculus.

3.2 Shallow Water Equations

3.2.1 Shallow water equations

Consider the situation that the wave length λ (as the horizontal length scale L) is much larger than the water depth H. Hence

$$0 < \epsilon = \left(\frac{H}{L}\right)^2 \ll 1. \tag{3.2.1}$$

Let us consider two-dimensional problems. The x-axis is along the horizontal direction, and the y-axis is along the vertical direction opposite to the gravitational force. Assume that the free surface is not subject to external forces. Then from section 3.1., we have the following mathematical setup of the problem:

$$u^*_{x^*} + v^*_{y^*} = 0, \tag{3.2.2}$$

$$u^*_{t^*} + u^* u^*_{x^*} + v^* u^*_{y^*} = -\frac{1}{\rho}\, p^*_{x^*}, \tag{3.2.3}$$

$$v^*_{t^*} + u^* v^*_{x^*} + v^* v^*_{y^*} = -\frac{1}{\rho}\, p^*_{y^*} - g, \tag{3.2.4}$$

$$u^*_{y^*} = v^*_{x^*} \qquad \text{(irrotational)}; \tag{3.2.5}$$

with boundary conditions

$$p^* = 0, \quad \eta^*_{t^*} + u^*\eta^*_{x^*} = v^* \quad \text{on } y^* = \eta^*(x^*, t^*); \tag{3.2.6}$$

$$u^* h^*_{x^*} + v^* = 0 \qquad \text{on } y^* = -h^*(x^*). \tag{3.2.7}$$

Here ρ is a constant, and "$*$ " signifies that the quantities are dimensional. To nondimensionalize the above equations, we introduce the following dimensionless quantities:

$$(x, y) = \left(\frac{H}{L}\frac{x^*}{H}, \frac{y^*}{H}\right), \quad t = \frac{H}{L}\frac{t^*}{\sqrt{H/g}},$$

$$(u, v) = \left(\frac{u^*}{\sqrt{gH}}, \frac{L}{H}\frac{v^*}{\sqrt{gH}}\right), \quad p = \frac{p^*}{\rho g H}. \tag{3.2.8}$$

In the above non-dimensionalization process, the length scale and the time scale are the most crucial. The condition of the horizontal length scale being not much greater than the vertical length scale implies that the fluid motion is quite violent. The vertical acceleration is important. In order to observe the relatively fast event, the time scale has to be short. The relative size of the vertical velocity with respect to the horizontal velocity is determined by the conservation of mass when the length scales are prescribed. The mass conservation equation suggests a relation

$$\frac{U^*}{L} + \frac{V^*}{L} = 0$$

where U^* and V^* are scales of the horizontal and vertical velocities respectively. Therefore, we have the relative size

$$\frac{V^*}{U^*} = \frac{H}{L}.$$

We carry out this process of nondimensionalizing the physical quantities in the above way because we would like to bring every dimensionless quantity to the size $O(1)$. The nondimensionalized form of (3.2.2) - (3.2.7) is

$$u_x + v_y = 0, \tag{3.2.9}$$

$$u_t + uu_x + vu_y = -p_x, \tag{3.2.10}$$

$$\epsilon\,(v_t + uv_x + vv_y) = -p_y - 1, \tag{3.2.11}$$

$$u_y = \epsilon v_x; \tag{3.2.12}$$

on the free surface $y = \eta(x,\, t)$:

$$p = 0\,, \qquad \eta_t + u\eta_x - v = 0; \tag{3.2.13}$$

and on the bottom $y = -h(x)$:

$$uh_x + v = 0. \tag{3.2.14}$$

Assume that the problem (3.2.9) - (3.2.14) has an asymptotic solution of the form

$$(u,\, v,\, \eta,\, p) = (u_0,\, v_0,\, \eta_0,\, p_0) + \epsilon(u_1,\, v_1,\, \eta_1,\, p_1) + 0\left(\epsilon^2\right)\,. \tag{3.2.15}$$

Substituting (3.2.15) into (3.2.9) - (3.2.14) and assembling the terms of like powers of ϵ gives

Order ϵ^0 :

$$u_{0x} + v_{0y} = 0, \tag{3.2.16}$$

$$u_{0t} + u_0 u_{0x} + v_0 u_{0y} = -p_{0x}, \tag{3.2.17}$$

$$p_{0y} = -1, \tag{3.2.18}$$

$$u_{0y} = 0; \tag{3.2.19}$$

$$\text{on } y = \eta_0 : \quad p_0 = 0, \quad \eta_{0t} + u_0\eta_{0x} = v_0; \tag{3.2.20}$$

$$\text{and on } y = -h(x) : \quad u_0 h_x + v_0 = 0. \tag{3.2.21}$$

In the above, the boundary conditions on the free surface are approximated by a Taylor expansion of (3.2.11) about $y = \eta_0$.

From equations (3.2.18) and (3.2.20),

$$p_0 = -y + \eta_0. \tag{3.2.22}$$

From equation (3.2.19), we find that u_0 is independent of y and hence a function of only x and t:

$$u_0 = f(x,\ t). \tag{3.2.23}$$

This equation clearly shows that the zeroth order horizontal velocity does not have a vertical structure. Intuitively, this is generally false when the moving water is very deep. Hence, equation (3.2.23) is a striking manifestation of the shallow water flow.

