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Resonant energy transfer in Bose–Einstein condensates

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Abstract

We consider the dynamics of a dilute, magnetically-trapped one-dimensional Bose–Einstein condensate whose scattering length is periodically modulated with a frequency that linearly *increases* in time. We show that the response frequency of the condensate locks to its eigenfrequency for appropriate ranges of the parameters. The locking sets in at resonance, *i.e.*, when the effective frequency of driving field is equal to the eigenfrequency, and is accompanied by a sudden increase of the oscillations amplitude due to resonant energy transfer. We show that the dynamics of the condensate is given, to leading order, by a driven harmonic oscillator on the time-dependent part of the width of the condensate. This equation captures accurately both the locking and the resonant energy transfer as it is evidenced by comparison with direct numerical simulations of original Gross–Pitaevskii equation.

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1. Introduction

Bose–Einstein condensates (BECs) [1] are one of the most appealing systems for nonlinear science due to their unprecedented experimental maneuverability and the supporting theoretical modeling. A BEC is comprised of a dilute gas of magnetically (or optically) trapped bosons that, when cooled to extremely low temperatures, occupy their lowest-energy quantum state. BECs can be manipulated in time and space. On the spatial side, one has good experimental control over the shape and the strength of the trapping potential, while on the temporal side one can modify in time the strength of the two-body scattering length [2,3] to the extent of probing the region between attractive and repulsive condensates. Theoretically, the dynamics of Bose-condensed gases is given by the so-called Gross–Pitaevskii (GP) equation [4,5], a cubic Schrödinger equation describing the $T = 0$ dynamics of the condensate. In one spatial dimension (with $\hbar = m = 1$, a convention followed throughout the rest of the paper, m being the mass of each boson) the adimensional GP equation is given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi + U|\psi|^2\psi, \quad (1)$$

where $V(x)$ is the trapping potential, here taken as $V(x) = \Omega^2 x^2/2$. The coefficient of the cubic term is $U = 4\pi a$, where a is the two-body scattering length. Notice that $N = \int |\psi(x)|^2$, the number of particles, is a dynamical invariant of Eq. (1) for general, time dependent, $V(x)$ and/or U. The GP Lagrangian is given by

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$$L(t)=\int_{-\infty}^{\infty}\mathcal{L}(t)\mathrm{d}t = \int_{-\infty}^{\infty}\left[\frac{\mathrm{i}}{2}\left(\psi\psi_t^*-\psi^*\psi_t\right)+\frac{1}{2}\left|\psi_x\right|^2+V(x)\left|\psi\right|^2 + \frac{U}{2}\left|\psi\right|^4\right]\mathrm{d}t. \tag{2}$$

Due to the advent of the Feshbach resonance [2,3] it is now possible to modulate in time the scattering length. While there are numerous results concerning the parametric resonances that take place when the scattering length is modulated as $a\left(1+\epsilon\sin\left(\omega t\right)\right)$, where $\omega=\omega_0$ is a fixed frequency (see Refs. [6–8] and references therein), very little is known about the case when the driving frequency is time *dependent*. Here we focus on the case when the driving frequency $\omega(t)$ increases linearly in time, *i.e.*, $\omega\left(t\right)=\gamma t$ with $\gamma>0$.

In this paper we show that, apart from a set of intrinsically nonlinear resonances exhibited by the Gross–Pitaevskii equation, the dynamics of the condensate is given to leading order by $\ddot{x}+x=\epsilon\sin(\tilde{\gamma}\tau^2)$, where x is a rescaled value of the time-dependent part of the width of the condensate, τ is the rescaled time and $\tilde{\gamma}$ is proportional to γ. Comparing this simplified equation with full GP numerics we see that for *small* ϵ and γ it captures accurately both the mode-locking and the resonant energy transfer that take place when the effective frequency of the driving field matches the eigenfrequency of the condensate, from now on called $\tilde{\Omega}$. More explicitly, for $t<\tilde{\Omega}/2\gamma$, the width of the condensate shows oscillations close to the effective frequency of the drive (*i.e.*, $\omega_{\mathrm{osc}}=2\gamma t$). At resonance (*i.e.*, $t\simeq\tilde{\Omega}/2\gamma$) there is a sudden increase of the amplitude of oscillations due to resonant energy transfer, while at later time the oscillations are on the condensate eigenfrequency and *not* that of the driving field (*i.e.*, $\omega_{\mathrm{osc}}=\tilde{\Omega}$). Drawing from previous studies on dissipative systems we refer to this phenomenon as *mode locking*.

