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Evidence of chaotic behaviour in Jordan–Brans–Dicke cosmology

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Abstract

We study numerically the properties of solutions of spatially homogeneous Bianchi-type IX cosmological models in the Jordan–Brans–Dicke theory of gravitation. Solutions are obtained in which the scale factors undergo irregular oscillations. The estimate of the maximum Lyapunov exponent is found to be positive in the cases studied. These results seem to be the first pieces of evidence in the literature of chaotic behaviour in Jordan–Brans–Dicke cosmology. The range of values of the Jordan–Brans–Dicke coupling parameter considered is $(-500, -1)$.

During the past decade there has been a growing interest in the nonlinear behaviour of cosmological models in general relativity (GR). Since the work of Barrow [1], various reports have appeared reporting chaotic behaviour in solutions of the Einstein field equations in specific cosmological models. These conclusions were obtained, either directly from the differential equations describing the model [2–6] or starting with a properly associated discrete mapping [7–9]. Recently, an ongoing discussion has started on the characterization of chaos in gravitational theories as there are seemingly contradictory pieces of evidence [5,10,11]. However, as it has been pinpointed previously and we see in this work, this problem is closely related with the freedom in choosing a time gauge in theories like the Jordan–Brans–Dicke (JBD) theory and GR. Despite all this interest, though, we do not know of any paper reporting chaotic behaviour in cosmological models based on the JBD theory of gravitation, notwithstanding the fact that this theory is one of the serious contenders of Einstein theory and in spite of the cosmological interest in models based on it [12–18]. Moreover, the (unfulfilled) possibility of finding a qualitatively different behaviour in similar models in both theories made the question extremely interesting, as the chaoticity of the GR model does not trivially imply similar behaviour in the analogous JBD model [19]. Although we do not pursue the question in this work, the stochastic behaviour of cosmological solutions may be important for devising, in the context of JBD, an explanation of the present state of the universe

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without invoking special initial conditions. On the other hand, our results for various values of the coupling constant ω may have some bearings with the extended inflation models which use a modified JBD theory in which ω is allowed to vary in an attempt to avoid the problems of standard inflationary models [20].

We report here numerical results on the dynamical behaviour of the spatially homogeneous Bianchi-type IX cosmological model in the JBD theory. This model may be considered a kind of JBD version of the "mixmaster" universe of GR [21]. In a synchronous system of coordinates this model is characterized by a line element synchronous system of coordinates this model is characterized by a line element of the form

$$ds^2 = -dt^2 = g_{ij}\omega^i\omega^j , \tag{1}$$

where ω^i are one-forms [22] expressing the symmetry properties of the three-space, the three-metric

$$g_{ij}(t) = \begin{pmatrix} [a_1(t)]^2 & 0 & 0 \\ 0 & [a_2(t)]^2 & 0 \\ 0 & 0 & [a_3(t)]^2 \end{pmatrix} \tag{2}$$

is a function only of the synchronous time t, and the $a_i(t)$ ($i=1, 2, 3$) are the scale factors of the model. With a metric of this form and assuming the universe filled with matter in the form of a barotropic fluid, $p=\beta\rho$, where p is the hydrodynamic pressure, ρ is the local density of the fluid and β is a constant, the JBD field equations lead to the following set of three equations for the scale factors,

$$\frac{\ddot{a}_i}{a_i} - \left(\frac{\dot{a}_i}{a_i}\right)^2 + \frac{\dot{a}_i}{a_i}\left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} + \frac{\dot{\phi}}{\phi}\right) + \frac{1}{a_i^2} + \frac{a_i^2}{2a_j^2a_k^2} - \frac{a_j^2}{2a_k^2a_i^2} - \frac{a_k^2}{2a_i^2a_j^2} = \mu , \tag{3}$$

where the subscripts $i, j, k=1, 2, 3$, should be taken in cyclic order, $\mu=\alpha[\rho+\omega(\rho-p)]/\phi$, $\alpha=8\pi/(3+2\omega)$, the overdot denotes a t-derivative and ω is the coupling parameter of the JBD theory. We have additionally an equation for the scalar field, ϕ, (this field may be considered proportional to the inverse of the time varying gravitational parameter ($\phi\sim 1/G$)) characterising the effect of the background universe on the inertial properties of matter or, in the context of extended inflation, as the field determining when inflation ends [20],