Integrating (3.2.16) with respect to y and using the bottom boundary condition (3.2.21), we have

$$v_0 = -f_x\, y - (fh)_x. \tag{3.2.24}$$

Then equations (3.2.17) and (3.2.20) become

$$f_t + f\, f_x + \eta_{0x} = 0,$$
$$\eta_{0t} + f\eta_{0x} + f_x\eta_0 + (fh)_x = 0.$$

These two equations can be written as

$$\eta_{0t} + \Big[f\big(\eta_0 + h\big)\Big]_x = 0, \tag{3.2.25}$$
$$f_t + f\, f_x + \eta_{0x} = 0. \tag{3.2.26}$$

Equations (3.2.25) - (3.2.26) are called the *shallow water equations.* These are the equations that the zeroth order asymptotic approximate solution (3.2.15) should satisfy. Equations (3.2.25)-(3.2.26) are in dimensionless variables. The dimensional form of these two equations (3.2.29)-(3.2.30) is going to be shown later.

If the asymptotic assumption (3.2.15) holds, it is clear that when $h=$ constant (flat bottom), then

$$\left|f(x,t) - \frac{1}{h}\int_{-h}^{0} u(x,\ y,\ t)dy\right| = O(\epsilon) \tag{3.2.27}$$

where $u(x, y, t)$ is from the exact solution of (3.2.9) - (3.2.14). Hence, $f(x, t)$ is approximately the average horizontal velocity.

However, the mathematical rigor of the asymptotic assumption (3.2.15) requires a proof.

Exercise: Suppose that h = *constant* and $u(x, y, t)$ satisfies the problem (3.2.9) - (3.2.14). Show that if η_0 and f solve the problem (3.2.25) -(3.2.26), then equation (3.2.27) holds.

A simpler way to derive the shallow water equations is from the physical point of view. The physics assumption for shallow water flow is that there is

no vertical acceleration. Since, in an inviscid fluid, the driving force for the acceleration in the vertical direction consists of only the pressure gradient and the gravity, the vanishing vertical acceleration of the fluid implies that the pressure gradient must be balanced by the gravity. Thus, the pressure is equal to the hydrostatic pressure. Namely,

$$p^* = \rho g(\eta^* - y^*). \tag{3.2.28}$$

Under the shallow water assumption, the continuity equation can be replaced by the conservation of mass flux

$$\eta^*_{t^*} + \left[u^*(h^* + \eta^*)\right]_{x^*} = 0. \tag{3.2.29}$$

In the Euler equations, there are two equations for the conservation of momentum. One is for the horizontal direction and the other one for the vertical direction. Since under the shallow water assumption the vertical acceleration is zero, only the conservation of the horizontal momentum plays a role. The conservation of the horizontal momentum is

$$u^*_{t^*} + u^* u^*_{x^*} = -p^*_{x^*}.$$

By equation (3.2.28), we have

$$u^*_{t^*} + u^* u^*_{x^*} + g\eta^*_{x^*} = 0. \tag{3.2.30}$$

Equations (3.2.29) and (3.2.30) are the dimensional form of the shallow water equations (3.2.25) and (3.2.26). The nondimensionalization rule is, of course, still according to (3.2.8).

From the above analysis, we have seen that the physical argument is simple and straightforward. In contrast, the mathematical analysis is tedious but shows explicitly the size of the terms thrown out in the approximation process. It seems that the physical analysis and the mathematical analysis are complimentary to each other and we should hesitate to suggest that one analysis be superior to the other.

If h is a constant, then equations (3.2.25) - (3.2.26) can be written into a conservation law

$$\begin{pmatrix} \eta_0 \\ f \end{pmatrix}_t + \begin{bmatrix} f & \eta_0 + h \\ 1 & f \end{bmatrix} \begin{pmatrix} \eta_0 \\ f \end{pmatrix}_x = 0. \tag{3.2.31}$$

The eigenvalues of the coefficient matrix are

$$\lambda_{1,2} = f \pm \sqrt{\eta_0 + h}. \tag{3.2.32}$$

Hence, equation (3.2.31) is a strictly hyperbolic system. The wave propagation speeds relative to the mean flow f are the differences between the characteristic speeds and the mean flow velocity. Namely,

$$U_0^{\pm} = \lambda_{1,2} - f = \pm\sqrt{\eta_0 + h} \tag{3.2.33}$$

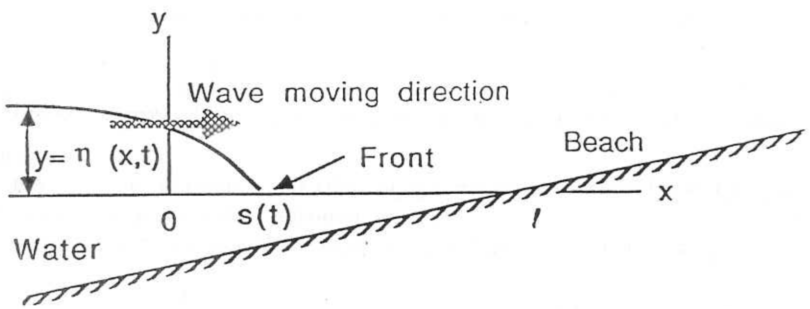

Figure 3.3: Wave breaking on a beach of uniform slope

and $U_0^{\pm}$ are called the the propagation speeds of shallow water waves. The "+" and "−" signs indicate that a disturbance in a shallow water propagates in both directions (upstream and downstream). This shallow water wave speed can be observed very easily by slightly disturbing a shallow water pond. The small wave propagates in the rest water at the shallow water speed $\sqrt{gH}$.

