HW#1 — Nonlinear Waves

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1. Consider the *bidirectional* linear wave equation

$$u_{tt} - c^2 u_{xx} = 0. \tag{1}$$

with $c \geq 0$.

a) Find the solution to (1) if $u(x,0) = \sin(x)/x$ and $u_t(x,0) = 0$.

b) Plot $u(x,t)$ in a) for $c = 1$ on $(x,t) \in [-25 : 25] \times [-10 : 10]$.

c) Find the solution to (1) if $u(x,0) = \exp(-x^2)$ and $u_t(x,0) = 1/(x^2+1)$.

d) Plot $u(x,t)$ in c) for $c = 1$ on $(x,t) \in [-15 : 15] \times [-10 : 10]$.

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2. This exercise aims to analyze the propagation of plane waves for the linear wave equation with dissipation *and* dispersion:

$$u_t + cu_x - \gamma u_{xx} + \delta u_{xxx} = 0, \tag{2}$$

with $c, \gamma, \delta \geq 0$.

a) Consider the plane wave $u(x,t) = A\exp[i(kx - \omega t)]$. Derive the dispersion relation for this plane wave.

b) Explain in detail what happens to plane wave solutions for the different choices of k, c, γ, δ

c) Plot, using Matlab, 4 different exact solutions for different choices of k, reflecting qualitatively different evolutions (i.e. $k = 0$, positive velocity, zero velocity and negative velocity), for $A = 1$, $c = 8$, $\delta = 2$, $\gamma = 0.5$, $x \in [-10, 10]$ and $t \in [0, 2]$. Also plot a nontrivial linear combination of these 4 solutions (cf. `dispersion.m`).

d) Rescale t, x, u in order to reduce the number of parameters (c,γ,δ) in Eq. (2). How many parameters the reduced system has? What are the implications?

e) Repeat d) for the *nonlinear* dispersive, dissipative wave equation:

$$u_t + cu_x + \beta uu_x - \gamma u_{xx} + \delta u_{xxx} = 0. \tag{3}$$

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3. Given the wave equation with dissipation *and* dispersion:

$$u_t + cu_x - \gamma u_{xx} + \delta u_{xxx} = 0, \tag{4}$$

with $c, \gamma, \delta \geq 0$.

a) Write an explicit solution for $u(x,t)$ if

$$u(x,0) = 3\cos^2(x) + \sin(x).$$

[Hint: is $\cos^2$ a plane wave? Can you make it one?]

b) Plot, using Matlab (see for example `dissipation.m`), the exact solution of a) with $c = 4$, $\delta = 1$, $\gamma = 0.25$, $x \in [0, 6\pi]$ and $t \in [0, 6]$.

c) Also plot this solution by integrating the wave equation using `wave_integrator.m`, and corroborate that you obtain the same behavior as in b).

d) The above solution seems to tend to a *single* damped plane wave displaced from zero. Explain why. Find this damped plane wave and plot it, using Matlab (see for example `dissipation.m`), for the same parameters as in b). **Caution:** When using the code `dissipation.m` you need to be aware that it is a spectral code and thus the solutions will be periodic. Therefore your initial condition must be periodic also and fit an integer number of periods in the domain. In the present case $u(x, 0)$ is indeed periodic of period 2π and this period does divide the total length of the domain (6π).

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4. Consider the nonlinear wave equation:

$$u_t + c(u)\, u_x = 0, \quad \text{with} \quad c(u) = \alpha\, u^n. \tag{5}$$

Repeat what we did in class to solve for the evolution of an initial wave $u(x, 0) = A \exp(-x^2)$ for general n. Specifically:

a) Solve Eq. (5) using the method of characteristics.

b) Find the general solution for the breaking time t_B as a function of the initial x-position ξ.

c) Find the minimum breaking time, $t_{\min}$, for the whole wave.

d) For $n = 2$ and $\alpha = 1$ give the expressions for t_B and $t_{\min}$.

e) For $n = 2$ and $\alpha = 1$ plot the solution $u(x, t)$ in the (x, t) plane. [Hint: you might want to use the code `characteristics.m`.]

f) For $A = 1$, $n = 2$ and $\alpha = 1$ modify the code `charac_example_pde.m` to integrate the solution and to show that the $t_{\min}$ found in c) is correct.

g) Do the same as in d)–f) for $n = 3$ and $\alpha = 1$. Contrast the differences/similarities between $n = 2$ and $n = 3$ and explain them using physical arguments. In particular, which case ($n = 2$ or $n = 3$) breaks earlier for $A = 1$. What if $A < 1$ or $A > 1$. Can you find a value for A such that the two cases break at the same time?