VA for bright Soliton (BS) w/ Gaussian ansatz

$$u_t = -\frac{1}{2}u_{xx} - |u|^2 u + Vu \qquad \text{(attractive)}$$

Lagrangian: $L[u] = \frac{i}{2}(u u_t^* - u^* u_t) + \frac{1}{2}|u_x|^2 - \frac{1}{2}|u|^4 + \cancel{V|u|^2}^{\,0}$

steady state Gaussian ansatz: $\boxed{u_A = A\, e^{-\frac{x^2}{2w^2}} e^{-i\mu t}}$; $\underbrace{\vec{p} = (A, w)}_{\text{time INDEP.}}$

$L_A = L[u_A]$ & $\overline{L_A} = \int_{-\infty}^{\infty} L_A\, dx$

- $|u| = Ae$ • $|u|^2 = A^2 e^2$ • $|u|^4 = A^4 e^4$ • $\begin{cases} e^2 = e^{-\frac{x^2}{w^2}} \\ e^4 = e^{-\frac{4x^2}{2w^2}} = e^{-\frac{x^2}{w^2/2}} \end{cases}$
- $u_t = \underbrace{-i\mu A e e^{-i}}$ • $|u_x| = -\frac{\cancel{2}x}{\cancel{2}w^2} A e$

[Here we would need also $\partial_t(x, w)$ if x, w were functions of t !!!]

$$\Rightarrow L_A = \frac{i}{2}\left[Ae\cancel{e^{i}}(+i\mu)Ae\cancel{e^{-i}} - Ae\cancel{e^{i}}(-i\mu)Ae\cancel{e^{-i}}\right] + \frac{1}{2}\frac{A^2}{w^4}x^2 e^2 - \frac{1}{2}A^4 e^4$$

$$= \frac{i}{\cancel{2}}A^2 \cancel{2} e^2 (i\mu) + \frac{1}{2}\frac{A^2}{w^4}x e^2 - \frac{1}{2}A^4 e^4$$

But $\int_{-\infty}^{\infty} e^{-\frac{x^2}{B}} dx = \sqrt{\pi}\sqrt{B}$ & $\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{B}} dx = \frac{1}{2}\sqrt{\pi}\, B^{3/2}$

$$\therefore \overline{L_A} = \int_{-\infty}^{\infty} L_A\, dx = -\mu A^2 \sqrt{\pi}\sqrt{w^2} + \frac{1}{2}\frac{A^2}{w^4}\frac{1}{2}\sqrt{\pi}(w^2)^{3/2} - \frac{1}{2}A^4 \frac{1}{2}\sqrt{\pi}\sqrt{w^2/2}$$

$$= \sqrt{\pi}\left[-\mu A^2 w + \frac{1}{4}\frac{A^2}{w} - \frac{\sqrt{2}}{4}A^4 w\right]$$

Euler-Lagrange

$\underline{A}$: $\underbrace{\frac{d}{dt}\frac{\partial \overline{L_A}}{\partial \dot{A}}}_{=0} = \frac{\partial \overline{L_A}}{\partial A} = \sqrt{\pi}\left[-2\mu A w + \frac{1}{2}\frac{A}{w} - \sqrt{2}A^3 w\right]$

$\boxed{w}$: $\underbrace{\frac{d}{dt}\frac{\partial \overline{L_A}}{\partial \dot{w}}}_{=0} = \frac{\partial \overline{L_A}}{\partial w} = \sqrt{\pi}\left[-\mu A^2 - \frac{1}{4}\frac{A^2}{w^2} - \frac{\sqrt{2}}{4}A^4\right]$

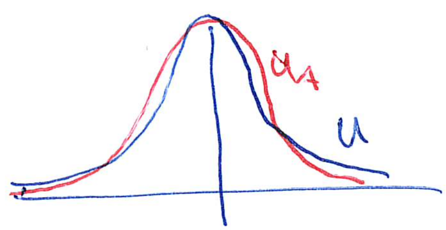
$$\begin{cases} =0 \;\Rightarrow\; \times \dfrac{w}{A}: & -2\mu w^2 + \dfrac{1}{2} - \sqrt{2}A^2 w^2 = 0 \quad (a) \\[2mm] =0 \;\times \dfrac{4w^2}{A^2}: & -4\mu w^2 - 1 - \sqrt{2}A^2 w^2 = 0 \quad (b) \end{cases}$$

$$(-) \qquad 2\mu w^2 + \frac{3}{2} = 0 \;\Rightarrow\; 4\mu w^2 = -3 \;\Rightarrow\; \boxed{\mu = -\frac{3}{4w^2}} \quad (1)$$

$$(1) \xrightarrow{b} \quad 3 - 1 - \sqrt{2}A^2 w^2 = 0 \;\Rightarrow\; 2 - \sqrt{2}A^2 w^2 = 0 \;\Rightarrow\; \boxed{A^2 w^2 = \sqrt{2}} \quad (2)$$

$$\therefore \quad u_A = A\, e^{-\frac{x^2}{2\sqrt{2}/A^2}}\, e^{+i\frac{3}{4\sqrt{2}}A^2 t} \qquad \frac{3}{4\sqrt{2}} = 0.5303 \approx 1/2$$

$$u = A \operatorname{sech} Ax \; e^{i\frac{A^2}{2}t}$$



<u>Dynamics</u>: let us try to get the internal oscillations of the BS

$$u_A = A\, e^{-\frac{x^2}{2w^2}}\, e^{-i\mu + i d x^2}$$

params: $\vec{p}(t) = (A(t), w(t), \mu(t), d(t))$

$$\ldots \quad L_A = \sqrt{\pi}\left[-A^2\mu w - \frac{1}{2}A^2 \dot{d} w^3 - \frac{1}{4}\frac{A^2}{w} - A^2 d^2 w^3 + \frac{\sqrt{2}}{4}A^4 w\right]$$

<u>Euler-Lagrange</u>: $\ldots$

$$\begin{cases} \dot{A} = -Ad \\ \dot{w} = 2wd \\ \dot{\mu} = \dfrac{1}{2w^2} - \dfrac{5\sqrt{2}}{8}A^2 \\ \dot{d} = \dfrac{1}{2w^4} - 2d^2 - \dfrac{\sqrt{2}}{4}\dfrac{A^2}{w^2} \end{cases}$$