Shallow water equations are particularly interesting when h is not a constant. Offshore engineers use them to study water motions on beaches, such as the well known running-up problem. In addition, the shallow water equations are of rich mathematical structure and applicable to many fields of sciences and engineering. Actually, the gas dynamics equations have the same mathematical structure as the shallow water equations (3.2.25) - (3.2.26). Therefore, it is not surprising for shallow water equations to have discontinuous solutions, which are known as the undular bores. As we discussed in Chapter 2, discontinuous solutions of gas dynamics equations exist and are called the shocks.

3.2.2 Wave breaking on a beach

Consider a surface wave of smooth profile moving toward the shoreline of a uniformly sloping beach (see Fig. 3.3). Our common sense suggests that this profile would soon or later become non-smooth and break on the beach.

The shallow water equations

$$\eta_t + \left[u(\eta + h)\right]_x = 0, \qquad (3.2.34)$$

$$u_t + u\,u_x + \eta_x = 0. \qquad (3.2.35)$$

are used as the governing equations. The x-axis is chosen to be the undisturbed free surface and $x = \ell$ is the fixed shoreline. The wave front $x = s(t)$ is defined by $u(x,\ t) = 0,\ \ \eta(x,\ t) = 0$ when $x > s(t)$. The following assumptions are taken:

(a) u and η are continuous functions;

(b) The first and the second partial derivatives of u and η suffer at most jump discontinuities;

(c) $u(x,t) = 0,\ \ \eta(x,t) = 0$ if $x > s(t)$.

Assumption (a) is reasonable since we are considering the wave motion before the breaking of the surface. An undular bore (a discontinuity) can form only after the breaking. Assumption (c) is consistent with the statement that "$x = \ell$ is a fixed shoreline." This is more or less true since waves break long before they reach the initial position of the shoreline. Let $f^- = \lim_{x \to s(t)-0} f(x)$ denote the value of the quantity f right behind the wave front. Then, assumptions (a) and (c) yield $(du/dt)^- = (d\eta/dt)^- = 0$, i.e.,

$$c\,(u_x)^- = -\,(u_t)^- \ , \quad c\,(\eta_x)^- = -\,(\eta_t)^- \qquad (3.2.36)$$

where

$$c = \frac{ds}{dt} \qquad (3.2.37)$$

is the velocity of the front. Just behind the wave front (i.e., $s - x = o+$), equations (3.2.34) - (3.2.35) become

$$(\eta_t)^- = -h\,(u_x)^- \ , \qquad (u_t)^- = -\,(\eta_x)^- \,. \qquad (3.2.38)$$

If we assume that $(\eta_x)^- \neq 0$, then (3.2.36) and (3.2.38) yield

$$c = \frac{ds}{dt} = \sqrt{h}. \qquad (3.2.39)$$

This is exactly the propagation speed of shallow water waves as we wanted to find.

We are mainly interested in the breaking, i.e., where and when $(\eta_x)^-$ becomes infinity. Let

$$a = a(x) = (\eta_x)^- \,. \qquad (3.2.40)$$

From (3.2.34) - (3.2.35) and the above derived results, we can have

$$c^2\,(u_{xx})^- - (u_{tt})^- + \frac{2h_x}{c}a + \frac{3a^2}{c} = 0. \qquad (3.2.41)$$

Exercise: Derive equation (3.2.41).

By

$$\frac{d}{dx}(u_x)^- = (u_{xx})^- + (u_{xt})^- \frac{dt}{dx},$$
$$\frac{d}{dx}(u_t)^- = (u_{tx})^- + (u_{tt})^- \frac{dt}{dx},$$

we have

$$c^2 \frac{d}{dx}(u_x)^- - c\frac{d}{dx}(u_t)^- = c^2\,(u_{xx})^- - (u_{tt})^- \,. \qquad (3.2.42)$$

From (3.2.38) - (3.2.42), one can derive

$$\frac{da}{dx} + \frac{3h_x}{4h}a + \frac{3}{2h}a^2 = 0. \qquad (3.2.43)$$

Exercise: Derive equation (3.2.43).

To integrate equation (3.2.43), let $a = 1/b$. Then

$$\frac{db}{dx} - \frac{3h_x}{4h}b - \frac{3}{2h} = 0. \tag{3.2.44}$$

This first order ODE can be solved by variation of constants. Finally,

$$b(x) = [h(0) + I(x)]\left(\frac{h(x)}{h(0)}\right)^{3/2} \tag{3.2.45}$$

where

$$I(x) = \frac{3h^{3/4}(0)}{2}\int_0^x h^{-7/4}(\sigma)\,d\sigma. \tag{3.2.46}$$

Whether a wave will break depends on whether $b(\xi) = 0$ for some ξ less than or equal to ℓ. There are two cases: a depressive wave $\big(a(0) = (\eta_x)^-(0) > 0\big)$ and an elevation wave $\big(a(0) = (\eta_x)^-(0) < 0\big)$. Elevation waves always break. But, depression waves will break if and only if $I(\ell) < \infty$ and $\xi = \ell$. If $|a(0)|I(\ell) > \ell$, then $\xi < \ell$, and the wave breaks before the shoreline. If $|a(0)|I(\ell) \leq 0$, then $\xi = \ell$, and the wave breaks at the shoreline.

