HW#5 — Nonlinear Waves

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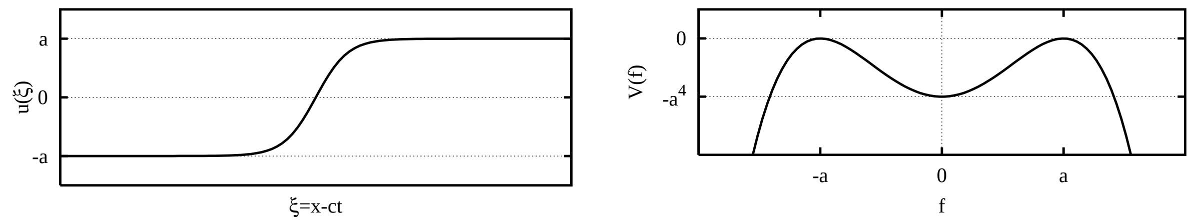
1. The aim of this exercise is to find a simple wave equation PDE model that propagates a *kink* solution with constant velocity and shape. The wave we are after looks like (left panel):



The method consists on defining a potential that generates a *heteroclinic* orbit connecting $\mp a$ with $\pm a$.

a) Consider the *quartic* potential $V(f)$ depicted on the right plot above.

  i) Sketch the phase portrait associated with $V(f)$.

  ii) Sketch all the qualitatively different orbits associated with $V(f)$.

  iii) Find the *simplest* quartic potential that satisfies the sketch in the right panel. Namely, find the coefficients A and B such that $V(f) = -f^4 + Af^2 + B$ passes through the particular points (with the right slopes) depicted in the right panel.

b) Consider now a particle of mass $m = 2$ subject to the potential $V(f)$ found in a.iii).

  i) Write the equation of motion using Newton's law ($mf'' = -dV/df$) for a particle of mass $m = 2$ at position f and time ξ under the potential $V(f)$ (i.e. f plays the role of x and ξ the role of t).

  ii) Differentiate this equation with respect to ξ to obtain a third order differential equation for f. Write a wave equation for $u(x,t) = f(\xi) = f(x - ct)$ with $c = 2a^2$ that leads to this third order differential equation for $f(\xi)$. Describe the terms you obtain (dissipation, dispersion, nonlinearity, etc..).

c) Now that you have the equation from b.ii), apply a similar analysis to the one done in class to obtain a kink solution connecting $u(-\infty, t) = -a$ and $u(+\infty, t) = +a$. Namely, you'll now check that the PDE you constructed does indeed support traveling kinks (as in the left panel of the figure). [Hint: every time you integrate w.r.t. ξ there is a constant that has to be determined by the boundary conditions. This is crucial to get the right solution. What are the boundary conditions for $f''(\pm\infty)$ and $f'(\pm\infty)$ and $f(\pm\infty)$?].

d) Check that the solution you obtained in c) does solve the wave equation obtained from b.ii).

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2. The aim of this exercise is to find approximate plane wave solutions to the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

a) Suppose that $u(x,t) = f(\xi) = f(x - vt)$ is a traveling wave solution for the KdV equation (1). Following the lecture notes, obtain a Newton's law-like equation for $f(\xi)$. Keep the integration constant intact in order to obtain the most general solution.

b) For this Newton's law-like equation, find the fixed points and study their stability. Perform a perturbation analysis around the neutrally stable fixed point f^*. Namely, consider $f(\xi) = f^* + \varepsilon g(\xi)$ with $|\varepsilon| \ll 1$. Write the differential equation for $g(\xi)$.

c) Suppose ε is small enough to neglect nonlinear terms. Write the linearized differential equation for $g(\xi)$ and solve it. Use this to find the oscillatory solution for f about the fixed point f^*. Write the approximate evolution for these waves.

d) Use `KdV.m` to integrate the approximate plane wave solution obtained in c) with amplitude=0.01 and $v = 2$ and the integration constant(s) equal to zero for $(x, t) \in [0 : 3 \text{ wave periods}] \times [0 : 5]$. Show all your plots. Does the wave evolve as expected (shape- and velocity-wise)? Perform the same simulation with amplitude=0.1 and amplitude=0.5. Notice any difference? Comment on your results. [Note: if the codes goes unstable (i.e. loads of spikes forming) reduce the time step size dt.]

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3. Consider the two-dimensional KdV equation:

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{2}$$

a) Show, using the traveling wave approach, that the two-dimensional KdV equation has the solitary wave solutions of the form:

$$u(x,t) = \frac{k^2}{2}\,\mathrm{sech}^2\left[\frac{1}{2}(kx + ly - \omega t)\right],$$

with $\omega = k^3 + 3l^2/k$.

b) Describe the evolution of such a solitary wave. What does it represent?