$$(\dot{\phi}a_1a_2a_3)^{\cdot} = \alpha(\rho-3p)a_1a_2a_3 . \tag{4}$$

These equations can be used to obtain the quantity

$$C := \frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{\dot{a}_1\dot{a}_3}{a_1a_3} + \frac{\dot{a}_2\dot{a}_3}{a_2a_3} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3}\right) - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{1}{2}\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2}\right) - \frac{1}{4}\left(\frac{a_1^2}{a_2^2a_3^2} + \frac{a_2^2}{a_3^2a_1^2} + \frac{a_3^2}{a_1^2a_2^2}\right). \tag{5}$$

In our case, C has the value $8\pi\rho/\phi$. This may be considered as a construction to be satisfied by the initial conditions. Obviously, C must vanish in a vacuum universe and, as can be easily verified by differentiating it with respect to t, in a vacuum C is a first integral of the system (3), (4). We consider starting conditions with different values of ω: we set $\omega=-500$ to be in agreement with the present day observational evidence [15,23]. We consider also other values of ω ($=-1, -5, -25, -50, -100$) to study what is the effect of varying ω in the behaviour of the model. In fact, in extended inflation models dealing with a Brans–Dicke scalar field, the interesting values are $|\omega|\leqslant 25$ and ω may be considered as a dynamical parameter [20]. In such models, there are ways of pinning down some features of the evolution – though abandoning pure JBD theory in the process – for avoiding conflicts with radar-ranging measurements which require $|\omega|\geqslant 500$ [15,16,20].

Eqs. (3) and (4) show that for the Bianchi IX model the JBD field equations become four coupled second-order differential equations for the scale factors plus the scalar field ϕ. Therefore, given a set of initial conditions ($a_i(0)$, ϕ, $\dot{a}_i$, $\dot{\phi}$, $i=1, 2, 3$), we can obtain numerically the evolution of the scale factors and the field ϕ. We choose the numerical approach given the unsuccessful analytical attempts to solve the unrestrained JBD Bianchi IX model [17]. On the other hand, there are results of the various numerical studies of Bianchi in GR [5,6] which have established the existence of stochastic behaviour in the Bianchi mixmaster GR model. We show here this kind of behaviour also occurs in Bianchi JBD models and thus we offer evidence of their very complex dynamics.

One of the ways in which chaos can be identified in a system of equations like (3) and (4) is the positivity of at least one of its Lyapunov exponents (LE). The maximum LE can be evaluated by linearizing the first-order system equivalent to (3), (4) around a fiducial solution $a_i^{\rm f}(t)$, $\dot{a}_i^{\rm f}(t)$, $\phi^{\rm f}(t)$, $\dot{\phi}^{\rm f}(t)$ of it, solving for the variations $\xi_i(t)$, $i=1, ..., 8$ ($\xi_i \equiv \delta a_i$ $i=1, 2, 3$, $\xi_4 \equiv \delta\phi$, $\xi_i \equiv \delta\dot{a}_i$, $i=5, 6, 7$, $\xi_8 \equiv \delta\dot{\phi}$) which measure the deviations from the fiducial solution and next considering the number

$$\lambda_t = \frac{1}{2t} \log \left[d_t(\xi)/d_{t0}(\xi) \right] , \qquad (6)$$