For more details on wave breaking on beaches, see Refs. [3] and [4] listed as additional reading materials at the end of this chapter.

3.3 Dispersive Water Waves

3.3.1 Dispersive waves

If waves of different wave lengths propagate at different speeds, then we say that the waves are *dispersive*. Whether propagated waves can be called dispersive waves depends on not only the wave maker but also the media in which the waves propagate. We can send dispersive waves through most materials in nature. Water is a dispersive medium. The following simple experiment can be done by everybody to demonstrate the dispersion behavior of the water waves of small amplitude.

Stand near a pond and get two stones in your hands, one of which is much larger than the other. Throw the smaller one into the pond. The stone generates water ripples, whose wave lengths are relatively short and which propagate at relatively low speeds. Then throw the larger piece into the pond. The larger stone generates water ripples whose wave lengths are relatively longer and which propagate at relatively higher speeds. In summary, the results of this experiment imply that the water waves with longer wave lengths propagate faster than those with shorter wave lengths.

To see the dispersive property of waves mathematically, let us look at the following three examples.

Example 1. Consider the Klein-Gordon equation

$$\phi_{tt} - \phi_{xx} + \phi = 0. \tag{3.3.1}$$

Assume this Klein-Gordon equation has a wave solution of the form

$$\phi = a\cos(kx - \omega t) \tag{3.3.2}$$

where ω is called the frequency, k is called the *wave number* and $\lambda = 1/k$ is called the *wave length*. Substituting (3.3.2) into (3.3.1) we have

$$\big(-\omega^2 + k^2 + 1\big)\cos(kx - \omega t) = 0.$$

Hence, expression (3.3.2) is a nontrivial solution of equation (3.3.1) if and only if

$$\omega^2 = k^2 + 1. \tag{3.3.3}$$

This equation is called a *dispersion relation* (or eikonal equation in optics, or Hamilton-Jacobi equation in the Hamiltonian mechanics). The observable velocity of a wave of frequency ω and wave length λ is $\omega\lambda$ (or ω/k) and is called the phase velocity. Hence, the phase velocity of the wave is

$$\frac{\omega}{k} = \pm\sqrt{1 + \frac{1}{k^2}} = \pm\sqrt{1 + \lambda^2}. \tag{3.3.4}$$

Therefore waves of different wave lengths propagate at different speeds. The larger the wave length, the higher the propagation speed. Furthermore, the dispersive wave (3.3.2) with a fixed wave length propagates in two directions since the phase velocity can be positive or negative according to (3.3.4).

Example 2. Consider the linear Korteweg-de Vries equation

$$\phi_t + \frac{3}{2}\phi_x + \frac{1}{6}\phi_{xxx} = 0. \tag{3.3.5}$$

Let

$$\phi = a\cos(kx - \omega t). \tag{3.3.6}$$

Inserting (3.3.6) in (3.3.5), we can obtain a dispersion relation

$$\omega = \frac{3}{2}k - \frac{1}{6}k^3. \tag{3.3.7}$$

Then the phase velocity is

$$\frac{\omega}{k} = \frac{3}{2} - \frac{1}{6}k^2. \tag{3.3.8}$$

Thus the wave (3.3.6) is dispersive and the wave with a fixed wave length propagates only in one direction since there is only one given sign for a given k.

Example 3. Consider the cubic nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2 u = 0, \quad -\infty < x, t < \infty, \tag{3.3.9}$$

where $i = \sqrt{-1}$ is the imaginary unit and $u(x,t)$ is a complex valued function.

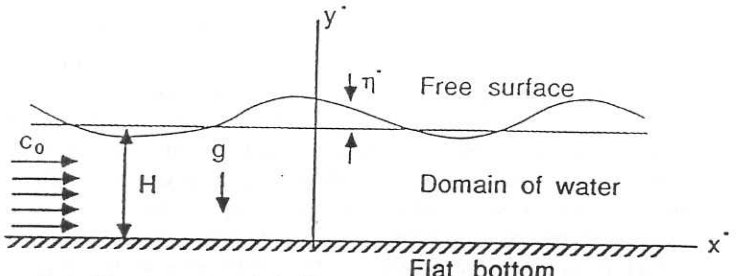

Figure 3.4: Two-imensional shallow water waves in an open ocean of flat bottom.

Let us look for a solution of the form

$$u = a\exp[i(kx+\omega t)], \quad a = \text{constant (may be complex valued)}. \tag{3.3.10}$$

Substituting (3.3.10) into equation (3.3.9), we have the following dispersion relation

$$\omega = |a|^2 - k^2. \tag{3.3.11}$$

This dispersion relation depends on the amplitude of the solution, and so the phase velocity also depends on the solution amplitude. The phase velocity is

$$\frac{\omega}{k} = \frac{|a|^2}{k} - k. \tag{3.3.12}$$

From this expression, we can see that a wave of very long wave length has a phase velocity which is proportional to the amplitude of the solution. But a wave of very short wave length has a phase velocity which is inversely propotional to the amplitude of the solution. This property should not be due to the wave maker. Instead, this property has something to do with the nature of the medium in which the wave propagates. We will discuss the cubic nonlinear Schrödinger equation and its solution properties further in Chapter 7.