We consider a dilute, magnetically-trapped condensate and a Gaussian wave-function ansatz (see Refs. [9–12] for the main results regarding the use of a Gaussian ansatz for the cubic nonlinear Schrödinger equation). After the usual variational recipe we linearize the ensuing ordinary differential equations around the equilibrium/ground-state value of the width of the condensate. The final equation (in the time-dependent part of the width of the condensate) is that of a driven harmonic oscillator. The rest of the paper is structured as follows. Section 2 is dedicated to the variational method that simplifies the condensate dynamics to an ODE. In Section 3 we analyze the equation of the driven harmonic oscillator, while in Section 4 we make a one-to-one comparison between the reduced ODE dynamics and full PDE numerics of the GP equation. Section 5 gathers our conclusions.

2. Variational recipe

We consider a dilute, magnetically-trapped BEC and a one-dimensional Gaussian-like profile as ansatz

$$\psi\left(x,t\right)=\frac{N^{1/2}}{\pi^{1/4}w\left(t\right)^{1/2}} \times\exp\left[-\frac{x^2}{2w\left(t\right)^2}+\mathrm{i}x^2\beta\left(t\right)+\mathrm{i}\phi\left(t\right)\right], \tag{3}$$

where N is the number of atoms in the cloud, ϕ is an overall phase (the canonical conjugate of N), w is the width of the condensate while β, the so-called chirp, is the canonical conjugate of w [9–12]. The above trial wave-function yields the Lagrangian

$$L(t)=\frac{N}{2}w^2\dot{\beta}+N\dot{\phi}+\frac{N^2U}{2\sqrt{2\pi}w}+\frac{N\Omega^2w^2}{4} +\frac{N}{4w^2}+Nw^2\beta^2.$$

The Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}}\right)=\frac{\partial L}{\partial q}, \tag{4}$$

give, for $q\in\{\phi,w,\beta\}$,

$$\frac{\mathrm{d}N}{\mathrm{d}t}=0, \tag{5}$$

i.e., the conservation of particles, for $q=\phi$,

$$w\dot{\beta}+\frac{\Omega^2w}{2}-\frac{1}{2w^3}-\frac{NU}{2\sqrt{2\pi}w^2}+2w\beta^2=0, \tag{6}$$

for $q=w$ and

$$\dot{w}=2w\beta, \tag{7}$$

for $q=\beta$. Following some straightforward algebraic manipulations, the previous can be combined into

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2}+\Omega^2w=\frac{1}{w^3}+\frac{UN}{\sqrt{2\pi}w^2}. \tag{8}$$

This equation also holds in the case of time-dependent magnetic traps and that of a time-dependent scattering length, *i.e.*, the equation is left unchanged by $U\to U(t)$ and $\Omega^2\to\Omega^2(t)$. Introducing now the rescaled variables $P=UN/\sqrt{2\pi}\Omega$, $\tau=\Omega t$ and $w=v/\sqrt{\Omega}$, Eq. (8) reads

$$\frac{\mathrm{d}^2v}{\mathrm{d}\tau^2}+v=\frac{1}{v^3}+\frac{P}{v^2}. \tag{9}$$

The equilibrium width is given by

$$\tilde{v}=\frac{1}{\tilde{v}^3}+\frac{P}{\tilde{v}^2}, \tag{10}$$

which has only one positive solution for $P>0$. Around the equilibrium point $\tilde{v}$, $v=\tilde{v}+\delta$, the dynamics for the perturbation δ is given by

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}\tau^2}+\delta\left(1+\frac{3}{\tilde{v}^4}+\frac{2P}{\tilde{v}^3}\right)=0, \tag{11}$$

indicating a period of $T=2\pi/\omega_P$ for the perturbation, where the natural eigenfrequency of the system is given by

$$\omega_P=\sqrt{1+\frac{3}{\tilde{v}^4}+\frac{2P}{\tilde{v}^3}}. \tag{12}$$

Modulating the scattering length as $a\,(1+\epsilon\sin(\omega\,(t)\,t))$, with $\omega\,(t) = \gamma t$, i.e., $U(t) = U(1+\epsilon\sin(\omega(t)t))$ and subsequently $P(\tau) = P(1+\epsilon\sin(\omega(\tau)\tau))$, and following the above steps one has