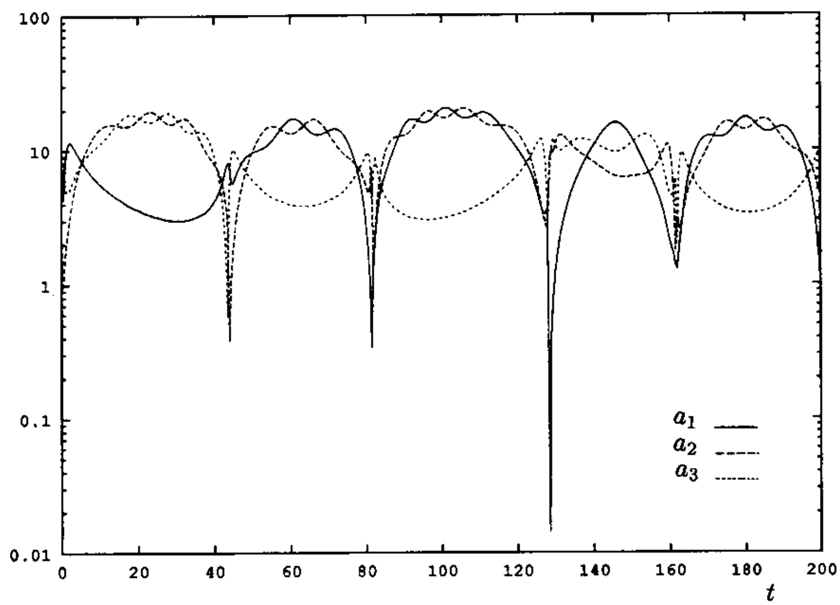

Fig. 1. Evolution of the scale factors a_1, a_2, a_3, corresponding to $\omega = -5$, plotted, in a logarithmic scale, against the synchronous time t.

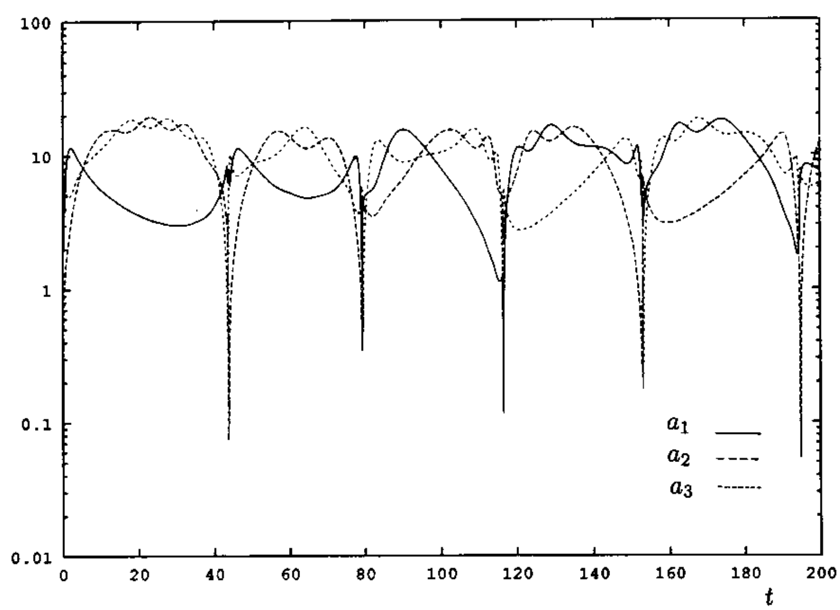

Fig. 2. Evolution of the scale factors a_1, a_2, a_3, corresponding to $\omega = -500$, plotted, in a logarithmic scale, against the synchronous time t.

where $d_t(\xi) \equiv \sum_{i=1}^{8} \xi_i^2(t)$ [24]. The Lyapunov exponent λ is defined as the limit $t \to \infty$ of Eq. (6). If λ is greater than zero, any solution in the vicinity of the fiducial one would diverge exponentially from it, thus implying that the fiducial solution itself will exhibit stochastic properties [1]. In an actual calculation, the number λ_t can serve [19,25] as an estimate of the maximum LE.

We choose for the study the starting values: $a_1 = 1.8540$, $a_2 = 0.4385$, $a_3 = 0.0854$, $\phi = 0.7129$, $\dot{a}_1 = -6.18191$, $\dot{a}_2 = -1.95165$, $\dot{a}_3 = 42.7131$ and $\dot{\phi}$ is calculated, for each ω value, from the constriction