3.3.2 Boussinesq equations and the KdV equation

Consider the long wave motion of water with flat bottom (see Fig. 3.4). Let L be the horizontal length scale which characterizes the typical wave length. Let H be the vertical scale which is the upstream mean depth. Assume the motion is irrotational. Then we have the following mathematical problem:

$$\Delta^*\phi^* = 0\,, \quad 0 < y^* < H + \eta^*, \tag{3.3.13}$$

on the free surface $y^* = H + \eta^*$,

$$\eta^*_{t^*} + \phi^*_{x^*}\eta^*_{x^*} - \phi^*_{y^*} = 0, \tag{3.3.14}$$

$$\eta^* + \phi^*_t + \frac{1}{2}(\nabla^*\phi^*)^2 = \text{constant}, \tag{3.3.15}$$

and on the bottom

$$\phi^*_{y^*} = 0. \tag{3.3.16}$$

We introduce the following dimensionless variables

$$x = \frac{x^*}{L}, \quad y = \frac{y^*}{H}, \quad t = \frac{\sqrt{gH}}{L}t^*,$$
$$\eta = \frac{\eta^*}{H}, \quad \phi = \epsilon^{-1}\frac{\phi^*}{\sqrt{gHL}},$$

and

$$\epsilon = \left(\frac{H}{L}\right)^2 \ll 1 \quad \text{(long wave assumption)}.$$

The time scale $L/\sqrt{gH}$ is the time needed for a linear shallow water wave to travel a distance L. This horizontal length scale may be regarded as a typical wave length for periodic waves. The horizontal velocity is $u^* = \phi^*_{x^*} \sim \epsilon L\sqrt{gH}/L = \epsilon\sqrt{gH}$. The vertical velocity scale is $v^* = \phi^*_{y^*} \sim \epsilon L\sqrt{gH}/H = \epsilon^{1/2}\sqrt{gH}$. From the above two statements it seems that the vertical velocity is in the order $\epsilon^{1/2}\sqrt{gH}$ and the horizontal velocity is in the order $\epsilon\sqrt{gH}$. Hence the vertical velocity is greater than the horizontal velocity. This is against our physical intuition and is an inappropriate interpretation of the two statements above. As we discussed in section 3.2 the relative size of the horizontal velocity with respect to the vertical velocity, i.e. the relationship $V^*/U^* = H/L$, should still hold. Hence, we expect that the leading order term of ϕ is independent of y in order to make the nondimensionalization consistent with the physics. Consequently, the vertical velocity is $v^* = \phi^*_{x^*} \sim \epsilon^{3/2}\sqrt{gH}$ which is consistent with the relation $V^*/U^* = H/L = \epsilon^{1/2}$. The nondimensionalized problem is

$$\epsilon\phi_{xx} + \phi_{yy} = 0\,, \quad 0 < y < 1 + \eta; \tag{3.3.17}$$

on $y = 1 + \eta$,

$$\epsilon\eta_t + \epsilon^2\phi_x\eta_x - \phi_y = 0, \tag{3.3.18}$$

$$\eta + \phi_t + \frac{\epsilon}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 = 0; \tag{3.3.19}$$

on $y = 0$,

$$\phi_y = 0. \tag{3.3.20}$$

Let

$$\phi = \sum_{n=0}^{\infty} y^n f_n(x,t). \tag{3.3.21}$$

Substituting (3.3.21) into (3.3.17) and using (3.3.20), we have

$$f_{2n-1} = 0, \quad n = 1, 2, 3, \cdots,$$
$$f_2 = -\frac{\epsilon}{2}f_{0xx}, \quad f_4 = \frac{\epsilon^2}{24}f_{0xxxx}, \cdots,$$
$$f_{2n} = \frac{(-1)^n\epsilon^n}{(2n)!}\frac{\partial^{2n}}{\partial x^{2n}}f_0, \cdots.$$

Let $f = f_0$ and take only quantities of order lower than $O(\epsilon^2)$ in (3.3.18) - (3.3.19). Then

$$\eta_t + \left[(1+\epsilon\eta)f_x\right]_x - \frac{\epsilon}{6} f_{xxxx} = 0\left(\epsilon^2\right), \tag{3.3.22}$$

$$\eta + f_t + \frac{\epsilon}{2} f_x^2 - \frac{\epsilon}{2} f_{xxt} = 0\left(\epsilon^2\right). \tag{3.3.23}$$

In the above, let $u = f_x$ and differentiate (3.3.23) with respect to x. This leads to

$$\eta_t + \left[(1+\epsilon\eta)u\right]_x - \frac{\epsilon}{6} u_{xxx} = 0, \tag{3.3.24}$$

$$u_t + \epsilon\, u\, u_x + \eta_x - \frac{\epsilon}{2} u_{xxt} = 0. \tag{3.3.25}$$