$$\ddot{x}+x=\epsilon\sin\left(\tilde{\gamma}\tilde{\tau}^2\right), \tag{13}$$

where $x = \delta\omega_P^2\tilde{v}^2/P$, $\dot{x} = \mathrm{d}x/\mathrm{d}\tilde{\tau}$, with $\tilde{\tau} = \omega_P\tau$, and $\tilde{\gamma}=\gamma/\Omega^2\omega_P^2$, and we have discarded a term proportional to ϵx as being second order, as both ϵ and x are small. The boundary conditions are taken as $x(0)=0$ and $\dot{x}(0)=0$, indicating that the condensate was initially at rest. For simplicity, we will omit in what follows the tilde on $\tilde{\tau}$ and $\tilde{\gamma}$ when referring to x.

Finally, let us notice that

$$w(t)=w_{\mathrm{eq}}+\frac{UNx(t)}{\sqrt{2\pi}\,\Omega\tilde{v}^2}\left(1+\frac{2}{\tilde{v}^4}+\frac{2P}{\tilde{v}^3}\right)^{-1}, \tag{14}$$

where w_{eq} is the equilibrium value of the width, *i.e.*, $\tilde{v}/\sqrt{\Omega}$. In order for the linear approximation to hold the second term on the right hand side of Eq. (14), *i.e.*, the deviation from the equilibrium value of the width of the condensate has to be numerically small.

3. Driven harmonic oscillator

3.1. Analytical solution

In this section we analyze the equation of the driven harmonic oscillator. The solution of Eq. (13) is given by

$$\begin{aligned} x = \frac{\epsilon\sqrt{2\pi}}{4\sqrt{\gamma}}\Bigg\{ & -2\cos\tau\cos\left(\frac{1}{4\gamma}\right)C\left(\frac{1}{\sqrt{2\pi\gamma}}\right) \\ & -2\cos\tau\sin\left(\frac{1}{4\gamma}\right)S\left(\frac{1}{\sqrt{2\pi\gamma}}\right) \\ & +\cos\tau\cos\left(\frac{1}{4\gamma}\right)C(z_+)-\cos\tau\cos\left(\frac{1}{4\gamma}\right)C(z_-) \\ & +\sin\tau\cos\left(\frac{1}{4\gamma}\right)S(z_+)+\sin\tau\cos\left(\frac{1}{4\gamma}\right)S(z_-) \\ & +\cos\tau\sin\left(\frac{1}{4\gamma}\right)S(z_+)-\cos\tau\sin\left(\frac{1}{4\gamma}\right)S(z_-) \\ & -\sin\tau\sin\left(\frac{1}{4\gamma}\right)C(z_+)-\sin\tau\sin\left(\frac{1}{4\gamma}\right)C(z_-)\Bigg\}, \end{aligned} \tag{15}$$

where $z_\pm \equiv (\pm 1+2\tau\gamma)/\sqrt{2\pi\gamma}$ and S and C are the well-known Fresnel functions defined as

$$S(z)=\int_0^z \sin\left(\frac{\pi}{2}t^2\right)\mathrm{d}t,$$

and

$$C(z)=\int_0^z \cos\left(\frac{\pi}{2}t^2\right)\mathrm{d}t.$$

Before stepping into the analysis of Eq. (15) let us first notice that in order for the linear analysis to hold we need

$$f=\frac{UN\epsilon}{4\sqrt{\gamma}\,\Omega\tilde{v}^2}\left(1+\frac{2}{\tilde{v}^4}+\frac{2P}{\tilde{v}^3}\right)^{-1} \tag{16}$$

to be numerically small. This follows trivially by replacing $x(t)$ in Eq. (14).

In the subsequent analysis, we will investigate the form of $x(t)$ for *small* values of γ. In all the cases detailed below this limit should be implicitly understood as follows: small values of γ and a sufficiently small value of ϵ such that f is numerically small, therefore the linear analysis holds.