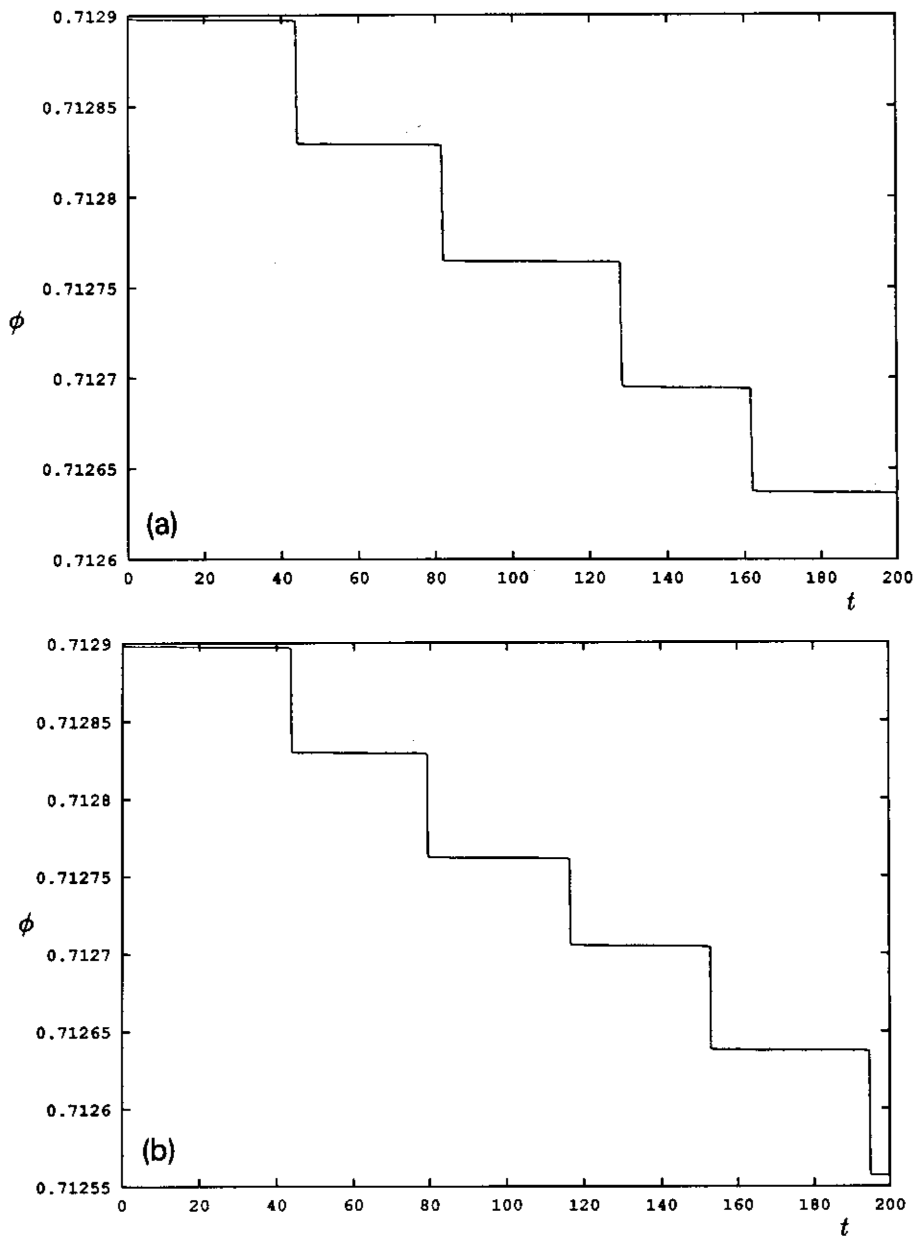

Fig. 3. The scalar field ϕ, (a) corresponding to the starting conditions with $\omega = -5$, (b) corresponding to $\omega = -500$, plotted against the synchronous time t. The staircase-like behaviour is partly due to the sampling frequency and the time scale selected for the plots.

(5). These initial conditions satisfy the constraint with $C=0$ and therefore correspond to a vacuum universe. In actual fact, the value of $\dot{\phi}$ is nearly the same ($\sim 0.1758\times 10^{-3}$) in all cases, but it has to be slightly adjusted for each one to assure $C=0$. The computations were performed on a Micro-Vax 3900 computer using extended precision arithmetic to minimize round-off errors. The algorithm used was a fifth-order Runge–Kutta one with typical time steps between 10^{-7} and 10^{-25}.