These equations are called the *Boussinesq equations*. Linearizing the above equations and dropping high order derivatives, we obtain

$$\eta_t + u_x = 0, \tag{3.3.26}$$

$$u_t + \eta_x = 0. \tag{3.3.27}$$

Hence

$$\eta_{tt} - \eta_{xx} = 0. \tag{3.3.28}$$

This equation admits waves traveling in two directions, $\eta(x,t) = \eta_R(x-t) + \eta_L(x+t)$. Furthermore, equations (3.3.26) - (3.3.27) can be satisfied with $u = \eta$. We may take this result as a hint and assume

$$u = \eta(x,\ t) + \epsilon A(x,\ t) + 0\left(\epsilon^2\right). \tag{3.3.29}$$

in (3.3.24) - (3.3.25). Substituting (3.3.29) into (3.3.24) - (3.3.25), we have

$$\eta_t + \eta_x + \epsilon\, A_x + 2\epsilon\eta\eta_x - \frac{\epsilon}{6}\eta_{xxx} - \frac{\epsilon}{6} A_{xxx} = 0,$$

$$\eta_t + \eta_x + \epsilon\, A_t + \epsilon\eta\eta_x - \frac{\epsilon}{2}\eta_{xxt} - \frac{\epsilon}{2} A_{xxt} = 0.$$

If we consider those waves traveling only to the right, then $\eta_t + \eta_x = 0$ is a constraint on the above two equations. We apply this constraint only in such a way that $\eta\,\eta_t = -\eta\,\eta_x$ and $\epsilon\eta_{xxt} = -\epsilon\eta_{xxx}$. The higher derivatives of the higher order terms $\frac{\epsilon}{2} A_{xxt}$ and $(\epsilon/6)A_{xxx}$ are omitted. With these adjustments, to make the two equations consistent, A may take

$$A = -\frac{\eta^2}{4} + \frac{\eta_{xx}}{3}.$$

In turn, η yields

$$\eta_t + \eta_x + \frac{3\epsilon}{2}\eta\,\eta_x + \frac{\epsilon}{6}\eta_{xxx} = 0. \tag{3.3.30}$$

Let us take the following transformation of coordinates

$$t \to \epsilon^{-1} t, \quad x \to x - (1-\epsilon)t. \tag{3.3.31}$$

Physically, this transformation puts the observer on a moving reference frame which moves at a speed equal to $(1-\epsilon)\sqrt{gH}$ to the right. At the same time, the time is enlarged by ϵ^{-1}. It implies that the physical phenomenon observed by this moving observer is going to be in a state of very fast motion. Equation (3.3.30) in the new coordinates becomes

$$\eta_t + \eta_x + \frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{xxx} = 0. \tag{3.3.32}$$

This is the so called *Korteweg-de Vries equation* (KdV in short) in terms of dimensionless variables. When it is expressed by dimensional quantities, the KdV becomes

$$\eta^*_{t^*} + \sqrt{gH}\eta^*_{x^*} + \frac{3}{2}\frac{\sqrt{gH}}{H}\eta^*\eta^*_{x^*} + \frac{1}{6}H^2\sqrt{gH}\eta^*_{x^*x^*x^*} = 0. \tag{3.3.33}$$

From Example 2 of section 3.3.1., we know that the linear KdV equation of (3.3.30) admits waves traveling only in one direction. So does the nonlinear KdV. The accuracy of the dimensional KdV (3.3.33) to serve as a model equation is discussed in an excellent paper by Hammack and Segur (1974). Their conclusion is that

> *the Korteweg-de Vries equation appears to provide an accurate model for determining the evolution from various set of initial data of gravity waves of moderate amplitude propagating in one direction in a non-dissipative or slightly dissipative fluid of uniform depth.*

There is another derivation of the Korteweg-de Vries equation shown in Appendix C, which is more systematical and possesses clear physical meanings.

Before we go to the next subsection, let us examine some conservation quantities. We do our inspection on a little faster moving reference frame than that defined by (3.3.31) for the KdV (3.3.32). This new frame is arranged in the following way:

$$t \to -t, \quad x \to x - 2\lambda t. \tag{3.3.34}$$

Then, the KdV (3.3.32) in this new reference frame becomes

$$\eta_t + \lambda\eta_x - \frac{3}{2}\eta\eta_x - \frac{1}{6}\eta_{xxx} = 0. \tag{3.3.35}$$

Recall that in Chapter 2 we called the PDE

$$\rho_t + q_x = 0, \quad -\infty < x, t < \infty, \quad q(\pm\infty) = 0 \tag{3.3.36}$$

a conservation law because the mass $\int_{-\infty}^{\infty} \rho(x,t)\,dx$ is a constant (i.e., conserved in time). It has been proved that the KdV (3.3.35) or (3.3.32) has infinitely many conservation laws. Namely, there are infinitely many quantities associated with (3.3.35) or (3.3.32) which are conserved during the motion of the fluid. The reader may see Whitham's book (p. 600) for more details. Not all the conserved quantities have physical meaning. But, the first few quantities have clear physical meanings such as mass, momentum, and energy.

It is straightforward to derive the following from the KdV (3.3.35):

$$\eta_t + \left[\lambda\eta - \frac{3}{4}\eta^2 - \frac{1}{6}\eta_{xx}\right]_x = 0, \tag{3.3.37}$$

$$\left(\frac{1}{2}\eta^2\right)_t + \left[\frac{\lambda\eta^2}{2} - \frac{1}{2}\eta^3 - \frac{1}{6}\eta\eta_{xx} + \frac{1}{3}\eta_x^2\right]_x = 0. \tag{3.3.38}$$

Since

$$\eta(\pm\infty) = \eta_x(\pm\infty) = \eta_{xx}(\pm\infty) = \eta_{xxx}(\pm\infty) = \eta_{xxxx}(\pm\infty) = 0,$$

we have

$$m = \int_{-\infty}^{\infty} \eta(x,t)\,dx = \text{const} \quad \text{(conservation of mass)}, \tag{3.3.39}$$

$$M = \int_{-\infty}^{\infty} \eta^2(x,t)\,dx = \text{const}$$

$$\text{(conservation of momentum)}. \tag{3.3.40}$$

Exercise: Derive the conservation law like (3.3.37) and (3.3.38) for the mechanical energy for the KdV (3.3.35) equation.