Analyzing Eq. (15) we see that for *small* γ and $\tau < 1/2\gamma$ the solution is dominated by the Fresnel functions, therefore the small-amplitude oscillations of x follow the frequency of the driving field. Following a set of elementary (though somewhat tedious) manipulations (detailed in the Appendix) one has that, for small γ and $\tau < 1/2\gamma$,

$$x=\frac{\epsilon}{2}\sin\left(\gamma\tau^2\right). \tag{17}$$

For $\tau > 1/2\gamma$ (with small γ) x reduces to

$$x=-\frac{\epsilon\sqrt{\pi}}{2\sqrt{2\gamma}}\left[\cos\left(\tau-\frac{1}{4\gamma}\right)-\sin\left(\tau-\frac{1}{4\gamma}\right)\right], \tag{18}$$

which is the so-called *mode-locked response* of the condensate [13]. We emphasize that this mode-locked response takes place for *small* values of γ (such that the above limits hold) *and* small values of f (such that the harmonic oscillator picture, *i.e.*, approximating Eq. (9) with Eq. (11), holds).

In the vicinity of $\tau = 1/2\gamma$ the Fresnel functions having $z_- = (-1+\tau 2\gamma)/\sqrt{2\pi\gamma}$ as argument resemble a step function and are responsible for the sudden increase in the oscillations amplitude. Physically, this is a resonant energy transfer that takes place at

$$t_c=\frac{\Omega\omega_P}{2\gamma}=\frac{\tilde{\Omega}}{2\gamma}, \tag{19}$$

where $\tilde{\Omega}=\Omega\omega_P$ is the eigenfrequency of the condensate.

Apart from the cumbersome full solution of the driven oscillator this later property can be understood through the so-called auto-resonance that is discussed in the next section.

3.2. Mode-locking

To explain why in the *small* γ limit the oscillator responds on its eigenfrequency for $\tau > 1/2\gamma$ we follow the recipe of auto-resonance phenomena put forward by Friedland and collaborators (see Ref. [14], and references therein). We will show (see also [15]) that close to resonance the second-order equation describing the oscillator is equivalent to those of a virtual particle trapped in a finite-depth energy-minimum of an effective potential.

Before stepping into the actual mode-locking analysis, let us stress that the skeleton of this approach is not new [15]. The novelty of this section comes, however, from our use of the Eq. (15), which substantiates an assumption that was previously based merely on numerical computations [cf., discarding the last term in Eq. (23)].

Taking $x = a\,(\tau)\sin\varphi\,(\tau)$ and discarding the second derivative of a with respect to τ, the so-called adiabatic assumption (*i.e.*, small γ), one has

$$\mathrm{i}2\dot{a}\dot{\varphi} + \mathrm{i}a\ddot{\varphi} - a\dot{\varphi}^2 + a = \epsilon \exp\left(\mathrm{i}\gamma\tau^2 - \mathrm{i}\varphi\right),$$

where $\dot{a} = \mathrm{d}a/\mathrm{d}\tau$. Equating real and imaginary parts we obtain

$$\begin{cases} a - a\dot{\varphi}^2 = \epsilon\cos(\gamma\tau^2 - \varphi), \\ 2\dot{a}\dot{\varphi} + a\ddot{\varphi} = \epsilon\sin(\gamma\tau^2 - \varphi). \end{cases} \quad (20)$$

Examining now the case close to the resonance, *i.e.*, we limit the analysis to a vicinity of τ_c such that

$$\dot{\varphi}(\tau_c) = 1,$$

the previous equations yield

$$\begin{cases} 1 - \dot{\varphi} = \dfrac{\epsilon}{2a}\cos(\gamma\tau^2 - \varphi), \\ \dfrac{\mathrm{d}a^2}{\mathrm{d}\tau} = a\epsilon\sin(\gamma\tau^2 - \varphi). \end{cases} \quad (21)$$

Defining the action $I = a^2$ and the phase mismatch $\Phi = \gamma\tau^2 - \varphi$ variables we can recast the previous equations as

$$\begin{cases} \dot{\Phi} = 2\gamma\tau - 1 + \dfrac{\epsilon}{2\sqrt{I}}\cos\Phi, \\ \dot{I} = \epsilon\sqrt{I}\sin\Phi. \end{cases} \quad (22)$$

In order for the condensate to stay mode-locked Φ must be close to 0 or π and the right hand side of the first equation of (22) should be equal to zero, *i.e.*,

$$\dot{\Phi} = 0 = 2\gamma\tau - 1 - \frac{\epsilon}{2\sqrt{I_0}}\cos\tilde{\Phi} \quad (23)$$

where I_0 is the usual equilibrium action while $\tilde{\Phi}$ is the equilibrium phase-mismatch. Notice that $\dot{\Phi} = 0$ amounts to τ_c roughly equal to $\omega_P/2\gamma$, as the last term in Eq. (23) can be neglected for small γ. This latter claim can be supported either by numerical arguments (as in Ref. [15]) or by analytical ones; in particular, using the analytical solution of x one has that $\sqrt{I_0}$ goes as $\epsilon/\sqrt{\gamma}$ (see the prefactor in Eq. (15)), therefore the last term in Eq. (23) goes as $\sqrt{\gamma}$ and can be safely ignored for small γ.