The calculated a_i are displayed in Fig. 1 for the starting values corresponding to $\omega=-5$, and in Fig. 2 for the starting values corresponding to $\omega=-500$. The evolution of the scalar field ϕ for both conditions is shown in Fig. 3. A preliminary discussion of these results was already presented in Ref. [26]. Notice that the staircase-like behaviour shown there is mainly a consequence of the scale used and of the sampling frequency of points in the numerical calculations, the graphs would become smoother if we use a contracted time scale or increase the sampling frequency. As Figs. 1 and 2 exhibit, the behaviour of the scale factors in the JBD Bianchi IX model has rather deep "bounces" in which the scale factors change by various orders of magnitude and in which the scalar field also varies, all in a very short time. But notice the smallness of the relative change in ϕ as compared with the change in a_i. Notice also a Kasner-epoch type of behaviour in which a scale factor governs the oscillations of the other two, which is analogous to that found in GR cosmologies. We have also investigated the dynamics of the model for $\omega=-1$, -25, -50, -100. We have found qualitative similarity between the results for the different values of the coupling parameter in the whole range studied. From this we may conclude that the behaviour is not strongly dependent on ω – although the value of λ depends on it, see Table 1. This is also the case for the several different initial conditions we have tried so far, thus stochastic behaviour seems to dominate the dynamics of the JBD Bianchi IX model.

The result of our estimations of the maximum LE, λ, are shown in Table 1 as a function of ω. As Table 1 illustrates, the maximum LE is positive and we must conclude that the model, though deterministic, behaves chaotically and thus that there is no possibility of forecasting (even numerically) its evolution for arbitrarily large time intervals, irrespective of the value selected for the coupling parameter. This follows since any uncertainty in the starting condition would grow exponentially with time (cf. Eq. (6)) filling the allowable phase space. Notice the possible consequences of this conclusion for early universe JBD cosmology; in the context of a more appropriate model, the chaotic irregularities would amplify any fluctuations which may seed the formation of structures. The chaotic properties may also help the model "to forget" its initial conditions.

We must point out that the main contributions to λ appear to come from the "bounces" in which one of the scale factors undergoes a fast decrease. The part of the behaviour in which this model universe shrinks and then reexpands is precisely what makes the Lyapunov exponent positive. This clearly emerges in Figs. 4a and 4b by comparing the times of occurrence of the bounces with the times at which positive contributions add to the Lyapunov exponent. A more detailed consideration of these results is in order, as the chaoticity of JBD cosmological solutions in such a large range of ω values may bear some consequences on the density perturbation spectrum of interest in inflationary cosmology [20,27]. But for these considerations to be meaningful, we have to consider to what extent this behaviour is representative of generic nonlinear effects in JBD cosmology. This is a question for which we do not have yet an answer.

On the other hand, all of these effects occur in the synchronous time t we are using in this work. But if we were to use the so-called intrinsic time Φ, defined by $d\Phi=(a_1a_2a_3)^{-1}\,dt$ [12,28], instead of the syn-

Table 1
The maximal Lyapunov exponent, λ, as a function of ω in the JBD Bianchi IX cosmological model (the relative error in λ is not easy to evaluate, but it can be estimated as at least 15%)