3.3.3 Solutions to Korteweg-de Vries equations

Consider the KdV equation

$$u_t + \lambda u_x + 2\alpha u u_x + \beta u_{xxx} = 0, \qquad -\infty < x,\ t < \infty \tag{3.3.41}$$

where λ, α and β are real constants. We look for its traveling solutions. Let

$$\xi = x - ct, \qquad u = u(\xi). \tag{3.3.42}$$

Then

$$(\lambda - c)u' + 2\alpha u u' + \beta u''' = 0. \tag{3.3.43}$$

The first integral of (3.3.43) gives

$$(\lambda - c)u + \alpha u^2 + \beta u'' + C = 0 \tag{3.3.44}$$

where C is the integration constant. Multiplying both sides of (3.3.44) by u' and integrating it, we can integrate one order further to obtain:

$$\frac{1}{2}(\lambda - c)u^2 + \frac{\alpha}{3}u^3 + \frac{\beta}{2}(u')^2 + Cu + D = 0 \tag{3.3.45}$$

where D is the integration constant. So

$$\frac{3}{2}\frac{\beta}{\alpha}(u')^2 = -u^3 - \frac{3}{2\alpha}(\lambda - c)\,u^2 - \frac{3C}{\alpha}u - \frac{3D}{\alpha} \equiv P(u). \tag{3.3.46}$$

If $\beta/\alpha > 0$, then solutions of (3.3.46) fall into one of the following three cases.

Case (i). If $P(u) = 0$ has three distinct real roots, then (3.3.46) has a cnoidal wave solution. In this case, we write (3.3.46) in the form

$$\frac{3}{2}\frac{\beta}{\alpha}(u')^2 = (r_1 - u)(u - r_2)(u - r_3), \quad r_1 > r_2 > r_3. \tag{3.3.47}$$

Let $v = u - r_2$, $\zeta = ((2\alpha)/(9\beta))^{1/2}\xi$, then it can be written as

$$\frac{1}{3}\left(\frac{dv}{d\zeta}\right)^2 = v(s_1 - v)(v - s_1 + s_2), \quad 0 < s_1 < s_2 \tag{3.3.48}$$

where

$$s_1 = r_1 - r_2 > 0, \quad s_2 = r_1 - r_3 > 0, \quad s_2 > s_1. \tag{3.3.49}$$

The solution of the differential equation (3.3.48) can be expressed by a Jacobi elliptic function

$$v = s_1 cn^2\sqrt{\frac{3s_2}{4}}\zeta. \tag{3.3.50}$$

This is a periodic function of ζ. Its wave length is

$$\lambda = \frac{4}{\sqrt{3s_2}}K(m) \tag{3.3.51}$$

where $m = \sqrt{\frac{s_1}{s_2}} < 1$ is called the modulus of the Jacobian elliptic function, and $K(m)$ is the complete elliptic integral of the first kind

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - m^2\sin^2 t}}. \tag{3.3.52}$$

Example: In equation (3.3.46), suppose that

$$\alpha = -9, \quad \beta = -2, \quad \lambda - c = 6, \quad C = -6, \quad D = 0.$$

Then, equation (3.3.46) becomes

$$\frac{1}{3}(u')^2 = u(2 - u)(u - 2 + 3). \tag{3.3.53}$$

So

$$u = 2cn^2\left(\frac{3}{2}x\right). \tag{3.3.54}$$

Its wave length is

$$\lambda = \frac{4}{3}K\left(\sqrt{\frac{2}{3}}\right) \approx 2.7. \tag{3.3.55}$$

The graph of the function (3.3.54) is shown in Fig. 3.5.

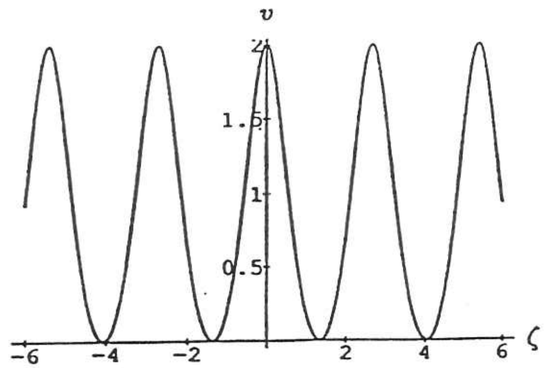

Figure 3.5: Cnoidal waves

Case (ii). If $P(u) = 0$ has a real double root, r_0, and r_0 is smaller than the third real root r, then (3.3.46) has a solitary wave solution. What we mean by a solitary wave solution of (3.3.46) is that

$$\lim_{|x|\to\infty} u(x) = 0 \quad \text{(or denoted by } u(\pm\infty) = 0).$$

In this situation, $r_1 = r, r_2 = r_3 = r_0, s_1 = r - r_0 > 0$ and $s_2 = r - r_0 = s_1$, and equation (3.3.47) becomes

$$\frac{1}{3}\left(\frac{dv}{d\zeta}\right)^2 = v^2\,(s_1 - v)\,. \tag{3.3.56}$$

This equation can be directly integrated by using techniques in elementary calculus:

$$v = s_1 \operatorname{sech}^2 \sqrt{\frac{3s_1}{4}}\zeta \tag{3.3.57}$$

(see Fig. 3.6).

