HW#4 — Nonlinear Waves

---

1. Variational approximation for sech solitons, gain and loss.
By using the NLS Lagrangian:

$$L = \frac{i}{2}(u^* u_t - u u_t^*) - \frac{1}{2}|u_x|^2 + \frac{1}{2}|u|^4 : \tag{1}$$

a) Obtain the equations of motion (ODEs) for the parameters $\{a(t), \xi(t), c(t), b(t)\}$ of the ansatz

$$u_A = a\,\mathrm{sech}(a(x-\xi)\ \exp\left[i(c(x-\xi)+b)\right], \tag{2}$$

using the Lagrangian variational approach for NLS described in the lectures.

Does the resulting evolution for the ansatz match your expectations? Elaborate.

Hint: you'll probably need the following integrals:

$I_1 = \int_{-\infty}^{\infty} \mathrm{sech}^2(ax)\ \tanh(ax)^2\,dx = \dfrac{2}{3a}$, $I_2 = \int_{-\infty}^{\infty} \mathrm{sech}^2(ax)\,dx = \dfrac{2}{a}$,
$I_3 = \int_{-\infty}^{\infty} \mathrm{sech}^4(ax)\,dx = \dfrac{4}{3a}$.

b) Perform a similar analysis, with same ansatz (2), for the NLS with loss.

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u + \epsilon i u = 0. \tag{3}$$

Describe the evolution of a typical soliton under this perturbed equation.

c) Compare ODE dynamics and PDE dynamics for a few examples in **b)**. Comment in detail what you observe!

Hints:

For the ODE numerics you can use `ode45` in Matlab. You can Google to find "simple example ode45" or go to this following site for some basic examples to use `ode45`: http://12000.org/my_notes/matlab_ODE/

On the other hand, for PDE numerics you can use the code `NLS_potential.m` and use the appropriate potential and/or add loss.

---

2. Variational approximation for LEFT-RIGHT oscillations of a Gaussian ansatz solitons inside a magnetic trap
Consider the behavior of a bright soliton trapped by a harmonic potential. This is the model for an attractive Bose-Einstein condensate (BEC) trapped inside a magnetic trap. The meanfield approximation for the BEC close to absolute zero temperature indicates that the wave function of the BEC must obey the following NLS with a potential term:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = V(x)\,u, \tag{4}$$

where

$$V(x) = \frac{\Omega^2}{2}\,x^2. \tag{5}$$

By using the Lagrangian:

$$L = \frac{i}{2}(u^* u_t - u u_t^*) - \frac{1}{2}|u_x|^2 + \frac{1}{2}|u|^4 - V(x)|u|^2 \tag{6}$$

with the ansatz

$$u_G = B\,\exp\left[-\frac{(x-\xi)^2}{2w^2}\right]\,\exp\left[i(b(t)+c(x-\xi)+d(x-\xi)^2)\right], \tag{7}$$

it is possible to find a set of coupled ODEs for all the ansatz parameters $\{B(t), w(t), \xi(t), c(t), b(t), d(t)\}$. In particular, after performing the integrations and uncoupling of the Euler-Lagrange equations one obtains the following ODEs for the parameters:

$$\begin{aligned}
\dot{B}(t) &= -B\,d, &(8)\\
\dot{w}(t) &= 2\,w\,d &(9)\\
\dot{\xi}(t) &= c &(10)\\
\dot{c}(t) &= -\Omega^2\,\xi &(11)\\
\dot{b}(t) &= -\frac{1}{2w^2}+\frac{5\sqrt{2}}{8}B+\frac{12}{2}c^2-\frac{1}{2}\Omega^2\,\xi^2 &(12)\\
\dot{d}(t) &= \frac{1}{2w^4}-2d^2-\frac{\sqrt{2}}{4}\frac{B^2}{w^2}-\frac{1}{2}\Omega^2 &(13)
\end{aligned}$$

a1) We are only interested in the ensuing LEFT-RIGHT oscillations of the soliton inside the harmonic potential. Write a single equation for the position $\xi(t)$ of the form:

$$\ddot{\xi}(t) = -\frac{d}{d\xi}V_{\text{eff}}(\xi) \qquad \text{where} \qquad V_{\text{eff}}(\xi) = \frac{\Omega_{\text{eff}}^2}{2}\,\xi^2, \tag{14}$$

where Ω_{eff} is the *effective* oscillation frequency felt by the bright soliton as a whole.

a2) What is the relation between Ω and Ω_{eff}? Interpret this result!

b) Compare the PDE dynamics with the ODE dynamics for a few parameter values of Ω. Comment in detail what you observe!

Hints:

For the ODE numerics you can use `ode45` in Matlab (you can Google to find "simple example ode45" or go to this site for some basic examples to use `ode45`: http://12000.org/my_notes/matlab_ODE/

For the PDE use the Matlab code `NLS_potential.m` posted in the Lectures (Nonlinear Waves) of our website.