ω	λ
-1	0.601
-5	0.650
-25	0.714
-50	0.947
-100	0.696
-500	0.680

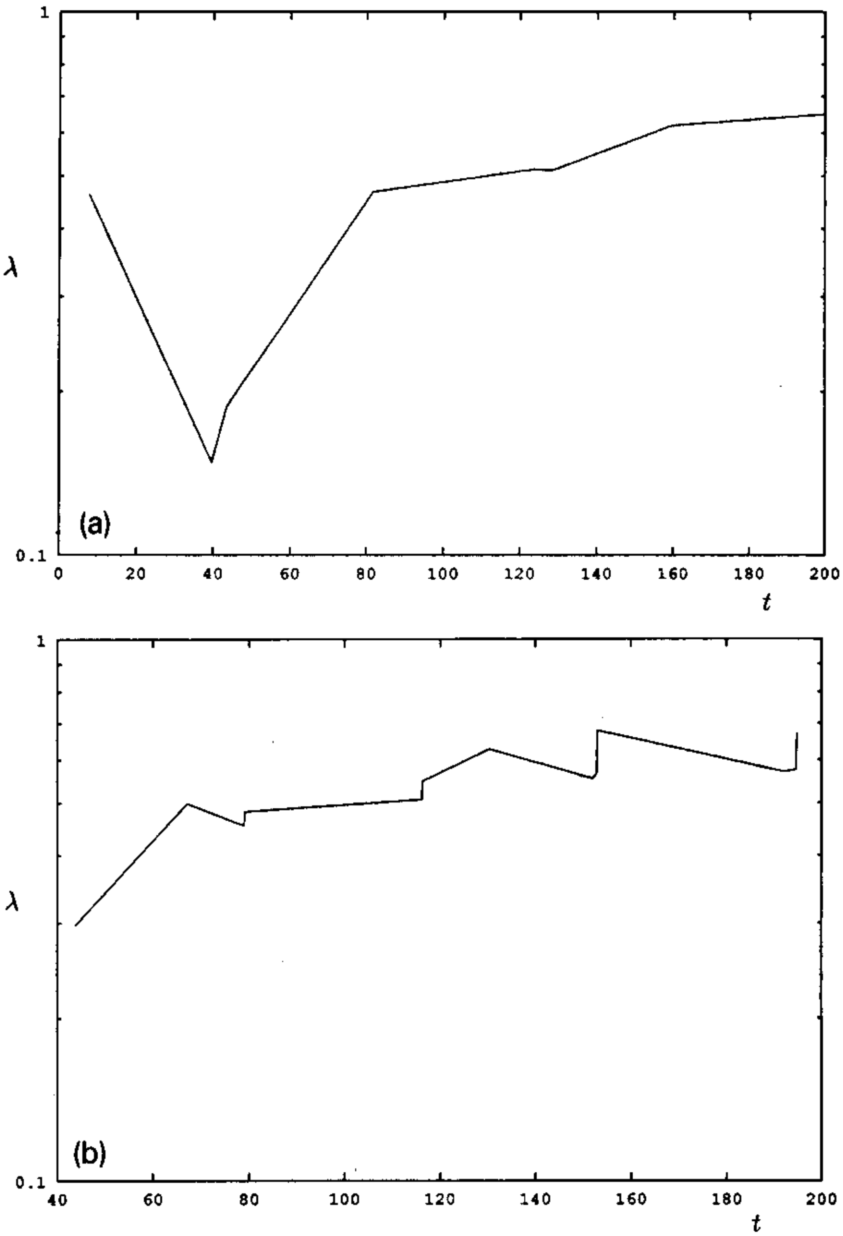

Fig. 4. Estimates of the maximum Lyapunov exponent, (a) for the starting conditions of Fig. 1, and (b) for the starting conditions of Fig. 2, as a function of the synchronous time t. Notice how the positive contributions to λ come from the bounce regions. Both plots show the positive tendency of λ_t, and thus of the maximum LE. We may safely conclude from the plots that the model is chaotic when its dynamics is studied in the synchronous time t.

chronous time t, the system would no longer behave chaotically. This fact can be understood by the following argument. From the depicted behaviour, we conclude the relationship between the two time coordinates, in the Kasner-like regimes, to be given by $\Phi \sim \log t$ [2], and since the maximum LE calculated in the synchronous time $\lambda^{(\mathrm{sinc})}=\alpha$ is positive, nearby trajectories diverge as $d_t \sim \exp(\alpha t)$. However, if we use the intrinsic time, Φ, the trajectories would diverge only as $d_\Phi \sim \Phi^\alpha$ and then the LE calculated using this time coordinate would vanish,

$$\lambda^{(\mathrm{int})}=\lim_{\Phi\to\infty}\frac{1}{\Phi}\log\frac{d_\Phi(y)}{d_{\Phi_0}(y)}\sim\lim_{\Phi\to\infty}\frac{\alpha\log\Phi}{\Phi}=0\,. \quad (7)$$

This conclusion is easily corroborated by a numerical calculation which shows that in the intrinsic time $\lambda\to 0$. This situation coincides with that found previously in GR models [5,11]. However, notice that this result *does not* necessarily imply the intergrability of the JBD Bianchi IX model and certainly it does not rule out completely the consequences which the stochastic behaviour may have for cosmological considerations. The results presented here corroborate a conclusion reached previously [11] on the concept of stochasticity in geometrical theories of gravity. According to this view, stochasticity in those theories is a concept which depends on the time scale used for describing the dynamics, thus pointing toward one of the main shortcomings of this and all other studies of gravitational stochasticity: a proper characterization of chaos in geometric theories of gravity is not yet available. This question is discussed more fully in Ref. [29].

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