The solution of interest is $\tilde{\Phi} = \pi$, for $\tilde{\Phi} = 0$ corresponds to an energy maximum, as will be shown below.

Let us set $I = I_0 + \Delta$ and $\Phi = \tilde{\Phi} + \phi$, where Δ and ϕ are small. It then yields from Eq. (23)

$$\frac{\mathrm{d}I_0}{\mathrm{d}\tau} = \frac{8\gamma I_0^{3/2}}{\epsilon\cos\tilde{\Phi}}. \quad (24)$$

Setting Φ to $\tilde{\Phi}$ on the right hand side of the first equation of (22) we have

$$\dot{\phi} = -\frac{\epsilon\Delta}{4I_0^{3/2}}\cos\tilde{\Phi}. \quad (25)$$

Finally, the dynamics around the equilibrium is given by the following Hamiltonian system

$$\begin{cases} \dot{\phi} = -\cos\tilde{\Phi}\Delta B, \\ \dot{\Delta} = A\cos\tilde{\Phi}\sin\phi - 2\gamma/B\cos\tilde{\Phi}, \end{cases}$$

where $A = \sqrt{I_0}\epsilon$ and $B = \epsilon/4I_0^{3/2}$. The associated Hamilton function is

$$\mathcal{H}(\Delta, \phi) = -\cos\tilde{\Phi}\frac{B\Delta^2}{2} + V(\phi),$$

where the potential is given by

$$V(\phi) = A\cos\tilde{\Phi}\cos\phi + \frac{2\gamma\phi}{B\cos\tilde{\Phi}}.$$

The mode-locking is now transparent: for $\tilde{\Phi} = \pi$ there is an energy minimum around $\phi = 0$ corresponding to the system oscillating on its eigenfrequency, while for $\tilde{\Phi} = 0$ there is an energy maximum of no physical interest around $\phi = 0$.

4. Numerical results

In this section we compare the width of the condensate obtained from the full partial-differential GP equation with the approximate formula derived from the harmonic-oscillator picture. Our main result is that for small amplitudes of the driving field the harmonic oscillator picture captures quantitatively both the mode-locking and the resonant energy transfer. In Fig. 1 we depict the dynamics of the width of a typical low-density condensate obtained from the full GP simulation (upper panel) and compare it with our analytical result, Eq. (15) —see the middle panel. The relative error (shown in the lower panel) is seen to be numerically small. As one can easily see by inspecting Eq. (15) the resonant energy transfer takes places at $t_c = \Omega\omega_P/2\gamma$ where the solution of the driven harmonic oscillator has a step-like behavior stemming from the Fresnel functions of argument $z_- = (-1 + 2\tau\gamma)/\sqrt{2\pi\gamma}$ (see Eq. (15)).

In Fig. 2 we show the quantitative agreement between the short- and long-time asymptotic of Eq. (15), *i.e.*, Eqs. (17) *and* (18), and the full GP simulation. The upper plot shows the region where the condensate responds on the effective frequency of the driving field, *i.e.*, $w \propto \sin\left(\gamma t^2\right)$, while the lower panel shows the region where the condensate responds on its natural frequency, *i.e.*, $w \propto \sin(\tilde{\Omega}t)$. Notice in both cases the good agreement between the full numerics and the simplified model. Naturally, higher values of ϵ show our model to be oversimplified as the wave-functions develops a visible non-Gaussian structure [16].

Investigating the dynamics of the condensate long after the first resonance one sees a set of intrinsically nonlinear resonances (not shown here—see Refs. [15,17]) which fall outside this simple model. The crucial dynamics, however, takes place *around* the first resonance when the condensates shifts from following the effective frequency of the driving field to responding on its natural frequency, a process that is referred to as *mode-locking* [15].

