HW#2 — Nonlinear Waves

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1. This exercise aims to analyze some generic aspects of the KdV equation:

$$\frac{1}{c}u_t + u_x + \frac{3}{2h}uu_x + \frac{h^2}{6}u_{xxx} = 0, \tag{1}$$

where $u(x,t)$ is the vertical displacement of the wave surface from its resting level, c is the speed of plane waves in the absence of nonlinearity and dispersion, and h is the resting depth of the water canal.

a) By the following transformations: $\xi = x - ct$, $\tilde{u} = \beta u$ and $y = \alpha\xi$, prove that the KdV equation (1) reduces to

$$\tilde{u}_t + 6\tilde{u}\tilde{u}_y + \tilde{u}_{yyy} = 0. \tag{2}$$

Give the explicit form for β and α in terms of h and c.

b) Prove that if $f(x - vt)$ is a solution for the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{3}$$

then $\lambda + f(x - \tilde{v}t)$ is also a solution of the KdV equation. Give the explicit expression for $\tilde{v}$ as a function of v and λ. Prove numerically this invariance by taking a `sech` solitary wave solution to the KdV and its transformed version using some value for λ (for example $\lambda = 1$) and integrating *numerically* both initial conditions. Do you get what you expected? Elaborate.

c) Show that the superposition principle is not valid for the KdV. I.e. show that if $f(x,t)$ and $g(x,t)$ are general solutions to (3) then $f + g$ is *not* a solution. Are there any particular cases when $f + g$ is indeed a solution (where f and g are solutions) for special choices of f and g (or combinations thereof)?

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2. The aim of the exercise is to use the traveling wave reduction ($u(x,t) = f(x - ct)$) to find ALL qualitatively different solutions using Newton's method + potential approach.

a) Modified KdV equation (mKdV)

$$u_t + 6u^2u_x + u_{xxx} = 0.$$

b) Generalized KdV equation

$$u_t + (n+1)(n+2)u^n u_x + u_{xxx} = 0, \quad \text{with} n = 1, 2, \ldots$$

Only for $n = 3$ and $n = 4$, find all qualitatively different solutions using the Newton's method + potential approach.

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3. For the following equations, find a traveling wave solution that satisfies the given conditions.
Hint: you might need the following integrals: $\displaystyle\int \frac{df}{f\sqrt{B - Af^n}} = \frac{-2}{n\sqrt{B}} \tanh^{-1}\left(\frac{\sqrt{B - Af^n}}{\sqrt{B}}\right)$,
and $\displaystyle\int \frac{df}{f(f - A)} = \frac{-2}{A} \tanh^{-1}\left(\frac{2f - A}{A}\right)$.

a) Burgers equation

$$u_t + uu_x = u_{xx}$$

with $u(+\infty, t) = 0$ and $u(-\infty, t) = a$ $(a > 0)$, and $u_x(\pm\infty, t) = 0$.

b) The elastic-medium equation

$$u_{tt} = u_{xx} + u_x u_{xx} + u_{xxxx}$$

with $u_x(\pm\infty, t) = 0$ and $u_{xx}(\pm\infty, t) = u_{xxx}(\pm\infty, t) = 0$. [Hint: integrate twice in the usual way and then note that the lowest order derivative is f' and thus, the change of coordinates $g = f'$ leads to a first order, separable, ODE. (do not forget that after solving for g you need to go back to f)].

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4. Consider the Boussinesq equation:

$$u_{tt} - u_{xx} + 3(u^2)_{xx} - u_{xxxx} = 0. \tag{4}$$

Note that there is a double time derivative and that $(u^2)_{xx} \neq (u_{xx})^2$. The Boussinesq equation is the analogue of the bidirectional wave equation for the KdV equation. Namely, the Boussinesq equation, supports wave propagation in both directions.

a) Follow the same methodology that we used in class for the KdV to find solitary wave solutions to the Boussinesq equation.

b) Show that these solitary waves can travel in both directions.