Example: In (3.3.46), assume

$$\beta = -\frac{1}{6}, \quad \alpha = -\frac{3}{4}, \quad \lambda - c < 0, \quad C = D = 0.$$

Then

$$\frac{1}{3}\,(u')^2 = -u^3 - 2\,(\lambda - c)\,u^2 \tag{3.3.58}$$

and

$$u = 2\,(c - \lambda)\operatorname{sech}^2\sqrt{\frac{3}{2}(c-\lambda)}\zeta. \tag{3.3.59}$$

This example comes from a fluid mechanical model of near critical flow of water in a two-dimensional channel.

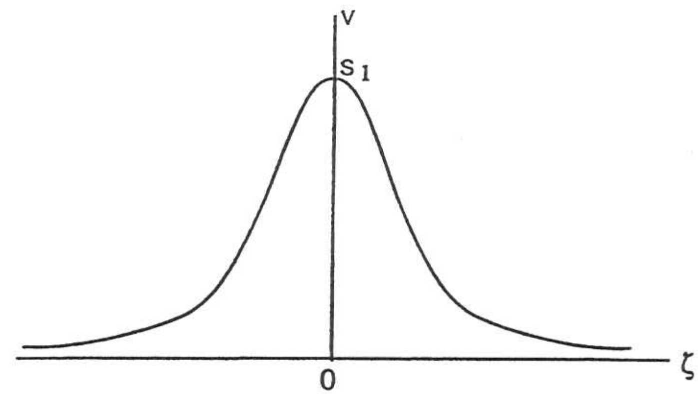

Figure 3.6: Solitary wave solution of the KdV.

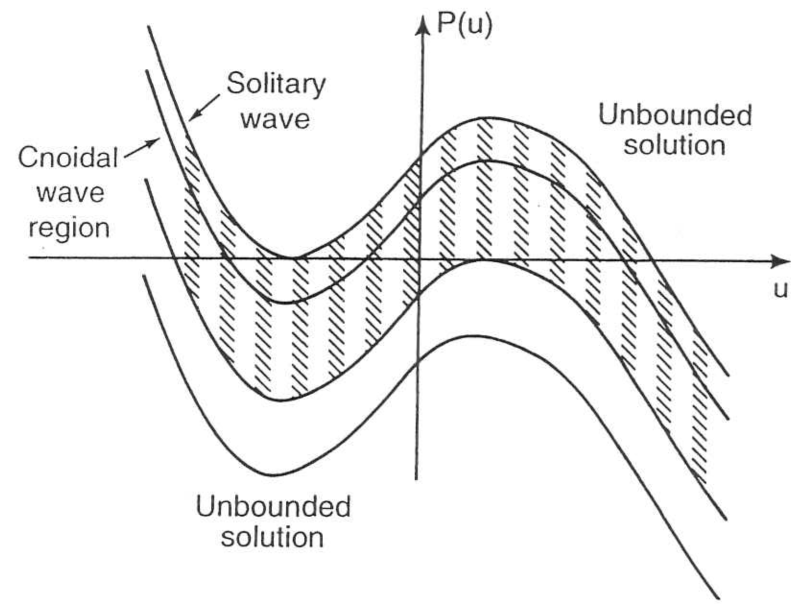

Figure 3.7: Regions of unbounded, cnoidal and solitary waves.

Case (iii). If $P(u) = 0$ has complex roots (There must be two complex roots. Why?), or if $P(u) = 0$ has a double real root which is larger than the third real root, then (3.3.46) does not have bounded solutions.

The above classification is shown in Fig. 3.7.

Additional Reading Materials

[1.] J. J. Stoker (1957), Water Waves: the Mathematical Theory with applications, Interscience, New York.

[2.] G. B. Whitham (1974), Linear and Nonlinear Waves, John Wiley, New York, Chapter 13.

[3.] M. E. Gurtin (1975), On the breaking of water waves on a sloping beach of arbitrary shape, Q. Appl. Math. **33**, 187-189.

[4.] M. C. Shen and R. E. Meyer (1963), Climb of a bore on a beach, J. Fluid Mech. **16**, 113-126.

[5.] J. L. Hammack and H. Segur (1974), The Korteweg-de Vries equation and water waves. Part 2. Comparison with experiments, J. Fluid Mech. **65**, 289-314.

Chapter 4

Scattering and Inverse Scattering

For a given potential, the scattering method has been commonly used to find the wave functions in quantum mechanics. An inverse process of this scattering is to find the potential from known scattering data. Such a process is called the *inverse scattering method.* If the potential satisfies a nonlinear evolution equation (the differential equation $u_t = E[u]$, where E is a nonlinear time independent operator), sometimes there exists a linear operator whose potential is $u(x,t)$ such that the spectrum of the linear operator is independent of time t. Hence the inverse scattering method can generate solutions to the nonlinear evolution equation by solving linear problems. This remarkable method that solves nonlinear evolution equations was invented by Kruskal, Greene, Gardner and Miura (1967), and it was first applied to find soliton solutions of an initial value problem for the Korteweg-de Vries equation. Later it was applied to