5. Conclusions

We have investigated the dynamics of a dilute, magnetically-trapped one-dimensional Bose–Einstein condensate whose scattering length is driven as $a(1 + \epsilon\sin(\omega(t)t))$, where $\omega(t)$ increases linearly in time, *i.e.*, $\omega(t) = \gamma t$. Solving numerically

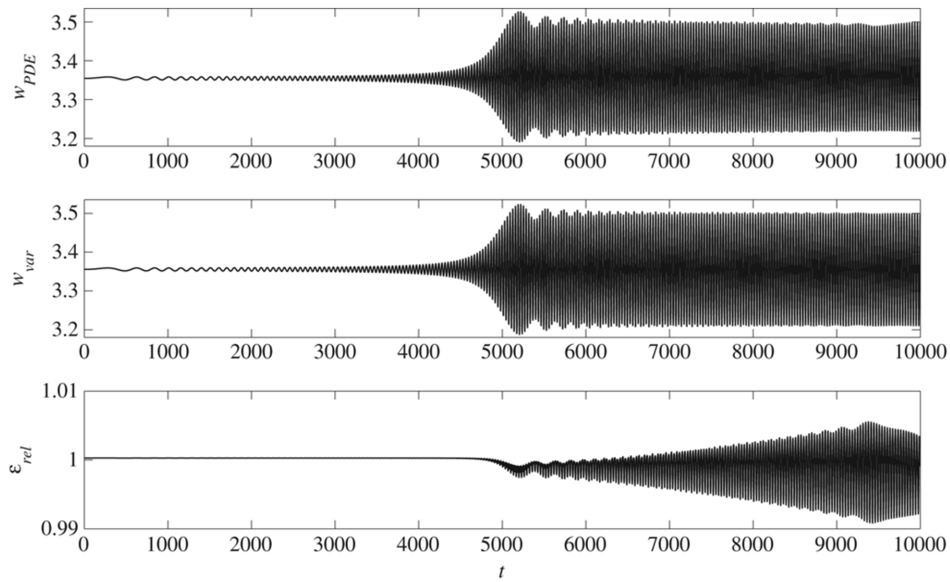

Fig. 1. Dynamics of the width of the condensate for $N = 0.2$, $U = 1$, $\Omega = 0.1$, $\epsilon = 0.02$, and $\gamma = 0.00002$. Using Eqs. (10), (12) and (16) one finds $f = 0.183$, therefore the harmonic oscillator picture is valid. In the upper panel we have depicted w_{PDE}, the width of the condensate obtained by fitting with a Gaussian function the density profile obtained from Eq. (1), while the middle panel shows w_{var}, the width of the condensate obtained from full numerical simulation of Eq. (15). The lower panel shows the relative error, *i.e.*, $\varepsilon_{\mathrm{rel}} = w_{\mathrm{PDE}}/w_{\mathrm{var}}$.

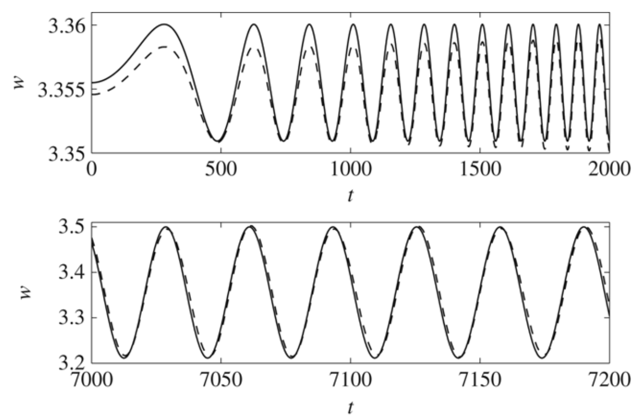

Fig. 2. Short- and long-time dynamics of the width of the condensate for $N = 0.2$, $U = 1$, $\Omega = 0.1$, $\epsilon = 0.02$, and $\gamma = 0.00002$. The upper panel depicts (in dashed line) the short-time dynamics of the width of the condensate obtained from Eq. (1) (upon fitting the density profile with a Gaussian) and the asymptotic value of the width as predicted by Eq. (17) (in full line). The lower panel depicts (in dashed line) the long-time dynamics of the width of the condensate obtained from Eq. (1) (upon fitting the density profile with a Gaussian) and the the asymptotic value of the width as predicted by Eq. (18) (in full line).

