HW#3 — Nonlinear Waves

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1. For the nonlinear repulive/defocusing Schrödinger equation:

$$iu_t + \frac{1}{2}u_{xx} - |u|^2 u = 0, \tag{1}$$

find the general form for a dark/grey soliton solution:

$$u(x,t) = u_0 \left\{B \tanh[u_0 B(x - Au_0 t - x_0)] + iA\right\} e^{-iu_0^2 t}, \tag{2}$$

where $A^2 + B^2 = 1$.

Hint: follow the notes where we split u as:

$$u(x,t) = [f(b\xi) + i\mathcal{A}]\, e^{i\phi(t)}, \tag{3}$$

where $\mathcal{A} = u_0\, A$, $\xi = x - ct$, and f and ϕ are real functions, and you will assume that $u(x = \pm\infty, t) = \pm u_0$.

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2. In this exercise you will obtain the basic conservation laws for the Nonlinear Schrödiger (NLS) equation:

$$iu_t + \frac{1}{2}u_{xx} \pm |u|^2 u = 0, \tag{4}$$

where $u = u(x,t)$ and $|u|^2$ describes a) the light intensity in nonlinear optics, or b) the atomic density in a Bose-Einstein condensate (BEC).

a) Show that the total power (optics) or mass (BEC), for localized solutions, is conserved. Namely, show that

$$M = \int_{-\infty}^{\infty} |u(x,t)|^2\, dx = \text{const.} \tag{5}$$

Hint: use $(4) \cdot u^* - (4)^* \cdot u$, where $(\cdot)^*$ means complex conjugation.

b) Show that the total energy is conserved. Namely, show that

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2}|u_x|^2 \mp \frac{1}{2}|u|^4\right) dx = \text{const.} \tag{6}$$

Hint: use $(4) \cdot u_t^* + (4)^* \cdot u_t$.

c) [**extra credit**] Using a couple of different initial conditions (ICs) that are not just constants nor single bright nor dark solitons (you can try a superposition of a few bright solitons or a sum of a few sines and cosines), numerically integrate the NLS and plot a time series of the above conserved quantities ($M(t)$ and $E(t)$) to verify they are conserved. Please also plot the relative error of the conservation (cf. $(M(t) - M(0))/M(0)$ and $(E(t) - E(0))/E(0)$) and discuss your results. You can use the spectral code `NLS.m` for this but remember that ICs have to be periodic in the domain!

d) [**extra-extra credit**] Show that the momentum is conserved. Namely, show that

$$P = i \int_{-\infty}^{\infty} (uu_x^* - u^* u_x)\, dx = \text{const.} \tag{7}$$

Hint: use $\{(4) \cdot u_x^* + (4)^* \cdot u_x\} - \{[(4)]_x \cdot u^* + [(4)^*]_x \cdot u\}$.

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3. This exercise aims at approximating the evolution of a bright NLS soliton trapped in a confining potential by exclusively using conserved quantities. As a reference please consult R. Scharf and A.R. Bishop, Phys. Rev. E **47** (1993) 1375 [Sections I–III].

Consider the self-focusing NLS with an external potential:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = V(x)\, u, \tag{8}$$

where the external potential $V(x)$ can represent (a) a transverse modulation of the index of refraction in an optical fiber trapping the optical beam or (b) an external magnetic or optical potential trapping a cloud of Bose-Einstein condensate (BEC) atoms. Consider that the evolution of a bright soliton placed inside the trap does not deviate too much from the shape of a standard NLS bright soliton. Then, one can approximate the shape of the evolving soliton by the following ansatz:

$$u(x,t) = a(t)\,\mathrm{sech}[a(t)\,(x-\xi(t))]\ \exp[i(c(t)\cdot x + d(t)\cdot t)], \tag{9}$$

where now the amplitude $a(t)$, position $\xi(t)$ and velocity, $c(t) = \dot{\xi}(t)$, and the so-called chirp, $d(t)$, are allowed to vary in time to accommodate for the perturbations in shape induced by the presence of the external potential.

A) Consider the following external potential

$$V(x) = -A\cos\left(\frac{2\pi}{\lambda}x\right). \tag{10}$$

This potential arises very naturally in a BEC trapped by a periodic potential induced by the interference pattern of two counter-propagating laser beams, the so-called optical lattice. By using the conservation of mass and the conservation of energy, that in this case with the additional trapping potential is:

$$E = \int_{-\infty}^{\infty}\left(\frac{1}{2}|u_x|^2 - \frac{1}{2}|u|^4 + V(x)\,|u|^2\right)dx = \text{const.} \tag{11}$$

perform the following tasks:

(a) Prove that the mass $M = \int_{-\infty}^{\infty}|u|^2\,dx$ and the energy (11) are indeed conserved quantities of the NLS when the external potential $V(x)$ is included [see Eq. (8)]. [Hint: use the results of the previous exercise].

(b) Using the conservation of M and E, with the asantz (9) for the bright soliton, find an ordinary differential equation (ODE) describing the motion of the center of the soliton. What does this equation represent? (Hint: Newton's law with effective potential, compare external to effective potential, elaborate). What about the relations obtained for $a(t)$ and $d(t)$? (What do they represent? Elaborate.)

Hint: you'll probably need the following integrals:

$I_1 = \int_{\infty}^{\infty} a^2\mathrm{sech}^2[a(x-\xi)]\cos(kx)\,dx = \dfrac{k\pi\cos(k\xi)}{\sinh(k\pi/(2a))}$ $I_2 = \int_{-\infty}^{\infty}\mathrm{sech}^2(ax)\,\tanh(ax)^2\,dx = \dfrac{2}{3a}$,

$I_3 = \int_{-\infty}^{\infty}\mathrm{sech}^2(ax)\,dx = \dfrac{2}{a}$, $I_4 = \int_{-\infty}^{\infty}\mathrm{sech}^4(ax)\,dx = \dfrac{4}{3a}$.

(c) What is the dynamics of a trapped soliton centered at $\xi_0 = \xi(t=0)$ with zero initial velocity, $c_0 = c(t=0) = 0$?

(d) If a soliton initially centered at $\xi_0 = 0$ is given an initial velocity $c_0 = c(t=0) > 0$, what is the the critical velocity (i.e. escape velocity), c_e, such that the soliton permanently leaves the central valley of the trap?

(e) Use your previous results to compare them with results from directly integrating the NLS. Please show several examples including: $(\xi_0, c_0) = (0,0)$, $(\xi_0, c_0) = (0, c_e/2)$, $(\xi_0, c_0) = (\lambda/2, 0)$, $(\xi_0, c_0) = (0, c_e)$ (and slight above and below c_e), $(\xi_0, c_0) = (0, c_0)$ with $c_0 = 2c_e$. Please perform the above for the following six setup combinations: $\lambda = 2$, $a(0) = 4, 8$ and $A = 0.01, 0.1, 0.5$. Please be critical about the comparison between the ODE approximation and the full PDE runs. Elaborate.

B) [**extra credit**] Consider now the following external potential

$$V(x) = A\tanh^2(\lambda x). \tag{12}$$

Try to repeat an analysis as the one above. Can you obtain an ODE in closed form? Even if you are not successful in obtaining a closed form for the ODE, perform several direct numerical experiments on NLS to study the behavior of the trapped soliton for at least 4 orbits: $(\xi_0, c_0) = (0,0)$, $(\xi_0, c_0) = (0, c_0)$ such that $c_0 < c_e$, $(\xi_0, c_0) = (0, c_0)$ such that c_0 is close to c_e, $(\xi_0, c_0) = (0, c_0)$ with c_0 larger that c_e.