the GP equation we have shown that the response frequency of the condensate locks to its eigenfrequency at resonance (for *small* values of ϵ and γ). The locking is accompanied by a sudden increase in the oscillations amplitude due to resonance energy transfer. We show, using a variational ansatz approach, that apart from a set of intrinsically nonlinear resonances (not shown here—see Refs. [15,17,18]), the dynamics of the condensate is given to leading order by $\ddot{x} + x = \epsilon \sin(\tilde{\gamma}\tau^2)$, where x is the rescaled value of the time-dependent part of the width of the condensate, τ is the rescaled time and $\tilde{\gamma}$ is proportional to γ. This equation captures accurately both the locking and the resonant energy transfer.

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Appendix. Short-time dynamics

Let us define

$$\begin{aligned}
S_1 &= -2\cos\tau\cos\left(\frac{1}{4\gamma}\right)C\left(\frac{1}{\sqrt{2\pi\gamma}}\right) \\
&\quad - \cos\tau\cos\left(\frac{1}{4\gamma}\right)\cos\left(\frac{-1+2\tau\gamma}{\sqrt{2\pi\gamma}}\right) \\
&\quad + \cos\tau\cos\left(\frac{1}{4\gamma}\right)\cos\left(\frac{1+2\tau\gamma}{\sqrt{2\pi\gamma}}\right), \\
S_2 &= \sin\tau\cos\left(\frac{1}{4\gamma}\right)S\left(\frac{-1+2\tau\gamma}{\sqrt{2\pi\gamma}}\right) \\
&\quad + \sin\tau\cos\left(\frac{1}{4\gamma}\right)S\left(\frac{1+2\tau\gamma}{\sqrt{2\pi\gamma}}\right), \\
S_3 &= -2\cos\tau\sin\left(\frac{1}{4\gamma}\right)S\left(\frac{1}{\sqrt{2\pi\gamma}}\right) \\
&\quad - \cos\tau\sin\left(\frac{1}{4\gamma}\right)S\left(\frac{-1+2\tau\gamma}{\sqrt{2\pi\gamma}}\right) \\
&\quad + \cos\tau\sin\left(\frac{1}{4\gamma}\right)S\left(\frac{1+2\tau\gamma}{\sqrt{2\pi\gamma}}\right), \\
S_4 &= -\sin\tau\sin\left(\frac{1}{4\gamma}\right)C\left(\frac{-1+2\tau\gamma}{\sqrt{2\pi\gamma}}\right) \\
&\quad - \sin\tau\sin\left(\frac{1}{4\gamma}\right)C\left(\frac{1+2\tau\gamma}{\sqrt{2\pi\gamma}}\right).
\end{aligned}$$

Expressing the Fresnel functions in their integral form one has after a few straightforward algebraic manipulations that

$$S = S_1 + S_2 + S_3 + S_4 = \int_{\frac{-1+2\tau\gamma}{\sqrt{2\pi\gamma}}}^{-\frac{1}{\sqrt{2\pi\gamma}}} \cos\left(\frac{\pi}{2}y^2 - \frac{1}{4\gamma} + \tau\right) \mathrm{d}y + \int_{\frac{1}{\sqrt{2\pi\gamma}}}^{\frac{1+2\tau\gamma}{\sqrt{2\pi\gamma}}} \cos\left(\frac{\pi}{2}y^2 - \frac{1}{4\gamma} - \tau\right) \mathrm{d}y.$$

Following a change of variables one has that

$$S = \frac{1}{\sqrt{2\pi\gamma}} \left(\int_{-1+2\gamma\tau}^{-1} \cos\left(\frac{y^2 - 1 + 4\gamma\tau}{4\gamma}\right) \mathrm{d}y + \int_{1}^{1+2\gamma\tau} \cos\left(\frac{y^2 - 1 - 4\gamma\tau}{4\gamma}\right) \mathrm{d}y \right).$$

For small values of γ and short times one has that $2\gamma\tau$ is small, therefore a good approximation of S is obtained by linearizing y^2 around $-1 + 2\gamma\tau$ and $1 + 2\gamma\tau$ (for the first and second integral respectively). After a trivial integration one has that

$$S \approx \sqrt{\frac{2}{\pi}} \sqrt{\gamma} \sin\left(\gamma\tau^2\right), \tag{A.1}$$

which is the short time asymptotic of Eq. (15).

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